Geometry of linear operators
Norm preserving operators

- Orthogonal $\leftrightarrow$ dot product preserving $\rightarrow$ angle preserving, orthogonality preserving

**Theorem 6.2.1** If $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear operator on $\mathbb{R}^n$, then the following statements are equivalent.

(a) $\|T(x)\| = \|x\|$ for all $x$ in $\mathbb{R}^n$. \[T \text{ orthogonal (i.e., length preserving)}\]

(b) $T(x) \cdot T(y) = x \cdot y$ for all $x$ and $y$ in $\mathbb{R}^n$. \[T \text{ is dot product preserving.}\]

**Proof:** (a)$\rightarrow$(b). $\|x+y\|^2 = (x+y) \cdot (x+y), \|x-y\|^2 = (x-y) \cdot (x-y)$.

- By adding, we obtain $\frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) = (x \cdot y)$.
- $T(x).T(y) = \frac{1}{4}(\|T(x+y)\|^2 - \|T(x-y)\|^2) = \frac{1}{4}(\|T(x+y)\|^2 - \|T(x-y)\|^2) = (x \cdot y)$.

(b)$\rightarrow$(a) omit
Orthogonal operators preserve angles and orthogonality

- $\Theta = \arccos(x.y/(||x|| ||y||))$.

- If $T$ is an orthogonal transformation $\mathbb{R}^n \rightarrow \mathbb{R}^n$, then
  $\angle(Tx,Ty) = \arccos(Tx.Ty/||Tx|| ||Ty||)$
  $= \arccos(x.y/||x|| ||y||) = \angle(x,y)$.

- Thus the orthogonal maps preserve angles and in particular orthogonal pair of vectors.

- Rotations and reflections are orthogonal maps.

- An orthogonal projection is not an orthogonal map.

- The angle preserving means $k$ times an orthogonal map.
Orthogonal matrices

**Definition 6.2.2** A square matrix $A$ is said to be *orthogonal* if $A^{-1} = A^T$. 

- Or $AA^T = I$ or $A^T A = I$. 
- Orthogonal matrix is always nonsingular. 
- Example: Rotation and reflection matrices.

$$
\begin{bmatrix}
  \cos \theta & -\sin \theta \\
  \sin \theta & \cos \theta 
\end{bmatrix}
\begin{bmatrix}
  \cos \theta & \sin \theta \\
  -\sin \theta & \cos \theta 
\end{bmatrix} = I
$$

- $T_A$ is orthogonal $\iff$ $A$ is orthogonal: to be proved later
Theorem 6.2.3

(a) The transpose of an orthogonal matrix is orthogonal.
(b) The inverse of an orthogonal matrix is orthogonal.
(c) A product of orthogonal matrices is orthogonal.
(d) If $A$ is orthogonal, then $\det(A) = 1$ or $\det(A) = -1$.

Proof

(a) $A^TA = I$. $A^T(A^T)^T = I$. $A^T$ is orthogonal.
(b) $(A^{-1})^T = (A^T)^T = A = (A^{-1})^{-1}$. $A^{-1}$ is orthogonal.
(c), (d) omit.
Theorem 6.2.4  If A is an m × n matrix, then the following statements are equivalent.

(a) \( A^T A = I \).
(b) \( \| A x \| = \| x \| \) for all \( x \) in \( \mathbb{R}^n \).
(c) \( A x \cdot A y = x \cdot y \) for all \( x \) and \( y \) in \( \mathbb{R}^n \).
(d) The column vectors of \( A \) are orthonormal.

Proof: (a)->(b): \( \| A x \|^2 = A x \cdot A x = x \cdot A^T A x = x \cdot x = \| x \|^2 \).
(b)->(c): Theorem 6.2.1. with \( T(x) = A x \).
(c)->(d): \( e_1, e_2, \ldots, e_n \) are orthonormal. Since \( A e_i \cdot A e_j = e_i \cdot e_j \) for all \( i \) and \( j \), \( A e_1, A e_2, \ldots, A e_n \) are orthonormal (see p.22-23). These are column vectors of \( A \).
(d)->(a): \( i j \)-th term of \( A^T A = a_i^T a_j = a_i \cdot a_j \) where \( a_i \) is the \( i \)th column of \( A \). This is 1 if \( i=j \). 0 otherwise.
Theorem 6.2.5  If $A$ is an $n \times n$ matrix, then the following statements are equivalent.

(a) $A$ is orthogonal.
(b) $\|Ax\| = \|x\|$ for all $x$ in $\mathbb{R}^n$.
(c) $Ax \cdot Ay = x \cdot y$ for all $x$ and $y$ in $\mathbb{R}^n$.
(d) The column vectors of $A$ are orthonormal.
(e) The row vectors of $A$ are orthonormal.

Proof: This is 6.2.4.

(e) Since the transpose of $A$ is also orthogonal.
An operator $T$ is orthogonal if and only if $\|T(x)\| = \|x\|$ for all $x$.

Thus, $\|Ax\| = \|x\|$ for all $x$ for the matrix $A$ of $T$.

Hence, we have by Theorem 6.2.5.

**Theorem 6.2.6** A linear operator $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is orthogonal if and only if its standard matrix is orthogonal.

**Theorem 6.2.7** If $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is an orthogonal linear operator, then the standard matrix for $T$ is expressible in the form

$$R_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \quad \text{or} \quad H_{\theta/2} = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix} \quad (10)$$

That is, $T$ is either a rotation about the origin or a reflection about a line through the origin.
Contraction and dilations of $\mathbb{R}^2$

- $T(x,y) = (kx, ky)$.
- $T$ is a contraction of $0 \leq k < 1$.
- $T$ is a dilation of $k > 1$.
- **Horizontal compression** with factor $k$: $T(x,y) = (kx, y)$ if $0 \leq k < 1$.
  - Horizontal expansion if $k > 1$.
- **Vertical compression**: $T(x,y) = (x, ky)$ if $0 \leq k < 1$.
  - Vertical expansion if $k > 1$. 
Shearing in the x-direction with factor k:

\[ T(x,y) = (x+ky, y) \]. This sends \((x,y)\) to \((x+ky, y)\).

Thus, it preserves y-coordinates and changes x coordinate by an amount proportional to y.

This sends a vertical line to a line of slope \(1/k\).

Shearing in the y-direction with factor k:

\[ T(x,y) = (x, y+kx) \]. This send \((x,y)\) to \((x, y+kx)\).

Thus it preseves x-coordinates and changes y-coordinates by an amount proportional to x.

This sends a horizontal line to a line of slope \(k\).

Example 6.
Linear operators on $\mathbb{R}^3$.

- A orthogonal transformations in $\mathbb{R}^3$ is classified:
  - A rotation about a line through the origin.
  - A reflection about a plane through the origin.
  - A rotation about a line $L$ through the origin composed with a reflection about the plane $P$ through the origin perpendicular to $L$.
- The first has $\det = 1$ and the other have determinant $-1$.
- Examples: Table 6.2.5.
- For rotations, the axis of rotation is the line fixed by the rotation. We obtain direction by $u = wxT(w)$ for $w$ in the perpendicular plane.
- Table 6.2.6.
General rotations

**Theorem 6.2.8** If \( \mathbf{u} = (a, b, c) \) is a unit vector, then the standard matrix \( R_{\mathbf{u}, \theta} \) for the rotation through the angle \( \theta \) about an axis through the origin with orientation \( \mathbf{u} \) is

\[
R_{\mathbf{u}, \theta} = \begin{bmatrix}
    a^2(1 - \cos \theta) + \cos \theta & ab(1 - \cos \theta) - c \sin \theta & ac(1 - \cos \theta) + b \sin \theta \\
    ab(1 - \cos \theta) + c \sin \theta & b^2(1 - \cos \theta) + \cos \theta & bc(1 - \cos \theta) - a \sin \theta \\
    ac(1 - \cos \theta) - b \sin \theta & bc(1 - \cos \theta) + a \sin \theta & c^2(1 - \cos \theta) + \cos \theta
\end{bmatrix}
\]

(13)
Suppose $A$ is a rotation matrix. To find out the axis of rotation, we need to solve $(I-A)x=0$.

- Once we know the line $L$ of fixed points, we find the perpendicular plane $P$ and a vector $w$ in it.
- Form $wxAw$. That is the direction of $L$.
- The angle of rotation is
  \[ \text{Angle}(w,Aw) = \text{ArcCos}\left(\frac{w.Aw}{||w||||Aw||}\right) \]
- This is always less than or equal to $\pi$.

Example 7.

- Actually, this is computable by $\cos \theta = (\text{tr}(A)-1)/2$ by using formular (13). Details omitted.
- We can also use $v = Ax + A^tx + [I-\text{tr}(A)]x$. $x$ any vector, $v$ is the axis direction.