# BASIC POLYNOMIAL INVARIANTS, FUNDAMENTAL REPRESENTATIONS AND THE CHERN CLASS MAP

S. BAEK, E. NEHER, AND K. ZAINOULLINE

## Introduction

Consider a crystallographic root system $\Phi$ together with its Weyl group $W$ acting on the weight lattice $\Lambda$ of $\Phi$. Let $\mathbb{Z}[\Lambda]^W$ and $S^*(\Lambda)^W$ be the $W$-invariant subrings of the integral group ring $\mathbb{Z}[\Lambda]$ and the symmetric algebra $S^*(\Lambda)$. A celebrated theorem of Chevalley says that $\mathbb{Z}[\Lambda]^W$ is a polynomial ring over $\mathbb{Z}$ in classes of fundamental representations $\rho_1, \ldots, \rho_n$ and $S^*(\Lambda)^W \otimes \mathbb{Q}$ is a polynomial ring over $\mathbb{Q}$ in basic polynomial invariants $q_1, \ldots, q_n$, where $n = \mathrm{rank}(\Phi)$. Another classical result due to Demazure says that the kernels of characteristic maps $\mathbb{Z}[\Lambda] \to K_0(X)$ and $S^*(\Lambda) \to \mathrm{CH}^*(X)$, where $X$ is the variety of Borel subgroups of the associated linear algebraic group, are generated by non-constant $W$-invariants. This fact provides a link between combinatorics of the $W$-action on $\mathbb{Z}[\Lambda]$ and $S^*(\Lambda)$ and the respective cohomology rings.

In the present paper we establish and investigate the relationship between $\rho_i$'s and $q_i$'s. To do this we introduce an equivariant analogue of the Chern class map $\phi_i$ that provides an isomorphism between the truncated rings $\mathbb{Z}[\Lambda]/I_m^j$ and $S^*(\Lambda)/I_a^j$ modulo powers of the respective augmentation ideals. This allows us to express basic polynomial invariants in terms of fundamental representations and vice versa, hence, relating the representation theory of the respective Lie algebra $\mathfrak{g}$ with the geometry of the variety of Borel subgroups $X$.

A multiple of $\phi_i$ restricted to the respective cohomology ($K_0$ and $\mathrm{CH}^*$) of $X$ gives the classical Chern class map $c_i\colon K_0(X) \to \mathrm{CH}^i(X)$. This geomeric interpretation provides a powerful tool to compute the annihilators of the torsion of the Grothendieck $\gamma$-filtration on $K_0$ of twisted forms of $X$ as well as a tool to estimate the torsion part of its Chow groups in small codimensions.

The paper is organized as follows. In the first section we introduce the $I$-adic filtrations on $\mathbb{Z}[\Lambda]$ and $S^*(\Lambda)$ together with an isomorphism $\phi_i$ on their truncations. Then we study the subrings of invariants and introduce the key notion of an exponent $\tau_i$ of a $W$-action on a free abelian group $\Lambda$. Roughly speaking, the integers $\tau_i$ measure how far is the ring $S^*(\Lambda)^W$ from being a polynomial ring in $q_i$'s. In section 5 we estimate all the exponents up to degree 4 and show that they all divide the Dynkin index of the Lie algebra $\mathfrak{g}$. We would like to stress that the procedure of estimating $\tau_i$-s has an algorithmic nature, i.e. given a group and an integer $i$ one can estimate $\tau_i$ for this group just using the explicit formulas for

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$W$-invariant polynomials. Finally, we apply the obtained results to estimate the torsion in Grothendieck $\gamma$-filtration of some twisted flag varieties.

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## 1. Two filtrations

Consider the two covariant functors $S^*(-)$ and $\mathbb{Z}[-]$ from the category of abelian groups to the category of commutative rings

$$S^*(-)\colon \Lambda \mapsto S^*(\Lambda) \text{ and } \mathbb{Z}[-]\colon \Lambda \mapsto \mathbb{Z}[\Lambda]$$

given by taking the symmetric algebra of an abelian group $\Lambda$ and the integral group ring of $\Lambda$ respectively. The $i$th graded component $S^i(\Lambda)$ is additively generated by monomials $\lambda_1\lambda_2\ldots\lambda_i$ with $\lambda_j \in \Lambda$ and the ring $\mathbb{Z}[\Lambda]$ is additively generated by exponents $e^\lambda$, $\lambda \in \Lambda$.

The trivial group homomorphism induces the ring homomorphisms

$$\epsilon_a\colon S^*(\Lambda) \to \mathbb{Z} \text{ and } \epsilon_m\colon \mathbb{Z}[\Lambda] \to \mathbb{Z}$$

called the augmentation maps. By definition $\epsilon_a$ sends every element of positive degree to 0 and $\epsilon_m$ sends every $e^\lambda$ to 1. Let $I_a$ and $I_m$ denote the kernels of $\epsilon_a$ and $\epsilon_m$ respectively. Observe that $I_a = S^{>0}(\Lambda)$ consists of elements of positive degree and $I_m$ is generated by differences $(1-e^{-\lambda})$, $\lambda \in \Lambda$. Consider the respective $I$-adic filtrations:

$$S^*(\Lambda) = I_a^0 \supseteq I_a \supseteq I_a^2 \supseteq \ldots \text{ and } \mathbb{Z}[\Lambda] = I_m^0 \supseteq I_m \supseteq I_m^2 \supseteq \ldots$$

and let

$$gr_a^*(\Lambda) = \bigoplus_{i\geq 0} I_a^i/I_a^{i+1} \text{ and } gr_m^*(\Lambda) = \bigoplus_{i\geq 0} I_m^i/I_m^{i+1}$$

denote the associated graded rings. Observe that $gr_a^*(\Lambda) = S^*(\Lambda)$.

1.1. **Example.** If $\Lambda \simeq \mathbb{Z}$, then the ring $S^*(\Lambda)$ can be identified with the polynomial ring in one variable $\mathbb{Z}[\omega]$, where $\omega$ is a generator of $\Lambda$ and the ring $\mathbb{Z}[\Lambda]$ can be identified with the Laurent polynomial ring $\mathbb{Z}[x, x^{-1}]$ where $x = e^\omega$. The augmentations $\epsilon_a$ and $\epsilon_m$ are given by

$$\epsilon_a\colon \omega \mapsto 0 \text{ and } \epsilon_m\colon x \mapsto 1.$$

We have $I_a = (\omega)$ and $I_m$ is additively generated by differences $(1-x^n)$, $n \in \mathbb{Z}$.

Note that the rings $\mathbb{Z}[\omega]$ and $\mathbb{Z}[x, x^{-1}]$ are not isomorphic, however they become isomorphic after the truncation. Namely for every $i \geq 0$ there is ring isomorphism

$$\phi_i\colon \mathbb{Z}[x, x^{-1}]/I_m^{i+1} \xrightarrow{\simeq} \mathbb{Z}[\omega]/I_a^{i+1}$$

defined by $\phi_i\colon x \mapsto (1-\omega)^{-1} = 1+\omega+\ldots+\omega^i$ with the inverse defined by $\phi_i^{-1}\colon \omega \mapsto 1-x^{-1}$. It is useful to keep the following picture in mind

$$\begin{tikzcd}
\mathbb{Z}[x,x^{-1}] \arrow[d] \arrow[dr] & \mathbb{Z}[\omega] \arrow[l] \arrow[d] \\
\mathbb{Z}[x,x^{-1}]/I_m^{i+1} \arrow[r, "\phi_i", "\simeq"'] & \mathbb{Z}[\omega]/I_a^{i+1}
\end{tikzcd}$$

observing that the inverse $\phi_i^{-1}$ can be lifted to the map $\mathbb{Z}[\omega] \to \mathbb{Z}[x,x^{-1}]$ but $\phi_i$ can't.

The example can be generalized as follows:

1.2. **Lemma.** [GZ10, 2.1] *Assume that $\Lambda$ is a free abelian group of finite rank $n$. The rings $\mathbb{Z}[\Lambda]$ and $S^*(\Lambda)$ become isomorphic after truncation. Namely, if $\{\omega_1,\ldots,\omega_n\}$ is a $\mathbb{Z}$-basis of $\Lambda$, then for every $i\geq 0$ there is a ring isomorphism*

$$\phi_i\colon \mathbb{Z}[\Lambda]/I_m^{i+1} \stackrel{\simeq}{\to} S^*(\Lambda)/I_a^{i+1}$$

*defined by $\phi_i(1)=1$ and*

$$\phi_i(e^{\sum_{j=1}^n a_j\omega_j}) = \prod_{j=1}^n (1-\omega_j)^{-a_j}$$

*with the inverse defined by $\phi_i^{-1}(\omega_j) = 1-e^{-\omega_j}$.*

Note that the map $\phi_i$ preserves the $I$-adic filtrations. Indeed, by definition $\phi_i(I_m^j)\subseteq I_a^j$ for every $0\leq j\leq i$. Moreover, we have the following

1.3. **Lemma.** *(cf.* [CPZ, 4.2]*) The isomorphism $\phi_i$ restricted to the subsequent quotients $I_m^i/I_m^{i+1}$ doesn't depend on the choice of a basis of $\Lambda$. Hence, there is an induced canonical isomorphism of graded rings*

$$\phi_* = \oplus_{i\geq 0}\phi_i \,:\, gr_m^*(\Lambda) \stackrel{\simeq}{\longrightarrow} gr_a^*(\Lambda) = S^*(\Lambda).$$

*Proof.* Indeed, in this case we can define the inverse $\phi_i^{-1}\colon I_a^i/I_a^{i+1} \to I_m^i/I_m^{i+1}$ by

$$\phi_i^{-1}(\lambda_1\lambda_2\ldots\lambda_i) = (1-e^{-\lambda_1})(1-e^{-\lambda_2})\ldots(1-e^{-\lambda_i}).$$

It is well-defined since $(1-e^{-\lambda-\lambda'}) = (1-e^{-\lambda}) + (1-e^{-\lambda'})$ modulo $I_m^2$. ☐

Consider the composite of the map $\phi_i$ with the projections

$$\phi^{(i)}\colon \mathbb{Z}[\Lambda] \to \mathbb{Z}[\Lambda]/I_m^{i+1} \xrightarrow{\phi_i} S^*(\Lambda)/I_a^{i+1} \to S^i(\Lambda).$$

The map $\phi^{(i)}$, and therefore $\phi_i$, can be computed on generators $e^\lambda$, $\lambda\in\Lambda$ as follows:

Let $f(z) = \prod_j (1-\omega_j z)^{-a_j}$, where $\lambda = \sum_j a_j\omega_j$. Then

$$\phi^{(i)}(e^{\sum_j a_j\omega_j}) = \frac{1}{i!}\frac{d^i f(z)}{dz^i}\Big|_{z=0}$$

To compute the derivatives of $f(z)$ we observe that $f'(z) = f(z)g(z)$, where $g(z) = \sum_j a_j\omega_j(1-\omega_j z)^{-1}$ and $\frac{d^i\, g(z)}{d\, z^i} = \sum_j \frac{i!\, a_j\omega_j^{i+1}}{(1-\omega_j z)^{i+1}}$. Hence, starting with $g_0 = 1$ we obtain the following recursive formulas

$$\frac{d^i\, f(z)}{d\, z^i} = f(z)\cdot g_i(z), \text{ where } g_i(z) = g(z)g_{i-1}(z) + g'_{i-1}(z).$$

1.4. **Example.** For small values of $i$ we obtain

| $i$ | $i!\cdot\phi^{(i)}(e^\lambda)=$ |
|---|---|
| 1 | $\lambda$ |
| 2 | $\lambda^2+\lambda(2)$ |
| 3 | $\lambda^3+3\lambda(2)\lambda+2\lambda(3)$ |
| 4 | $\lambda^4+6\lambda(4)+6\lambda(2)\lambda^2+8\lambda(3)\lambda+3\lambda(2)^2$ |

where given a presentation $\lambda=\sum_{j=1}^n a_{j,\lambda}\omega_j,\ a_{j,\lambda}\in\mathbb{Z}$ in terms of the basis $\{\omega_1,\omega_2,\ldots.\omega_n\}$ we set $\lambda(m)=\sum_{j=1}^n a_{j,\lambda}\omega_j^m$ for $m\geq 1$.

## 2. Invariants and exponents

Let $W$ be a finite group which acts on a free abelian group $\Lambda$ of finite rank by $\mathbb{Z}$-linear automorphisms. Consider the induced action of $W$ on $\mathbb{Z}[\Lambda]$ and $S^*(\Lambda)$. Observe that it is compatible with the $I$-adic filtrations, i.e. $W(I_m^i)\subseteq I_m^i$ and $W(I_a^i)\subseteq I_a^i$ for every $i\geq 0$.

Note that the isomorphisms $\phi_i$ and $\phi_i^{-1}$ are not necessarily $W$-equivariant. However, by Lemma 1.3 their restrictions to the subsequent quotients $I_m^i/I_m^{i+1}$ and $I_a^i/I_a^{i+1}=S^i(\Lambda)$ are $W$-equivariant and we have

$$(I_m^i/I_m^{i+1})^W\simeq (I_a^i/I_a^{i+1})^W.$$

Let $I_m^W$ denote the ideal of $\mathbb{Z}[\Lambda]$ generated by $W$-invariant elements from the augmentation ideal $I_m$, i.e., by elements from $\mathbb{Z}[\Lambda]^W\cap I_m$. Similarly, let $I_a^W$ denote the ideal of $S^*(\Lambda)$ generated by $W$-invariant elements from $I_a$, i.e., by elements from $S^*(\Lambda)^W\cap I_a$.

For each $\chi\in\Lambda$ let $\rho(\chi)=\sum_{\lambda\in W(\chi)}e^\lambda$ denote the sum over all elements of the $W$-orbit of $\chi$. Every element in $I_m^W$ can be written as a finite linear combination with integer coefficients of the elements $\hat\rho(\chi)=\rho(\chi)-\epsilon_m(\rho(\chi))$, $\chi\in\Lambda$. Therefore, the ideal $I_m^W$ is generated by the elements $\hat\rho(\chi)$, i.e.,

$$I_m^W=\langle\hat\rho(\chi)\mid\chi\in\Lambda\rangle.$$

The image of $I_m^W$ by means of the composite

$$\mathbb{Z}[\Lambda]\to\mathbb{Z}[\Lambda]/I_m^{i+1}\xrightarrow{\phi_i}S^*(\Lambda)/I_a^{i+1}\,.$$

is an ideal in $S^*(\Lambda)/I_a^{i+1}$ generated by the elements $\phi_i(\hat\rho(\chi))$, $\chi\in\Lambda$. Therefore, the image of $I_m^W$ in $S^i(\Lambda)$ is the $i$th homogeneous component of the ideal generated by $\phi^{(j)}(\hat\rho(\chi))$, where $1\leq j\leq i$, $\chi\in\Lambda$, i.e.

$$\phi^{(i)}(I_m^W)=\langle f\cdot\phi^{(j)}(\hat\rho(\chi))\mid 1\leq j\leq i,\ f\in S^{i-j}(\Lambda),\ \chi\in\Lambda\rangle_{\mathbb{Z}}\,.$$

We are ready now to introduce the central notion of the present paper:

2.1. **Definition.** We say that an action of $W$ on $\Lambda$ *has finite exponent in degree $i$* if there exists a non-zero integer $N_i$ such that

$$N_i\cdot(I_a^W)^{(i)}\subseteq\phi^{(i)}(I_m^W),$$

where $(I_a^W)^{(i)}=I_a^W\cap S^i(\Lambda)$. In this case the g.c.d. of all such $N_i$s will be called the *$i$-th exponent* of the $W$-action and will be denoted by $\tau_i$.

Observe that if $\phi^{(i)}(I_m^W)$ is a subgroup of finite index in $(I_a^W)^{(i)}$, then $\tau_i$ is simply the exponent of $\phi^{(i)}(I_m^W)$ in $(I_a^W)^{(i)}$. Note also that by the very definition $\tau_0=1$.

2.2. **Example.** Consider $\Lambda = \mathbb{Z}\cdot\omega$ with the action $\omega \mapsto -\omega$ of $W = \mathbb{Z}/2\mathbb{Z}$. Then $(I_a^W)$ is generated by $\omega^2, \omega^4, \cdots$, hence $(I_a^W)^{(i)} = \mathbb{Z}\cdot\omega^i$ if $i$ is even, 0 otherwise. On the other hand, $\phi^{(i)}(I_m^W)$ is generated by $\phi^{(i)}(\hat\rho(\omega)) = \phi^{(i)}(e^\omega + e^{-\omega} - 2) = \omega^i$ if $i \geq 2$, 0 otherwise. Therefore, we have $\tau_i = 1$ for every $i \geq 0$.

## 3. Essential actions

In the present section we study $W$-actions that have no $W$-invariant linear forms, i.e. we assume that $\Lambda^W = 0$. In the theory of reflection groups such actions are called *essential* (see [B4-6, V, §3.7] or [Hu]). Note that this immediately implies that $\tau_1 = 1$.

3.1. **Lemma.** *For every $\chi \in \Lambda$ and $m \in \mathbb{N}_+$ we have $\sum_{\lambda\in W(\chi)} \lambda(m) = 0$.*

*Proof.* Let $\omega_1, \omega_2, \ldots .\omega_n$ be a $\mathbb{Z}$-basis of $\Lambda$. For $m \in \mathbb{N}_+$ we have

$$\sum_{\lambda\in W(\chi)} \lambda(m) = \sum_{\lambda\in W(\chi)} \big(\sum_{j=1}^n a_{j,\lambda}\omega_j^m\big) = \sum_{j=1}^n \big(\sum_{\lambda\in W(\chi)} a_{j,\lambda}\big)\omega_j^m.$$

In particular, for $m = 1$ we obtain

$$\sum_{\lambda\in W(\chi)} \lambda = \sum_{j=1}^n \big(\sum_{\lambda\in W(\chi)} a_{j,\lambda}\big)\omega_i.$$

Since $\Lambda^W = 0$, we have $\sum_{\lambda\in W(\chi)} \lambda = 0$. Since $\omega_j$, $1 \leq j \leq n$ are $\mathbb{Z}$-free, we have $\sum_{\lambda\in W(\chi)} a_{j,\lambda} = 0$ for all $1 \leq j \leq n$. $\square$

3.2. **Corollary.** *For every $\chi \in \Lambda$ we have*

$$\phi^{(2)}(\rho(\chi)) = \tfrac{1}{2}\sum_{\lambda\in W(\chi)} \lambda^2.$$

*In particular, the quadratic form $\phi^{(2)}(\rho(\chi))$ is $W$-invariant, i.e.*

$$\phi^{(2)}(\rho(\chi)) \in S^2(\Lambda)^W.$$

*Proof.* By the formula for $\phi^{(2)}$ in Example 1.4 and by Lemma 3.1 we obtain that

$$\phi^{(2)}\big(\sum_{\lambda\in W(\chi)} e^\lambda\big) = \tfrac{1}{2}\sum_{\lambda\in W(\chi)} (\lambda^2 + \lambda(2)) = \tfrac{1}{2}\sum_{\lambda\in W(\chi)} \lambda^2. \quad \square$$

3.3. **Corollary.** *If $S^2(\Lambda)^W = \langle q\rangle$ for some $q$, then $\phi^{(2)}(I_m^W)$ is a subgroup of finite index in $(I_a^W)^{(2)}$.*

*Proof.* The image of the ideal $I_m^W$ is generated by $\phi^{(1)}(\rho(\chi))$ and $\phi^{(2)}(\rho(\chi))$. Since $\Lambda^W = 0$, $\phi^{(1)}(\rho(\chi)) = \sum_{\lambda\in W(\chi)} \lambda = 0$ and by Corollary 3.2, $\phi^{(2)}(I_m^W)$ is generated only by the $W$-invariant quadratic forms $\phi^{(2)}(\rho(\chi))$. For every $\chi \in \Lambda$ let

$$\phi^{(2)}(\rho(\chi)) = N_\chi \cdot q,\ N_\chi \in \mathbb{N}. \tag{1}$$

Then the subgroup $\phi^{(2)}(I_m^W)$ is a subgroup of $(I_a^W)^{(2)}$ of exponent

$$\tau_2 = \gcd_{\chi\in\Lambda} N_\chi. \quad \square$$

We now investigate the invariants of degree 3 and 4.

3.4. **Lemma.** *For every $\chi \in \Lambda$ we have*

$$\phi^{(3)}(\rho(\chi)) = \tfrac{1}{6} \sum_{\lambda \in W(\chi)} (\lambda^3 + 3\lambda(2)\lambda).$$

*Proof.* By the formula for $\phi^{(3)}$ in Example 1.4 and by Lemma 3.1 we obtain that

$$\phi^{(3)}(\rho(\chi)) = \tfrac{1}{6} \sum_{\lambda \in W(\chi)} (\lambda^3 + 3\lambda(2)\lambda + 2\lambda(3)) = \tfrac{1}{6} \sum_{\lambda \in W(\chi)} (\lambda^3 + 3\lambda(2)\lambda). \quad \square$$

3.5. **Lemma.** *For every $\chi \in \Lambda$ we have*

$$\phi^{(4)}(\rho(\chi)) = \tfrac{1}{24} \sum_{\lambda \in W(\chi)} [\lambda^4 + 6\lambda(2)\lambda^2 + 8\lambda(3)\lambda + 3\lambda(2)^2].$$

*Proof.* It follows from Example 1.4 and Lemma 3.1. $\square$

## 4. The Dynkin index

In the present section we show that the action of the Weyl group $W$ of a crystallographic root system $\Phi$ on the weight lattice $\Lambda$ has finite exponent in degree 2 which coincides with the Dynkin index of the respective Lie algebra.

Let $W$ be the Weyl group of a crystallographic root system $\Phi$ and let $\Lambda$ be its weight lattice as defined in [Hu, §2.9]. Let $\{\omega_1, \ldots, \omega_n\}$ be a basis of $\Lambda$ consisting of fundamental weights (here $n$ is the rank of $\Phi$).

The Weyl group $W$ acts on $\lambda \in \Lambda$ by means of simple reflections

$$s_j(\lambda) = \lambda - \langle \alpha_j^\vee, \lambda \rangle \cdot \alpha_j, \quad j = 1 \ldots n$$

where $\alpha_j^\vee$ is the $j$-th simple coroot and $\langle -, - \rangle$ is the usual pairing. Note that $\langle \alpha_j^\vee, \omega_i \rangle = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker symbol.

The subring of invariants $\mathbb{Z}[\Lambda]^W$ is the representation ring of the respective Lie algebra $\mathfrak{g}$. By a theorem of Chevalley it is the polynomial ring in classes of fundamental representations $\mathrm{ch}(V_j) \in \mathbb{Z}[\Lambda]^W$, i.e.

$$\mathbb{Z}[\Lambda]^W \simeq \mathbb{Z}[\mathrm{ch}(V_1), \ldots, \mathrm{ch}(V_n)].$$

Note that every $\mathrm{ch}(V_l)$ is a sum of $W$-orbits $\rho(\chi)$ with some multiplicities.

Therefore, the image $\phi^{(i)}(I_m^W)$ is the $i$-th homogeneous component of the ideal generated by $\phi^{(j)}(\mathrm{ch}(V_l))$, $1 \le j \le i$, $l = 1 \ldots n$.

4.1. **Lemma.** *We have $\Lambda^W = 0$ and hence also*

$$\phi^{(1)}(\mathbb{Z}[\Lambda]^W) = \phi^{(1)}(I_m^W) = 0.$$

*Proof.* Let $\eta \in \Lambda^W$. Since $\eta = s_{\alpha_j}(\eta) = \eta - \langle \eta, \alpha_j^\vee \rangle \alpha_j$ we have $\langle \eta, \alpha_j^\vee \rangle = \frac{2(\alpha_j, \eta)}{(\alpha_j, \alpha_j)} = 0$ for all simple roots $\alpha_j$ which implies that $\eta = 0$. $\square$

4.2. **Lemma.** *We have $S^2(\Lambda)^W = \langle q \rangle$.*

*Proof.* By [GN04, Prop. 4] there exists an integer valued $W$-invariant quadratic form on $\Lambda$ which has value 1 on short coroots. As the group $S^2(\Lambda)^W$ is identical to the group of all integral $W$-invariant quadratic forms on $T_* \otimes \mathbb{R}$, the result follows. $\square$

4.3. **Corollary.** *The image $\phi^{(2)}(I_m^W)$ is a subgroup of $(I_a^W)^{(2)}$ of finite index.*

*Proof.* This follows from Corollary 3.3 and Lemma 4.1. □

We recall briefly the notion of indices of representations introduced by Dynkin [Dy57, §2] (See also [Br91]).

Let $f : \mathfrak{g} \to \mathfrak{g}'$ be a morphism between simple Lie algebras. Then there exists a unique number $j_f \in \mathbb{C}$, called the *Dynkin index of* $f$, satisfying

$$(f(x), f(y)) = j_f(x, y),$$

for all $x, y \in \mathfrak{g}$, where (–,–) is the Killing form on $\mathfrak{g}$ and $\mathfrak{g}'$ normalized such that $(\alpha, \alpha) = 2$ for any long root $\alpha$. In particular, if $f : \mathfrak{g} \to \mathfrak{sl}(V)$ is a linear representation, $j_f$ is a positive integer, called the *Dynkin index of the linear representation* $f$, defined by

$$\mathrm{tr}(f(x), f(y)) = j_f(x, y).$$

The *Dynkin index of* $\mathfrak{g}$ is defined to be the greatest common divisor of all the Dynkin indices of all linear representations of $\mathfrak{g}$. By [Dy57, (2.24) and (2.25)], the Dynkin index of $\mathfrak{g}$ is the greatest common divisor of the Dynkin indexes $j_l$ of its fundamental representations $V_l$, $l = 1 \dots m$. All the Dynkin indexes $j_l$ were calculated in [Dy57, Table 5]. We provide below the list of Dynkin indexes taken from [LS97, Prop. 2.6]:

| type of $\mathfrak{g}$ | $A$ or $C$ | $B_n$ $(n \geq 3)$, $D_n$ $(n \geq 4)$, $G_2$ | $F_4$ or $E_6$ | $E_7$ | $E_8$ |
|---|---|---|---|---|---|
| Dynkin index | 1 | 2 | 6 | 12 | 60 |

Using the $\mathfrak{sl}_2$-representation theory, the Dynkin index of a linear representation $f : \mathfrak{g} \to \mathfrak{sl}(V)$ can be described as follows. Let $\alpha$ be a long root. For the formal character $\mathrm{ch}(V) = \sum_\lambda n_\lambda e^\lambda$, one has (see [LS97, Lemma 2.4] or [KNR, 5.1 and Lemma 5.2])

$$j_f = \frac{1}{2} \sum_\lambda n_\lambda \langle \lambda, \alpha^\vee \rangle^2.$$

4.4. **Theorem.** *The second exponent equals the Dynkin index of* $\mathfrak{g}$.

*Proof.* As explained at the beginning of this section, the image $\phi^{(2)}(I_m^W)$ is spanned by $\phi^{(2)}(\mathrm{ch}(V_l))$, where $V_l$ is the $l$-th fundamental representation. It follows that $\tau_2$ is the greatest common divisor of the integers $N_l$ defined by $\phi^{(2)}(\mathrm{ch}(V_l)) = N_l \cdot q$ as in Corollary 3.3.

To find the precise value of $\tau_2$ we use the explicit formula for $\phi^{(2)}(\rho(\chi))$ given in Corollary 3.2, that is

$$\phi^{(2)}(\rho(\chi)) = \tfrac{1}{2} \sum_{\lambda \in W(\chi)} \lambda^2.$$

Recall that $\mathrm{ch}(V_l)$ is a sum of $W$-orbits $\rho(\chi)$ of some $\chi \in \Lambda$ with some multiplicities. Evaluating $\phi^{(2)}(\mathrm{ch}(V_l))$ (considered as a linear combination of $\phi^{(2)}(\rho(\chi))$) at $\alpha^\vee$, where $\alpha$ is long, we obtain that $j_l = N_l q(\alpha^\vee) = N_l$. Therefore, $gcd(j_1, \dots, j_n) = gcd(N_1, \dots, N_n) = \tau_2$. □

We note that Theorem 4.4 was shown in [GZ10, §2] with a different proof.

## 5. Exponents of degrees 3 and 4

In the present section we show that $\tau_2 = N_3 = N_4$ for all crystallographic root systems, i.e. that the exponents $\tau_3$ and $\tau_4$ divide the Dynkin index of $G$.

Let $S = \{\lambda_1, \dots, \lambda_r\}$ be a finite set of weights. We denote by $-S$ the set of opposite weights $\{-\lambda_1, \dots, -\lambda_r\}$, by $S_+$ the set of sums $\{\lambda_i + \lambda_j\}_{i<j}$, by $S_-$ the set of differences $\{\lambda_i - \lambda_j\}_{i<j}$ and by $S_\pm$ the disjoint union $S_+ \amalg S_-$. By definition we have $|S_+| = |S_-| = \binom{r}{2}$.

Using the fact that $(\lambda + \lambda')(m) = \lambda(m) + \lambda'(m)$ for every $\lambda, \lambda' \in \Lambda$ and $m \geq 0$ we obtain the following lemma which will be extensively used in the computations

5.1. **Lemma.** *(i) For every integer $m_1, m_2, x, y \geq 0$ and a finite subset $S \subset \Lambda$ we have*

$$\sum_{\lambda \in S \amalg -S} \lambda(m_1)^x \lambda(m_2)^y = (1 + (-1)^{x+y}) \sum_{\lambda \in S} \lambda(m_1)^x \lambda(m_2)^y.$$

*In particular, $\sum_{\lambda \in S \amalg -S} \lambda(2)\lambda^2 = 0$.*

*(ii) For every subset $S \subset \Lambda$ with $|S| = r$ and for every $m_1, m_2 \geq 0$ we have*

$$\sum_{\lambda \in S_+} \lambda(m_1)\lambda(m_2) = (r-1) \sum_{\lambda \in S} \lambda(m_1)\lambda(m_2) + \sum_{i \neq j} \lambda_i(m_1)\lambda_j(m_2) \text{ and}$$

$$\sum_{\lambda \in S_-} \lambda(m_1)\lambda(m_2) = (r-1) \sum_{\lambda \in S} \lambda(m_1)\lambda(m_2) - \sum_{i \neq j} \lambda_i(m_1)\lambda_j(m_2).$$

*In particular, this implies that $\sum_{\lambda \in S_\pm} \lambda(m_1)\lambda(m_2) = 2(r-1) \sum_{\lambda \in S} \lambda(m_1)\lambda(m_2)$.*

**$A_n$-case.** Let $\Phi$ be of type $A_n$ for $n \geq 3$. We denote the canonical basis of $\mathbb{R}^{n+1}$ by $e_i$ with $1 \leq i \leq n+1$. According to [Hu, §3.5 and §3.12] the basic polynomial invariants of the $W$-action on $\Lambda$ (algebraically independent homogeneous generators of $S^*(\Lambda)^W$ as a $\mathbb{Q}$-algebra) are given by the symmetric power sums

$$q_i := e_1^i + \cdots + e_{n+1}^i, \quad 2 \leq i \leq n+1.$$

Let $s_i$ denote the $i$th elementary symmetric function in $e_1, \dots, e_{n+1}$. Using the classical identities

$$q_1 = s_1, \quad q_i = s_1 q_{i-1} - s_2 q_{i-2} + \ldots + (-1)^i s_{i-1} q_1 + (-1)^{i+1} i \cdot s_i, \quad 1 < i < n+1$$

and the fact that $s_1 = 0$, we obtain that

$$q_2/2 = -s_2, \; q_3/3 = s_3, \text{ and } q_4/2 = s_2^2 - 2s_4.$$

generate (with integral coefficients) the ideal $I_a^W$ up to degree 4.

The fundamental weights of $\Phi$ can be expressed as follows

$$\omega_1 = e_1, \; \omega_2 = e_1 + e_2, \; \ldots \; , \omega_{n-1} = e_1 + \ldots + e_{n-1}, \; \omega_n = -e_{n+1},$$

where $e_1 + e_2 + \ldots + e_{n+1} = 0$. The orbits of $\omega_1$, $\omega_1 + \omega_n$, $\omega_n$ and $\omega_2$, $\omega_{n-1}$ under the action of the Weyl group $W = S_{n+1}$ are given by

$$W(\omega_1) = \{e_1, \dots, e_{n+1}\} = -W(\omega_n), \; W(\omega_1 + \omega_n) = \{e_i - e_j\}_{i \neq j} \text{ and}$$

$$W(\omega_2) = \{e_i + e_j\}_{i<j} = -W(\omega_{n-1}).$$

Therefore, $W(\omega_1 + \omega_n) = S_- \amalg -S_-$ and $W(\omega_2) = S_+$, where $S = W(\omega_1)$.

Applying Lemma 3.5 and Lemma 5.1 we obtain that

$$\phi^{(4)}(\rho(\omega_1)+\rho(\omega_n)) = \tfrac{1}{12}\sum_{\lambda\in S}(\lambda^4+8\lambda(3)\lambda+3\lambda(2)^2) \text{ and}$$

$$\phi^{(4)}(\rho(\omega_1+\omega_n)+\rho(\omega_2)+\rho(\omega_{n-1})) = \tfrac{1}{24}\sum_{\lambda\in S_\pm\amalg -S_\pm}(\lambda^4+8\lambda(3)\lambda+3\lambda(2)^2) =$$

$$= \tfrac{1}{24}\sum_{\lambda\in S_\pm\amalg -S_\pm}\lambda^4 + \tfrac{n}{6}\sum_{\lambda\in S}(8\lambda(3)\lambda+3\lambda(2)^2).$$

Then the difference

$$\phi^{(4)}(\rho(\omega_1+\omega_n)+\rho(\omega_2)+\rho(\omega_{n-1})) - 2n\cdot\phi^{(4)}(\rho(\omega_1)+\rho(\omega_n)) =$$

$$= \tfrac{1}{24}\sum_{\lambda\in S_\pm\amalg -S_\pm}\lambda^4 - \tfrac{n}{6}\sum_{\lambda\in S}\lambda^4 = \qquad (2)$$

is a symmetric function in $e_1,\ldots,e_{n+1}$ and, therefore, it can be always written as a polynomial in $q_i$s. Indeed, since

$$\sum_{\lambda\in S_\pm\amalg -S_\pm}\lambda^4 = 2\sum_{i<j}((e_i+e_j)^4+(e_i-e_j)^4) = 4n\sum_{\lambda\in S}\lambda^4 + 24\sum_{i<j}e_i^2e_j^2,$$

the difference (2) equals

$$= \sum_{i<j}e_i^2e_j^2 = (q_2^2-q_4)/2.$$

5.2. **Lemma.** *For a root system of type $A_n$, $n\geq 2$, we have $\tau_2=\tau_3=\tau_4=1$.*

*Proof.* It is enough to show that the generators $q_2/2$, $q_3/3$ and $q_4/2$ are in the ideal generated by the image of $\phi^{(i)}$, $i\leq 4$.

By Corollary 3.2 we have $\phi^{(2)}(\rho(\omega_1)) = \frac{1}{2}\sum_{\lambda\in S}\lambda^2 = q_2/2$. By Lemma 3.4 we have $q_3/3 = \phi^{(3)}(\rho(\omega_1)) - \phi^{(3)}(\rho(\omega_n))$ (see also [GZ10, §1C]). If $\Phi$ is of type $A_2$, then $s_4=0$ and, hence, $q_4=q_2^2/2$. If $\Phi$ is of type $A_n$, $n\geq 3$, then by (2) the generator $q_4/2$ belongs to the ideal generated by the images of $\phi^{(2)}$ and $\phi^{(4)}$. $\square$

**$B_n$, $C_n$ and $D_n$ cases.** Let $\Phi$ be of type $B_n$ or $C_n$ for $n\geq 2$ or of type $D_n$ for $n\geq 4$. We denote the canonical basis of $\mathbb{R}^n$ by $e_i$ with $1\leq i\leq n$. By [Hu, §3.5 and §3.12] the basic polynomial invariants of the $W$-action on $\Lambda$ are given by even power sums

$$q_{2i} := e_1^{2i}+\cdots+e_n^{2i},\quad 1\leq i\leq n,$$

together with $p_n := e_1\cdots e_n$ if $\Phi$ is of type $D_n$.

The first two fundamental weights of $\Phi$ are given by $\omega_1=e_1$, $\omega_2=e_1+e_2$ and their $W$-orbits are

$$W(\omega_1)=\{\pm e_1,\ldots,\pm e_n\} \text{ and } W(\omega_2)=\{\pm e_i\pm e_j\}_{i<j}.$$

Hence $W(\omega_1)=S\amalg -S$ and $W(\omega_2)=S_\pm\amalg -S_\pm$, where $S=\{e_1,\ldots,e_n\}$.

Applying Lemma 3.5 and Lemma 5.1 we obtain that

$$\phi^{(4)}(\rho(\omega_1)) = \tfrac{1}{12}\sum_{\lambda\in S}\lambda^4 + \tfrac{1}{12}\sum_{\lambda\in S}(8\lambda(3)\lambda+3\lambda(2)^2) \text{ and}$$

$$\phi^{(4)}(\rho(\omega_2)) = \tfrac{1}{24}\sum_{\lambda\in S_\pm\amalg -S_\pm}\lambda^4 + \tfrac{n-1}{6}\sum_{\lambda\in S}(8\lambda(3)\lambda+3\lambda(2)^2).$$

Then similar to the $A_n$-case we obtain

$$\phi^{(4)}(\rho(\omega_2)) - 2(n-1)\phi^{(4)}(\rho(\omega_1)) = (q_2^2 - q_4)/2, \tag{3}$$

where $q_i = e_1^i + \ldots + e_n^i$ and

$$-\phi^{(4)}(\rho(\omega_3)) + \phi^{(4)}(\rho(\omega_4)) = p_4, \tag{4}$$

if $\Phi$ is of type $D_4$.

5.3. **Lemma.** *For a root system of type $B_n$ or $C_n$, $n \geq 2$ or $D_n$, $n \geq 4$ the exponents $\tau_3$ and $\tau_4$ divide the Dynkin index $\tau_2$.*

*Proof.* Since there are no basic polynomial invariants in degree 3 [Hu, §3.7 Table 1] we have $\tau_3 \mid \tau_2 = 2$. For $D_4$, by (4) the invariant $p_4$ is in the ideal generated by the image of $\phi^{(4)}$. Hence, to show that $\tau_4 \mid \tau_2$ it is enough to show that $q_4/2$ is in the ideal generated by the image of $\phi^{(2)}$ and $\phi^{(4)}$. Indeed, by Corollary 3.2 we have $\phi^{(2)}(\rho(\omega_1)) = \sum_{\lambda \in S} \lambda^2 = q_2$. Therefore, by (3)

$$q_4/2 = (q_2/2) \cdot \phi^{(2)}(\rho(\omega_1)) - \phi^{(4)}(\rho(\omega_2)) + 2(n-1)\phi^{(4)}(\rho(\omega_1)). \quad \square$$

5.4. **Theorem.** *For every crystallographic root system $\Phi$ the exponents $\tau_3$ and $\tau_4$ divide the Dynkin index $\tau_2$.*

*Proof.* If $\Phi$ is of type $A_n$, this follows from Lemma 5.2. If $\Phi$ is of type $B_n$, $C_n$ or $D_n$ this follows from Lemma 5.3; for all other types $\tau_3$ and $\tau_4$ divide $\tau_2$ since there are no basic polynomial invariants of degree 3 and 4 (see [Hu, §3.7 Table 1]). $\square$

## 6. Torsion in the Grothendieck $\gamma$-filtration

The goal of the present section is to provide geometric interpretation (see (6)) of the map $\phi_i$ and the exponents $\tau_i$.

Let $G$ be a split simple simply-connected group over a field $k$. We fix a maximal split torus $T$ of $G$ and a Borel subgroup $B \supset T$. Let $\Lambda$ be the group of characters of $T$. Since $G$ is simply-connected, $\Lambda$ coincides with the weight lattice of $G$.

Let $X$ denote the variety of Borel subgroups of $G$ (conjugate to $B$). Consider the Chow ring $\mathrm{CH}^*(X)$ of algebraic cycles modulo rational equivalence and the Grothendieck ring $K_0(X)$. Following [De74, §1] to every character $\lambda \in \Lambda$ we may associate the line bundle $\mathcal{L}(\lambda)$ over $X$. It induces the ring homomorphisms (called the characteristic maps)

$$\mathfrak{c}_a \colon S^*(\Lambda) \to \mathrm{CH}^*(X) \text{ and } \mathfrak{c}_m \colon \mathbb{Z}[\Lambda] \twoheadrightarrow K_0(X)$$

by sending $\lambda \mapsto c_1(\mathcal{L}(\lambda))$ and $e^\lambda \mapsto [\mathcal{L}(\lambda)]$ respectively. Note that the map $\mathfrak{c}_a$ is an isomorphism in codimension one, hence, giving

$$\mathfrak{c}_a \colon S^1(\Lambda) = \Lambda \xrightarrow{\simeq} Pic(X) = \mathrm{CH}^1(X)$$

and the map $\mathfrak{c}_m$ is surjective. Let $W$ be the Weyl group and let $I_a^W$ and $I_m^W$ denote the respective $W$-invariant ideals. Then according to [De73, §4 Cor.2,§9] and [CPZ, §6]

$$\ker \mathfrak{c}_m = I_m^W \tag{5}$$

and $\ker \mathfrak{c}_a$ is generated by elements of $S^*(\Lambda)$ such that their multiples are in $I_a^W$.

Consider the Grothendieck $\gamma$-filtration on $K_0(X)$ (see [GZ10, §1]). Its $i$th term is an ideal generated by products

$$\gamma^i(X) := \langle (1-[\mathcal{L}_1^\vee])(1-[\mathcal{L}_2^\vee])\cdot\ldots\cdot(1-[\mathcal{L}_i^\vee])\rangle,$$

where $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_i$ are line bundles over $X$. Consider the $i$th subsequent quotient $\gamma^i(X)/\gamma^{i+1}(X)$. The usual Chern class $c_i$ induces a group homomorphism $c_i\colon \gamma^i(X)/\gamma^{i+1}(X) \to \mathrm{CH}^i(X)$.

6.1. **Proposition.** *For every $i \geq 0$ there is a commutative diagram of group homomorphisms*

$$\begin{tikzcd} I_m^i/I_m^{i+1} \arrow[r, "(-1)^{i-1}(i-1)!\cdot\phi_i"] \arrow[d, "\mathfrak{c}_m"] & S^i(\Lambda) \arrow[d, "\mathfrak{c}_a"] \\ \gamma^i(X)/\gamma^{i+1}(X) \arrow[r, "c_i"] & \mathrm{CH}^i(X) \end{tikzcd} \tag{6}$$

*Proof.* Indeed, the $\gamma$-filtration on $K_0(X)$ is the image of the $I_m$-adic filtration on $\mathbb{Z}[\Lambda]$, i.e. $\gamma^i(X) = \mathfrak{c}_m(I_m^i)$ for every $i \geq 0$. The Proposition then follows from the identity

$$c_i\Big((1-[\mathcal{L}_1^\vee])(1-[\mathcal{L}_2^\vee])\ldots(1-[\mathcal{L}_i^\vee])\Big) = (-1)^{i-1}(i-1)!\cdot c_1(\mathcal{L}_1)c_1(\mathcal{L}_2)\ldots c_1(\mathcal{L}_i),$$

where $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_i$ are line bundles over $X$ and $\mathcal{L}_i^\vee$ denotes the dual of $\mathcal{L}_i$. □

6.2. **Remark.** Note that $\mathbb{Z}[\Lambda]$ can be identitfied with the $T$-equivariant $K_0$ of a point $pt = Spec\ k$ and $S^*(\Lambda)$ with the $T$-equivariant CH of a point (see [GZ11]). The maps $\mathfrak{c}_a$ and $\mathfrak{c}_m$ then can be identified with the pull-backs $K_0^T(pt) \to K_0^T(G)$ and $\mathrm{CH}_T(pt) \to \mathrm{CH}_T(G)$ induced by the structure map $G \to pt$.

In view of these identifications the map $\phi_i$ can be viewed as an equivariant analogue of the Chern class map $c_i$.

Consider the diagram (6) with $\mathbb{Q}$-coefficients. In this case the Chern class map $c_i$ will become an isomorphism (by the Riemann-Roch theorem), the characteristic map $\mathfrak{c}_a$ will turn into a surjection and the map $(-1)^{i-1}(i-1)!\cdot\phi_i$ will be an isomorphism as well. In view of (5) we obtain an isomorphism

$$\phi^{(i)} \otimes \mathbb{Q}\colon I_m^W \cap I_m^i / I_m^W \cap I_m^{i+1} \otimes \mathbb{Q} \longrightarrow (I_a^W)^{(i)} \otimes \mathbb{Q}$$

on the kernels of $\mathfrak{c}_m$ and $\mathfrak{c}_a$. By the very definition of the exponents $\tau_i$ this implies that

6.3. **Corollary.** *The action of the Weyl group of a crystallograhic root system has finite exponent $\tau_i$ for every $i$.*

6.4. **Lemma.** *We have $(\ker\mathfrak{c}_a)^{(i)} = (I_a^W)^{(i)}$ for each $i \leq 4$ except the case $i = 4$ and $G$ is of type $B_n$ ($n \geq 3$) or $D_n$ ($n \geq 5$) where we have $2(\ker\mathfrak{c}_a)^{(4)} \subseteq (I_a^W)^{(4)}$.*

*Proof.* The statement follows by the same analysis as in [GZ10, §1B]. For the exception it is enough to show that the polynomial $P = q\cdot f_2 + d\cdot(q_4/2)$ in $\omega_i$-s is not divisible by 4, where $d \in \mathbb{Z}$, $f_2$ is a polynomial of degree 2, $q_4/2$ is the basic polynomial invariant of degree 4 and $g.c.d.(f_2, d) = 1$.

Assume that $4 \mid P$, we claim that in this case $g.c.d.(f_2, d) = 2$. Indeed, let $f_2 = \sum_{i=1}^n a_i\omega_i^2 + \sum_{i<j} a_{ij}\omega_i\omega_j$, $a_i, a_{ij} \in \mathbb{Z}$. Take $\omega_i$ and $\omega_j$ corresponding to

adjacent long roots. Set $\omega_k = 0$ for $k \neq i, j$. Then the congruence $P \equiv 0\,(mod\ 4)$ turns into

$$(\omega_i^2 - \omega_i\omega_j + \omega_j^2)(a_i\omega_i^2 + a_{ij}\omega_i\omega_j + a_j\omega_j^2) + d(\omega_i^4 - 2\omega_i^3\omega_j + 3\omega_i^2\omega_j^2 - 2\omega_i\omega_j^3 + \omega_j^4) \equiv 0$$

which gives $a_i \equiv a_j \equiv -d$, $a_{ij} - a_i \equiv a_{ij} - a_j \equiv -2d$ and $a_i - a_{ij} + a_j \equiv 3d$. This implies that $2d \equiv 0$, therefore, $2 \mid d$. Finally, since $q$ is indivisible, $2 \mid f_2$.

In the $D_4$-case let $Q = q \cdot f_2 + d \cdot (q_4/2) + e \cdot p_4$ with $g.c.d.(f_2, d, e) = 1$. If $4 \mid Q$, then we have $d \equiv a_i \equiv 0\,(mod\ 2)$ by the same argument. Hence, $2 \mid q \cdot f_2 + e \cdot p_4$. Set $\omega_2 = 0$. Then we have

$$(\omega_1^2 + \omega_3^2 + \omega_4^2) f_2 \mid_{\omega_2=0} + e(\omega_1^2\omega_3^2 - \omega_1^2\omega_4^2) \equiv 0\,(mod\ 2).$$

In particular, $2 \mid a_1 + a_3 + e$. As $2 \mid a_i$, we have $2 \mid e$, which implies that $2 \mid f_2$. □

We are now ready to prove the main result of this section

6.5. **Theorem.** *The integer $\tau_i \cdot (i-1)!$ annihilates the torsion of the ith subsequent quotient $\gamma^i(X)/\gamma^{i+1}(X)$ of the $\gamma$-filtration on $K_0(X)$ for $i = 2, 3, 4$ except the case $i = 4$ and $G$ is of type $B_n$ ($n \geq 3$) or $D_n$ ($n \geq 5$) where the torsion of $\gamma^4(X)/\gamma^5(X)$ is annihilated by 24.*

6.6. **Remark.** Note that by [SGA6, Exposé XIV, 4.5] for groups of types $A_n$ and $C_n$ the quotients $\gamma^i(X)/\gamma^{i+1}(X)$ have no torsion.

*Proof.* Assume that $\alpha$ is a torsion element in $\gamma^i(X)/\gamma^{i+1}(X)$. Then $c_i(\alpha) = 0$ since $\mathrm{CH}^i(G/B)$ has no torsion. Let $\tilde{\alpha}$ be a preimage of $\alpha$ via $\mathfrak{c}_m$ in $I_m^i/I_m^{i+1} \subseteq \mathbb{Z}[\Lambda]/I_m^{i+1}$. By (6) we obtain that

$$(i-1)!\,\phi_i(\tilde{\alpha}) \in (\ker \mathfrak{c}_a)^{(i)}$$

where $(\ker \mathfrak{c}_a)^{(i)}$ coincides with $(I_a^W)^{(i)}$ up to a multiple (see Lemma 6.4). By definition of the index $\tau_i$ we have

$$\tau_i \cdot (i-1)!\,\phi_i(\tilde{\alpha}) = \phi_i(\beta), \text{ where } \beta \in I_m^W / I_m^{i+1} \cap I_m^W.$$

Applying $\phi_i^{-1}$ to the both sides we obtain

$$\tau_i \cdot (i-1)! \cdot \tilde{\alpha} = \beta \in I_m^W / I_m^{i+1} \cap I_m^W$$

Applying $\mathfrak{c}_m$ to the both sides and observing that $I_m^W = \ker \mathfrak{c}_m$ we obtain that $\tau_i \cdot (i-1)! \cdot \alpha = 0$. □

Let ${}_\xi X$ be a twisted form of the variety $X$ by means of a cocycle $\xi \in Z^1(k, G)$. By [Pa94, Thm. 2.2.(2)] the restriction map $K_0({}_\xi X) \to K_0(X)$ (here we identify $K_0(X)$ with the $K_0(X \times_k \bar{k})$ over the algebraic closure $\bar{k}$) is an isomorphism. Since the characteristic classes commute with restrictions, this induces an isomorphism between the $\gamma$-filtrations, i.e. $\gamma^i({}_\xi X) \simeq \gamma^i(X)$ for every $i \geq 0$, and between the respective quotients

$$\gamma^i({}_\xi X)/\gamma^{i+1}({}_\xi X) \simeq \gamma^i(X)/\gamma^{i+1}(X) \quad \text{for every } i \geq 0.$$

In view of this fact Theorem 6.5 implies that

6.7. **Corollary.** *Let $G$ be a split simple simply connected group of type $B_n$ ($n \geq 3$) or $D_n$ ($n \geq 4$). Then for every $\xi \in Z^1(k, G)$ the torsion in $\gamma^4({}_\xi X)/\gamma^5({}_\xi X)$ is annihilated by 24.*

Consider the topological filtration on $K_0(Y)$, where $Y$ is a smooth projective variety, given by the ideals

$$\tau^i(Y) := \langle [\mathcal{O}_V] \mid V \hookrightarrow Y,\, codim_V Y \geq i\rangle.$$

It is known (see [FuLa, Ch.V, Thm. 3.9]) that $\gamma^i(Y) \subseteq \tau^i(Y)$ for every $i \geq 0$.

Given an Abelian group $M$ let $e(M)$ denote the exponent of its torsion subgroup. The following exact sequences of Abelian groups

$$\text{(i) } \gamma^i/\gamma^{i+1} \hookrightarrow \tau^i/\gamma^{i+1} \twoheadrightarrow \tau^i/\gamma^i \text{ and (ii) } \tau^{i+1}/\gamma^{i+1} \hookrightarrow \tau^i/\gamma^{i+1} \twoheadrightarrow \tau^i/\tau^{i+1}, \quad (7)$$

where $\tau^i = \tau^i(Y)$, $\gamma^i = \gamma^i(Y)$, lead to the recursive divisibility for each $i \geq 1$

$$e(\tau^i/\gamma^{i+1}) \mid e(\gamma^i/\gamma^{i+1}) \cdot e(\tau^i/\gamma^i) \mid e(\gamma^i/\gamma^{i+1}) \cdot e(\tau^{i-1}/\gamma^i)$$

which gives

$$e(\tau^i/\gamma^{i+1}) \mid e(\gamma^i/\gamma^{i+1}) \cdot e(\gamma^{i-1}/\gamma^i) \cdot \ldots \cdot e(\gamma^1/\gamma^2). \quad (8)$$

By the Riemann-Roch theorem [Fu, Ex.15.3.6], the composition

$$\mathrm{CH}^i(Y) \to \tau^i/\tau^{i+1} \xrightarrow{c_i} \mathrm{CH}^i(Y)$$

is the multiplication by $(-1)^{i-1}(i-1)!$, therefore, by (7).(ii) the torsion subgroup of $\mathrm{CH}^i(Y)$ is annihilated by $(i-1)! \cdot e(\tau^i/\tau^{i+1}) \mid (i-1)! \cdot e(\tau^i/\gamma^{i+1})$. Combining this with the formula (8) and Theorem 6.5 we obtain

6.8. **Corollary.** *Let $G$ be a split simple simply connected group. Then for every $\xi \in Z^1(k,G)$ the torsion in $\mathrm{CH}^i({}_\xi X)$ for $i = 2,3,4$ is annihilated by the integer*

$$(i-1)! \cdot \prod_{j=2}^{i} \tau_j (j-1)!$$

*except for $i = 4$ and $G$ is of type $B_n$ $(n \geq 3)$ or $D_n$ $(n \geq 5)$ where it is annihilated by $2^7$.*

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Sanghoon Baek, Department of Mathematics and Statistics, University of Ottawa
*E-mail address*: sbaek@uottawa.ca

Erhard Neher, Department of Mathematics and Statistics, University of Ottawa
*E-mail address*: erhard.neher@uottawa.ca

Kirill Zainoulline, Department of Mathematics and Statistics, University of Ottawa, 585 King Edward, Ottawa ON K1N 6N5, Canada
*E-mail address*: kirill@uottawa.ca