

RANK-WIDTH AND WELL-QUASI-ORDERING

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ABSTRACT. Robertson and Seymour (1990) proved that graphs of bounded tree-width are well-quasi-ordered by the graph minor relation. By extending their arguments, Geelen, Gerards, and Whittle (2002) proved that binary matroids of bounded branch-width are well-quasi-ordered by the matroid minor relation. We prove another theorem of this kind in terms of *rank-width* and *vertex-minors*. For a graph $G = (V, E)$ and a vertex v of G , a *local complementation* at v is an operation that replaces the subgraph induced by the neighbors of v with its complement graph. A graph H is called a *vertex-minor* of G if H can be obtained from G by applying a sequence of vertex deletions and local complementations. Rank-width was defined by Oum and Seymour (2006) to investigate clique-width; they showed that graphs have bounded rank-width if and only if they have bounded clique-width. We prove that graphs of bounded rank-width are well-quasi-ordered by the vertex-minor relation; in other words, for every infinite sequence G_1, G_2, \dots of graphs of rank-width (or clique-width) at most k , there exist $i < j$ such that G_i is isomorphic to a vertex-minor of G_j . This implies that there is a finite list of graphs such that a graph has rank-width at most k if and only if it contains no one in the list as a vertex-minor. The proof uses the notion of *isotropic systems* defined by Bouchet.

1. INTRODUCTION

Oum and Seymour [12] defined *rank-width* to investigate *clique-width*, which was defined by Courcelle and Olariu [5]. Rank-width is a complexity measure of graph in a kind of tree-structure, called a *rank-decomposition*. Oum and Seymour showed that graphs have bounded rank-width if and only if they have bounded clique-width and they used it to find an approximation algorithm for clique-width.

Later the author [11] showed that the notion of rank-width of graphs has an interesting intersection with the notion of *branch-width* of matroids. More specifically, the branch-width of a binary matroid is exactly one more than the rank-width of its fundamental graph.

The branch-width of matroids generalizes the branch-width of graphs, both defined by Robertson and Seymour [14]. Branch-width of graphs is approximately equal to *tree-width*, also defined by Robertson and Seymour. Informally speaking, tree-width is a measure describing how close a graph is to being a forest.

Both tree-width and branch-width are interesting with respect to the *graph minor* relation. A *contraction* of an edge e is an operation, that deletes e and identifies the ends of e . A graph H is a *minor* of a graph G if H can be obtained from G by a sequence of contractions, vertex deletions, and edge deletions. If H is a minor of G , then the tree-width of H is at most that of G and the same is true for branch-width. This implies that a set of graphs of bounded tree-width is closed under the graph minor relation.

In this paper, we consider vertex-minors of graphs, previously called *l -reductions* by Bouchet [4]. For a graph G and a vertex v of G , let $G * v$ be the graph, obtained by

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local complementation at v , that is by replacing the subgraph induced by the neighbors of v with its complement graph. We say that G is *locally equivalent* to H if H can be obtained from G by applying a sequence of local complementations. A graph H is a *vertex-minor* of G if H can be obtained from G by applying a sequence of vertex deletions and local complementations. A simple fact is that if H is a vertex-minor of G , then the rank-width of H is at most that of G . For an edge uv of G , *pivoting* on uv , denoted by $G \wedge uv$, is the composition of three local complementations, $G \wedge uv = G * u * v * u$. It is an easy exercise to show that $G * u * v * u = G * v * u * v$ if uv is an edge of G . We say that H is a *pivot-minor* of G if H can be obtained from G by applying a sequence of vertex deletions and pivots. Every pivot-minor of G is a vertex-minor of G , but not vice versa.

In this paper, we prove the following.

Theorem 4.1. *Let k be a constant. If $\{G_1, G_2, G_3, \dots\}$ is an infinite sequence of graphs of rank-width at most k , then there exist $i < j$ such that G_i is isomorphic to a pivot-minor of G_j , and therefore isomorphic to a vertex-minor of G_j .*

This theorem is motivated by the following two theorems. The first one is for graphs of bounded tree-width, proved by Robertson and Seymour [13].

Theorem 1.1. *Let k be a constant. If $\{G_1, G_2, G_3, \dots\}$ is an infinite sequence of graphs of tree-width at most k , then there exist $i < j$ such that G_i is isomorphic to a minor of G_j .*

The next one, generalizing the previous one, is due to Geelen, Gerards, and Whittle [8].

Theorem 1.2. *Let k be a constant. Let \mathbb{F} be a finite field. If $\{M_1, M_2, M_3, \dots\}$ is an infinite sequence of \mathbb{F} -representable matroids of branch-width at most k , then there exist $i < j$ such that M_i is isomorphic to a minor of M_j .*

It is straightforward to prove (Section 11) that Theorem 1.2 restricted to binary matroids is equivalent to Theorem 4.1 restricted to bipartite graphs. In fact, the main idea of proving Theorem 1.2 remains in our paper, although we have to go through a different technical notion. In the original proof of Theorem 1.2, the authors use “configurations” to represent \mathbb{F} -representable matroids, and then convert the matroid problem into a vector space problem. In our proof, we use the similar approach but a different notion. Bouchet defined *isotropic systems* and developed their minors and their relation to graphs in a series of papers [1, 2, 3]. Informally speaking, an isotropic system can be considered as an equivalence class of graphs by *local equivalence*. Isotropic systems also generalize pairs of a binary matroid and its dual. A detailed definition will be reviewed in Section 3.

In sections 2 and 3, we review some notions and results. The overview of the proof is given in Section 4. Then in sections 5 till 8, we prove several lemmas, including an extension of Tutte’s linking theorem to isotropic systems, in Section 7. Then in Section 9 we prove that isotropic systems of bounded branch-width are well-quasi-ordered under the “isotropic system minor” order and in Section 10 we prove that the theorem implies the well-quasi-ordering of graphs under the pivot-minor order. In Section 11 we explain why the result restricted to bipartite graphs is the same as Theorem 1.2 restricted to binary matroids.

2. REVIEW ON RANK-WIDTH

In this paper, we assume that graphs are simple, undirected, and finite. Let us review the definition of rank-width, introduced by Oum and Seymour [12]. We will first describe the branch-width of connectivity functions and then use this to define the rank-width.

For a finite set V , an integer-valued function c on subsets of V is called a *connectivity function* if $c(X) + c(Y) \geq c(X \cap Y) + c(X \cup Y)$ for all $X, Y \subseteq V$, $c(X) = c(V \setminus X)$ for all $X \subseteq V$, and $c(\emptyset) = 0$. A *subcubic tree* is a tree such that every vertex has exactly one or three incident edges. We call (T, \mathcal{L}) a *branch-decomposition* of a connectivity function c on subsets of V if T is a subcubic tree and \mathcal{L} is a bijection from V to the set of all leaves of T . For an edge e of T , connected components of $T \setminus e$ induce a partition (X, Y) of the set of leaves of T . The *width* of an edge e of a branch-decomposition (T, \mathcal{L}) is $c(\mathcal{L}^{-1}(X))$. The *width* of (T, \mathcal{L}) is the maximum width of all edges of T . The *branch-width* $\text{bw}(c)$ of c is the minimum width of a branch-decomposition of c . (If $|V| \leq 1$, we define that $\text{bw}(c) = c(\emptyset)$.)

For a branch-decomposition (T, \mathcal{L}) of c , let e and f be two edges of T . Let E be the set of leaves of T in the component of $T \setminus e$ not containing f , and let F be the set of leaves of T in the component of $T \setminus f$ not containing e . Let P be the shortest path in T containing e and f . We call e and f *linked* if

$$\min_{h \in E(P)} (\text{width of } h \text{ of } (T, \mathcal{L})) = \min_{\mathcal{L}^{-1}(E) \subseteq Z \subseteq V \setminus \mathcal{L}^{-1}(F)} c(Z).$$

We call a branch-decomposition (T, \mathcal{L}) *linked* if each pair of edges of T is linked.

For a matrix $M = (m_{ij} : i \in R, j \in C)$ over a field \mathbb{F} , if $X \subseteq R$ and $Y \subseteq C$, let $M[X, Y]$ denote the submatrix $(m_{ij} : i \in X, j \in Y)$. For a graph G , let A_G be its adjacency matrix over the binary field $\text{GF}(2)$.

For a graph $G = (V, E)$ and two disjoint subsets $X, Y \subseteq V$, we define $\rho_G^*(X, Y) = \text{rk}(A_G[X, Y])$ where rk is the matrix rank function; and we define the *cut-rank* function ρ_G of G by $\rho_G(X) = \rho_G^*(X, V(G) \setminus X)$ for $X \subseteq V$.

It is well-known that the cut-rank function is an instance of a connectivity function (see [12]). A *rank-decomposition* of a graph is a branch-decomposition of its cut-rank function and the *rank-width* of a graph is the branch-width of its cut-rank function.

3. REVIEW ON THE ISOTROPIC SYSTEM

In this section, the notion of the isotropic system and a few useful theorems will be reviewed. All material is from Bouchet's papers [1, 2, 3]. We change some notations. The author's thesis [10] also contains the proofs of all theorems in this section.

3.1. Definition of the isotropic system. For a vector space W with a bilinear form $\langle \cdot, \cdot \rangle$, a subspace L of W is called *totally isotropic* if and only if $\langle x, y \rangle = 0$ for all $x, y \in L$.

Let $K = \{0, \alpha, \beta, \gamma\}$ be the 2-dimensional vector space over $\text{GF}(2)$. So this means that $\alpha + \beta + \gamma = \alpha + \alpha = \beta + \beta = \gamma + \gamma = 0$. Moreover, let $\langle \cdot, \cdot \rangle$ be the bilinear form on K defined by

$$\langle x, y \rangle = \begin{cases} 1 & \text{if } x \neq y \text{ and } x, y \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Let V be a finite set. Let K^V be the set of functions from V to K , and so K^V is a vector space over $\text{GF}(2)$. We attach the following bilinear form to K^V :

$$\text{for } x, y \in K^V, \quad \langle x, y \rangle = \sum_{v \in V} \langle x(v), y(v) \rangle.$$

Definition 3.1 (Bouchet [1]). An *isotropic system* is a pair $S = (V, L)$ of a finite set V and a totally isotropic subspace L of K^V with $\dim(L) = |V|$. V is called the *element set* of S .

For $X \subseteq V$, let $p_X : K^V \rightarrow K^X$ be the *canonical projection* such that

$$(p_X(a))(v) = a(v) \quad \text{for all } v \in X.$$

For $a \in K^V$ and $X \subseteq V$, $a[X]$ is the vector in K^V such that

$$a[X](v) = \begin{cases} a(v) & \text{if } v \in X, \\ 0 & \text{if } v \in V \setminus X. \end{cases}$$

Let L be a subspace of K^V and $v \in V$. Let $x \in K \setminus \{0\} = \{\alpha, \beta, \gamma\}$.

- Let L^\perp be the subspace of K^V such that $L^\perp = \{z \in K^V : \langle z, y \rangle = 0 \text{ for all } y \in L\}$.
- Let $L|_x^v$ be the subspace of $K^{V \setminus \{v\}}$ such that $L|_x^v = \{p_{V \setminus \{v\}}(a) : a \in L, a(v) = 0 \text{ or } x\}$.
- Let $L|_{\subseteq X}$, $L|_X$ be the subspaces of K^X such that

$$\begin{aligned} L|_{\subseteq X} &= \{p_X(a) : a \in L, a(v) = 0 \text{ for all } v \notin X\} \\ L|_X &= \{p_X(a) : a \in L\} \end{aligned}$$

Two vectors $a, b \in K^V$ are called *supplementary* if $\langle a(v), b(v) \rangle = 1$ for all $v \in V$. We call $a \in K^V$ *complete* if $a(v) \neq 0$ for all $v \in V$. For $X \subseteq V$ and a complete vector a of K^X , $L|_a^X$ is the subspace of $K^{V \setminus X}$ such that

$$L|_a^X = \{p_{V \setminus X}(b) : b \in L, b(v) \in \{a(v), 0\} \text{ for all } v \in X\}.$$

Note that $L|_{x_1}^{v_1} |_{x_2}^{v_2} |_{x_3}^{v_3} \cdots |_{x_k}^{v_k} = L|_x^{\{v_1, v_2, \dots, v_k\}}$ where $x \in K^{\{v_1, v_2, \dots, v_k\}}$ such that $x(v_i) = x_i$.

Definition 3.2. Let $S = (V, L)$ be an isotropic system and $v \in V$. For $x \in K \setminus \{0\}$, $S|_x^v = (V \setminus \{v\}, L|_x^v)$ is called an *elementary minor* of S . An isotropic system S' is called a *minor* of S if S' can be obtained from S by applying a sequence of elementary minor operations; in other words, $S' = S|_{x_1}^{v_1} |_{x_2}^{v_2} |_{x_3}^{v_3} \cdots |_{x_k}^{v_k}$ for $v_i \in V$ and $x_i \in K \setminus \{0\}$.

Bouchet [1, (8.1)] proved that an elementary minor of an isotropic system is again an isotropic system and therefore a minor of an isotropic system is an isotropic system.

3.2. Fundamental basis and fundamental graphs. The connection between isotropic systems and graphs was also studied by Bouchet [2].

Definition 3.3. We call $x \in K^V$ an *Eulerian vector* of an isotropic system $S = (V, L)$ if $x[P] \notin L$ for every nonempty subset P of V .

Proposition 3.4 (Bouchet [2, (4.1)]). *For every complete vector c of K^V , there is an Eulerian vector a of S , supplementary to c .*

Proposition 3.5 (Bouchet [2, (4.3)]). *Let a be an Eulerian vector of an isotropic system $S = (V, L)$. For every $v \in V$, there exists a unique vector $b_v \in L$ such that*

$$(1) \quad b_v(v) \neq 0,$$

(2) $b_v(w) \in \{0, a(w)\}$ for $w \neq v$.

Moreover, the set $\{b_v : v \in V\}$ is a basis of L .

We call $\{b_v : v \in V\}$ the *fundamental basis* of L with respect to a .

It is straightforward to construct an isotropic system from every graph. Let $G = (V, E)$ be a graph and a, b be a pair of supplementary vectors of K^V . Let $n_G(v)$ be the set of neighbors of v . Then, we may construct an isotropic system $S = (V, L)$ [2, (3.1)] by letting L be the subspace of K^V spanned by

$$\{a[n_G(v)] + b[\{v\}] : v \in V\}.$$

We call (G, a, b) a *graphic presentation* of S .

It is interesting that the reverse direction also works. Suppose an isotropic system $S = (V, L)$ is given with an Eulerian vector a . Let $\{b_v : v \in V\}$ be the fundamental basis of $S = (V, L)$ with respect to a . Let $G = (V, E)$ be the graph such that $vw \in E$ if and only if $v \neq w$ and $b_v(w) \neq 0$. Since $\langle b_v, b_w \rangle = 0$ implies $b_v(w) \neq 0 \Leftrightarrow b_w(v) \neq 0$, G is undirected. We call G the *fundamental graph* of S with respect to a . In fact, if S has a graphic presentation (G, a, b) , then G is the fundamental graph of S with respect to a .

Bouchet [2, (7.6)] showed that if (G, a, b) is a graphic presentation of an isotropic system $S = (V, L)$ and $v \in V$, then

$$(G * v, a + b[\{v\}], a[n_G(v)] + b)$$

is also a graphic presentation of S . Thus, local complementations preserve the associated isotropic system. If G and H are locally equivalent, associated isotropic systems can be chosen to be same by an appropriate choice of supplementary vectors. We note that if $uv \in E(G)$, then

$$(G \wedge uv, a[V \setminus \{u, v\}] + b[\{u, v\}], b[V \setminus \{u, v\}] + a[\{u, v\}])$$

is a graphic presentation of S . This fact will be used in Section 10.

Proposition 3.6 ([2, (4.5) and (7.1)]). *If G_1 and G_2 are fundamental graphs of an isotropic system S , then G_1 and G_2 are locally equivalent.*

A minor of an isotropic system is closely related to a vertex-minor of its fundamental graph as follows.

Proposition 3.7 ([2, (9.1)]). *Let $G = (V, E)$ be a graph. Let $v \in v(G)$ and $x \in K \setminus \{0\}$. If (G, a, b) is a graphic presentation of an isotropic system $S = (V, L)$, then the elementary minor $S|_x^v$ has a graphic presentation, that is*

- (i) $(G \setminus v, p_{V \setminus \{v\}}(a), p_{V \setminus \{v\}}(b))$ if either $x = a(v)$, or $x = b(v)$ and v is an isolated vertex of G ,
- (ii) $(G \wedge vw \setminus v, p_{V \setminus \{v\}}(a[V \setminus \{v, w\}] + b[\{v, w\}]), p_{V \setminus \{v\}}(b[V \setminus \{v, w\}] + a[\{v, w\}]))$ if $x = b(v)$ and v has a neighbor w in G ,
- (iii) $(G * v \setminus v, p_{V \setminus \{v\}}(a), p_{V \setminus \{v\}}(b + a[n_G(v)]))$ otherwise.

Corollary 3.8. *Suppose G_1 and G_2 are fundamental graphs of isotropic systems S_1 and S_2 respectively. If S_1 is a minor of S_2 , then G_1 is a vertex-minor of G_2 .*

Note that the choice of w in Proposition 3.7 does not affect the isotropic system because of the following proposition, proved in [11].

Proposition 3.9. *If $vv_1, vv_2 \in E(G)$ are two distinct edges incident with v , then, $G \wedge vv_1 \wedge v_1v_2 = G \wedge vv_2$, and therefore $G \wedge vv_1 \setminus v$ is locally equivalent to $G \wedge vv_2 \setminus v$.*

3.3. Connectivity. For a subspace L of K^V , let $\lambda(L) = |V| - \dim(L)$.

Definition 3.10. For an isotropic system $S = (V, L)$, we call $c : V \rightarrow \mathbb{Z}$ the *connectivity function* of S if $c(X) = \lambda(L|_{\subseteq X}) = |X| - \dim(L|_{\subseteq X})$.

If L is a totally isotropic subspace of K^V , then $L|_{\subseteq X}$ is also a totally isotropic subspace of K^X . Thus, $\dim(L|_{\subseteq X}) \leq |X|$, and therefore $c(X) \geq 0$.

Bouchet [3] observed that the connectivity function of an isotropic system is equal to the cut-rank function of its fundamental graph.

Proposition 3.11 ([3, Theorem 6]). *Let a be an Eulerian vector of an isotropic system $S = (V, L)$ and let c be the connectivity function of S . Let G be the fundamental graph of S with respect to a . Then, $c(X) = \rho_G(X)$ for all $X \subseteq V$.*

By Proposition 3.11, the connectivity function of an isotropic system $S = (V, L)$ is indeed a connectivity function, as defined in Section 2. A *branch-decomposition* and the *branch-width* of an isotropic system are defined as a branch-decomposition and the branch-width of its connectivity function, respectively. By Proposition 3.11, the branch-width of an isotropic system is equal to the rank-width of its fundamental graph.

4. OVERVIEW OF THE MAIN PROOF

Our main objective is to prove the following.

Theorem 4.1. *Let k be a constant. If $\{G_1, G_2, G_3, \dots\}$ is an infinite sequence of graphs of rank-width at most k , then there exist $i < j$ such that G_i is isomorphic to a pivot-minor of G_j , and therefore isomorphic to a vertex-minor of G_j .*

In general, we say that a binary relation \leq on a set X is a *quasi-order* if it is reflexive and transitive. For a quasi-order \leq , we say “ \leq is a *well-quasi-ordering*” or “ X is *well-quasi-ordered* by \leq ” if for every infinite sequence a_1, a_2, \dots of elements of X , there exist $i < j$ such that $a_i \leq a_j$. We may reiterate Theorem 4.1 as follows: a set of graphs of bounded rank-width is well-quasi-ordered up to isomorphism by the vertex-minor relation and also by the pivot-minor relation.

Here is a corollary of Theorem 4.1. Note that this corollary has an elementary proof in [11], and is used [6] to show the existence of the polynomial-time algorithm to decide whether rank-width is at most k for a fixed k .

Corollary 4.2. *For a fixed k , there is a finite list of graphs G_1, G_2, \dots, G_m such that for every graph H , the rank-width of H is at most k if and only if G_i is not isomorphic to a vertex-minor of H for all i .*

Proof. Let $X = \{G_1, G_2, \dots\}$ be a set of graphs satisfying that for every graph H , the rank-width of a graph H is at most k if and only if G_i is not isomorphic to a vertex-minor of H for all i . We choose X minimal by set inclusion. There are no $G_i, G_j \in X$ such that G_i is isomorphic to a vertex-minor of G_j , because if so, then we may remove G_j from X . By assumption, the rank-width of $G \setminus v$ for $v \in V(G)$ is at most k , and therefore the rank-width of G_i is at most $k + 1$. By Theorem 4.1, X is finite. \square

We say that an isotropic system $S_1 = (V_1, L_1)$ is *simply isomorphic* to another isotropic system $S_2 = (V_2, L_2)$ if there exists a bijection $\mu : V_1 \rightarrow V_2$ such that $L_1 = \{a \circ \mu : a \in L_2\}$. A bijection μ is called a *simple isomorphism*. It is clear that if S_1 is simply isomorphic to S_2 , then every fundamental graph of S_1 is isomorphic to a graph locally equivalent to a fundamental graph of S_2 .

We say that an isotropic system S_1 is an $\alpha\beta$ -*minor* of an isotropic system S if there is $a \in K^X$ with $S_1 = S|_a^X$ and $a(v) \in \{\alpha, \beta\}$ for all $v \in X$. Every $\alpha\beta$ -minor of an isotropic system S is a minor of S , but not vice versa. Pivot-minors of graphs are closely related to $\alpha\beta$ -minors of isotropic systems as the follows.

Lemma 10.2. *For $i \in \{1, 2\}$, let S_i be an isotropic system with a graphic presentation (G_i, a_i, b_i) such that*

$$a_i(v), b_i(v) \in \{\alpha, \beta\}$$

for all $v \in V(G_i)$. If S_1 is an $\alpha\beta$ -minor of S_2 , then G_1 is a pivot-minor of G_2 .

Instead of dealing with graphs, we show the following stronger proposition on isotropic systems.

Proposition 9.1. *Let k be a constant. If $\{S_1, S_2, S_3, \dots\}$ is an infinite sequence of isotropic systems of branch-width at most k , then there exist $i < j$ such that S_i is simply isomorphic to an $\alpha\beta$ -minor of S_j .*

We deduce Theorem 4.1 from Proposition 9.1.

Proof of Theorem 4.1. Let S_i be the isotropic system with the graphic presentation (G_i, a_i, b_i) where $a_i(v) = \alpha$, $b_i(v) = \beta$ for all $v \in V(G_i)$. Each S_i has branch-width at most k , since its branch-width is equal to the rank-width of G_i . By Proposition 9.1, there exist $i < j$ such that S_i is simply isomorphic to an $\alpha\beta$ -minor of S_j , and therefore by Lemma 10.2, G_i is isomorphic to a pivot-minor of G_j . \square

The following theorem was shown by Geelen, Gerards, and Whittle [8]. It was the first step to prove well-quasi-ordering of matroids representable over a fixed finite field having bounded branch-width. Its analogous result by Thomas [15] was used to prove well-quasi-ordering of graphs of bounded tree-width in [13].

Theorem 4.3 (Geelen et al. [8, 9, Theorem (2.1)]). *A connectivity function with branch-width n has a linked branch decomposition of width n .*

Corollary 4.4. *An isotropic system of branch-width n has a linked branch-decomposition of width n . Equivalently, a graph of rank-width n has a linked rank-decomposition of width n .*

We also use Robertson and Seymour’s “lemma on trees,” proved in [13]. It enabled them to prove that a set of graphs of bounded tree-width is well-quasi-ordered by the graph minor relation. Geelen, Gerards, and Whittle [8] used it to prove that a set of matroids representable over a fixed finite field and having bounded branch-width is well-quasi-ordered by the matroid minor relation. We need a special case of the “lemma on trees,” namely for subcubic forests, that was also used in [8].

The following definitions are as in [8]. A *rooted tree* is a finite directed tree where all but one of the vertices have indegree 1. A *rooted forest* is a collection of countably many vertex-disjoint rooted trees. Its vertices with indegree 0 are called *roots* and those with outdegree 0 are called *leaves*. Edges leaving a root are *root edges* and those entering a leaf are *leaf edges*.

An n -edge labeling of a graph F is a map from the set of edges of F to the set $\{0, 1, \dots, n\}$. Let λ be an n -edge labeling of a rooted forest F and let e and f be edges in F . We say that e is λ -linked to f if F contains a directed path P starting with e and ending with f such that $\lambda(g) \geq \lambda(e) = \lambda(f)$ for every edge g on P .

A *binary forest* is a rooted orientation of a subcubic forest with a distinction between left and right outgoing edges. More precisely, we call a triple (F, l, r) a *binary forest* if F is a rooted forest where roots have outdegree 1 and l and r are functions defined on non-leaf edges of F , such that the head of each non-leaf edge e of F has exactly two outgoing edges, namely $l(e)$ and $r(e)$.

Lemma 4.5 ([8, (3.2) Lemma on Subcubic Trees]). *Let (F, l, r) be an infinite binary forest with an n -edge labeling λ . Moreover, let \leq be a quasi-order on the set of edges of F with no infinite strictly descending sequences, such that $e \leq f$ whenever f is λ -linked to e . If the set of leaf edges of F is well-quasi-ordered by \leq but the set of root edges of F is not, then F contains an infinite sequence (e_0, e_1, \dots) of non-leaf edges such that*

- (i) $\{e_0, e_1, \dots\}$ is an antichain with respect to \leq ,
- (ii) $l(e_0) \leq l(e_1) \leq l(e_2) \leq \dots$,
- (iii) $r(e_0) \leq r(e_1) \leq r(e_2) \leq \dots$.

Informally speaking, at the last stage of proving Proposition 9.1, we need an object describing a piece of isotropic systems such that the number of ways to merge two of such objects into one isotropic system is finite up to simple isomorphisms. More precisely, we call a triple $P = (V, L, B)$ a *scrap* if V is a finite set, L is a totally isotropic subspace of K^V , and B is an ordered basis of L^\perp/L . An *ordered basis* is a basis with a linear ordering, and therefore B is of the form $\{b_1 + L, b_2 + L, \dots, b_k + L\}$ with $b_i \in L^\perp$. We denote $V(P) = V$. Note that L^\perp/L is a vector space containing vectors of the form $a + L$ with $a \in L^\perp$ and $a + L = b + L$ if and only if $a - b \in L$. Also note that $|B| = \dim(L^\perp/L) = \dim(L^\perp) - \dim(L) = 2(|V| - \dim(L)) = 2\lambda(L)$.

Two scraps $P_1 = (V, L, B)$ and $P_2 = (V', L', B')$ are called *isomorphic* if there exists a bijection $\mu : V \rightarrow V'$ such that $L = \{a \circ \mu : a \in L'\}$ and $b_i + L = (b'_i \circ \mu) + L$ where $B = \{b_1 + L, b_2 + L, \dots, b_k + L\}$ and $B' = \{b'_1 + L', b'_2 + L', \dots, b'_k + L'\}$.

For $x \in K \setminus \{0\}$ and $v \in V$, let $\delta_x^v \in K^V$ such that $\delta_x^v(v) = x$ and $\delta_x^v(w) = 0$ for all $w \neq v$. We often write δ_x^v without referring to V if that is not ambiguous. If $P = (V, L, B)$ is a scrap and $\delta_x^v \notin L^\perp \setminus L$, we define

$$P|_x^v = (V \setminus \{x\}, L|_x^v, \{p_{V \setminus \{v\}}(b_i) + L|_x^v\}_i)$$

where each $b_i \in L^\perp$ is chosen to satisfy that $B = \{b_i + L\}_i$ and $b_i(v) \in \{0, x\}$. We will prove that $P|_x^v$ is a well-defined scrap in Proposition 6.2. Note that $P|_x^v$ is only defined when $\delta_x^v \notin L^\perp \setminus L$.

A scrap P' is called a *minor* of a scrap P if $P' = P|_{x_1}^{v_1} |_{x_2}^{v_2} \dots |_{x_l}^{v_l}$ for some v_i and x_i . Similarly a scrap P' is called an $\alpha\beta$ -*minor* of a scrap P if $P' = P|_{x_1}^{v_1} |_{x_2}^{v_2} \dots |_{x_l}^{v_l}$ for some v_i and $x_i \in \{\alpha, \beta\}$.

Two scraps $P_1 = (V, L, B)$ and $P_2 = (V', L', B')$ are called *disjoint* if $V \cap V' = \emptyset$. A scrap $P = (V, L, B)$ is called a *sum* of two disjoint scraps $P_1 = (V_1, L_1, B_1)$ and $P_2 = (V_2, L_2, B_2)$ if

$$V = V_1 \cup V_2, \quad L_1 = L|_{\subseteq V_1}, \quad \text{and} \quad L_2 = L|_{\subseteq V_2}.$$

A sum of two disjoint scraps is not uniquely determined; we, however, will define the *connection types* that will determine a sum of two disjoint scraps such that there are only finitely many connection types. Moreover, we will prove the following.

Lemma 8.5. *Let P_1, P_2, Q_1, Q_2 be scraps. Let P be the sum of P_1 and P_2 and Q be the sum of Q_1 and Q_2 . If P_i is a minor of Q_i for $i = 1, 2$ and the connection type of P_1 and P_2 is equal to the connection type of Q_1 and Q_2 , then P is a minor of Q .*

Moreover, if P_i is an $\alpha\beta$ -minor of Q_i for $i \in \{1, 2\}$ and the connection type of P_1 and P_2 is equal to the connection type of Q_1 and Q_2 , then P is an $\alpha\beta$ -minor of Q .

Another requirement to apply Lemma 4.5 is that $e \leq f$ whenever f is λ -linked to e . This condition will be satisfied by the following lemma, which is an extension of Tutte's linking theorem. Tutte's linking theorem for matroids was used by Geelen, Gerards, and Whittle [8] and is an extension of Menger's theorem. Robertson and Seymour also used Menger's theorem in [13].

Theorem 7.2. *Let V be a finite set and X be a subset of V . Let L be a totally isotropic subspace of K^V . Let k be a constant. Let b be a complete vector of $K^{V \setminus X}$.*

Then $\lambda(L|_{\subseteq Z}) \geq k$ for all $Z \supseteq X$ if and only if there is a complete vector $a \in K^{V \setminus X}$ such that $\lambda(L|_a^{V \setminus X}) \geq k$ and $a(v) \neq b(v)$ for all $v \in V \setminus X$.

The actual proof of Proposition 9.1 is based on a construction of a forest with a certain k -labeling from branch-decompositions of isotropic systems, and applying the lemmas described above. In the subsequent sections, we will prove these lemmas.

5. LEMMAS ON TOTALLY ISOTROPIC SUBSPACES

In this section, L is a totally isotropic subspace of K^V , not necessarily $\dim(L) = |V|$. We prove some general results on totally isotropic subspaces.

Lemma 5.1. *Let L be a totally isotropic subspace of K^V and $v \in V, x \in K \setminus \{0\}$. Then,*

$$(L|_x^v)^\perp = L^\perp|_x^v.$$

Proof. Suppose that $y \in L^\perp|_x^v$. There exists $\bar{y} \in L^\perp$ such that $\bar{y}(v) \in \{0, x\}$ and $y = p_{V \setminus \{v\}}(\bar{y})$. For every $z \in L|_x^v$, there exists $\bar{z} \in L$ such that $\bar{z}(v) \in \{0, x\}$ and $p_{V \setminus \{v\}}(\bar{z}) = z$. Since $\langle y, z \rangle = \langle \bar{y}, \bar{z} \rangle - \langle \bar{y}(v), \bar{z}(v) \rangle = 0$, we have $y \in (L|_x^v)^\perp$.

Conversely, suppose that $y \notin L^\perp|_x^v$. Let $y \oplus x \in K^V$ be such that $p_{V \setminus \{v\}}(y \oplus x) = y$ and $(y \oplus x)(v) = x$. By assumption, $y \oplus x \notin L^\perp$. Therefore, there exists $z \in L$ such that

$$\langle y, p_{V \setminus \{v\}}(z) \rangle + \langle x, z(v) \rangle = \langle y \oplus x, z \rangle = 1.$$

If $\langle x, z(v) \rangle = 0$, then $p_{V \setminus \{v\}}(z) \in L|_x^v$ and $\langle y, p_{V \setminus \{v\}}(z) \rangle = 1$, and therefore $y \notin (L|_x^v)^\perp$. So, we may assume that $\langle x, z(v) \rangle = 1$.

Let $y \oplus 0 \in K^V$ such that $p_{V \setminus \{v\}}(y \oplus 0) = y$ and $(y \oplus 0)(v) = 0$. By assumption, $y \oplus 0 \notin L^\perp$. Therefore, there exists $w \in L$ such that $\langle y, p_{V \setminus \{v\}}(w) \rangle = \langle y \oplus 0, w \rangle = 1$. If $w(v) \in \{0, x\}$, then $p_{V \setminus \{v\}}(w) \in L|_x^v$ and $y \notin (L|_x^v)^\perp$. Hence we may assume that $\langle x, w(v) \rangle = 1$.

Now, we obtain that $\langle x, w(v) + z(v) \rangle = 0$, and so $w(v) + z(v) \in \{0, x\}$. Therefore $p_{V \setminus \{v\}}(w + z) \in L|_x^v$. Furthermore $\langle p_{V \setminus \{v\}}(w + z), y \rangle = 1$. So, $y \notin (L|_x^v)^\perp$. \square

Lemma 5.2. *If L is a totally isotropic subspace of K^V and $X \subseteq V$, then*

$$(L|_{\subseteq X})^\perp = L^\perp|_X.$$

Proof. We use induction on $|V \setminus X|$. If $|X| < |V| - 1$, then we pick $v \notin X$, and deduce that $(L|_{\subseteq V \setminus \{v\}}|_{\subseteq X})^\perp = (L|_{\subseteq V \setminus \{v\}})^\perp|_X = L^\perp|_{V \setminus \{v\}}|_X = L^\perp|_X$. Therefore we may assume that $V \setminus X = \{v\}$.

For $x \in K^X$ and $y \in K$, we let $x \oplus y$ denote the vector in K^V such that $p_X(x \oplus y) = x$ and $(x \oplus y)(v) = y$.

(1) We claim that $L^\perp|_X \subseteq (L|_{\subseteq X})^\perp$.

Suppose that $a \in L^\perp|_X$. Then there exists $b \in K$ such that $a \oplus b \in L^\perp$. For any $c \in L|_{\subseteq X}$, $\langle a \oplus b, c \oplus 0 \rangle = 0$, and therefore $\langle a, c \rangle = 0$. Thus, $a \in (L|_{\subseteq X})^\perp$.

(2) We claim that $(L|_{\subseteq X})^\perp \subseteq L^\perp|_X$.

Suppose that $a \in (L|_{\subseteq X})^\perp$ and $a \notin L^\perp|_X$. We have $a \oplus x \notin L^\perp$ and therefore for every $x \in K$, there exists $a_x \oplus c_x \in L$ such that $\langle a_x, a \rangle + \langle c_x, x \rangle = \langle a_x \oplus c_x, a \oplus x \rangle = 1$. In particular, $\langle a_0, a \rangle = 1$.

If $c_x = 0$, then $a_x \in L|_{\subseteq X}$ and so $\langle a_x, a \rangle = 0$ and $\langle c_x, x \rangle = 0$, contrary to the fact that $\langle a_x, a \rangle + \langle c_x, x \rangle = 1$. Therefore $c_x \neq 0$ for all $x \in K$.

If $c_x = c_y$ for $x \neq y$, then $a_x + a_y \in L|_{\subseteq X}$. Thus, $\langle c_x, x + y \rangle = 1 + \langle c_x, x \rangle + 1 + \langle c_y, y \rangle = \langle a_x + a_y, a \rangle = 0$. Since $c_x \neq 0$ and $x + y \neq 0$, we have $c_x = c_y = x + y$ and $\langle a_x, a \rangle = 1 + \langle x + y, x \rangle = 1 + \langle x, y \rangle$.

If $c_x = c_y = c_z$ for distinct x, y, z , then $x + y = y + z = z + x$. So, $x = y = z$, which is a contradiction.

If $c_x = c_y, c_z = c_w$ for distinct x, y, z, w , then $c_x = x + y = z + w = c_z$. So, $x = y = z = w$. This is a contradiction.

Therefore, there is exactly one pair $x, y \in K$ such that $c_x = c_y$. Let $\{z, w\} = K \setminus \{x, y\}$.

Since $c_z \neq c_w$ and $c_z, c_w \in K \setminus \{0, x + y\}$, we have $c_z + c_w = x + y = c_x = c_y$. Therefore, $a_z + a_w + a_x \in L|_{\subseteq X}$ and $\langle a_z + a_w + a_x, a \rangle = 0$. Since $\langle a_z, a \rangle + \langle a_w, a \rangle = \langle c_z, z \rangle + \langle c_w, w \rangle$, we have

$$\langle c_z, z \rangle + \langle c_w, w \rangle = \langle a_z + a_w + a_x, a \rangle + \langle a_x, a \rangle = 1 + \langle x, y \rangle.$$

If $x = 0$, then $c_z + c_w = y$, hence $c_z, c_w \in \{z, w\}$. So, $\langle c_z, z \rangle + \langle c_w, w \rangle = 0$. A contradiction.

So we may assume that $x \neq 0, y \neq 0, z = 0$, and then $x + y = w$ and $\langle c_w, w \rangle = 0$. But, this implies that $c_w = w = x + y = c_x$. A contradiction. \square

Proposition 5.3. *Let V be a finite set and L be a totally isotropic subspace of K^V and $v \in V, x \in K \setminus \{0\}$. Then,*

$$\dim(L|_x^v) = \begin{cases} \dim(L) & \text{if } \delta_x^v \in L^\perp \setminus L, \\ \dim(L) - 1 & \text{otherwise.} \end{cases}$$

$$\text{In other words, } \lambda(L|_x^v) = \begin{cases} \lambda(L) & \text{if } \delta_x^v \notin L^\perp \setminus L, \\ \lambda(L) - 1 & \text{otherwise.} \end{cases}$$

Proof. Let $L' = \{a \in L : a(v) \in \{0, x\}\}$. Then clearly $L' = L$ if $a(v) \in \{0, x\}$ for each $a \in L$, and $\dim(L') = \dim(L) - 1$ otherwise. Also, $\dim(L|_x^v) = \dim(L')$ when $\delta_x^v \notin L'$, and $\dim(L|_x^v) = \dim(L') - 1$ otherwise. Moreover, obviously, “ $a(v) \in \{0, x\}$ for each $a \in L$ ” is equivalent with “ $\delta_x^v \in L^\perp$ ”; and “ $\delta_x^v \notin L'$ ” is equivalent with “ $\delta_x^v \notin L$ ”. As $L \subseteq L^\perp$, all this implies this proposition. \square

Corollary 5.4. *Let V be a finite set and L be a totally isotropic subspace of K^V and $v \in V$. Let $C \subseteq K \setminus \{0\}$, $|C| = 2$. Then, either there is $x \in C$ such that $\lambda(L|_x^v) = \lambda(L)$ or for all $y \in K \setminus \{0\}$,*

$$L|_y^v = L|_{\subseteq V \setminus \{v\}} \quad \text{and} \quad \lambda(L|_y^v) = \lambda(L) - 1.$$

Proof. Let $C = \{a, b\}$. Suppose there is no such $x \in C$. Then δ_a^v, δ_b^v are in $L^\perp \setminus L$. Therefore, $z(v) = 0$ for all $z \in L$. Thus, $L|_y^v = L|_{\subseteq V \setminus \{v\}}$ and $\lambda(L|_y^v) = \lambda(L) - 1$ for all $y \in K \setminus \{0\}$. \square

6. SCRAPS

In this section, we prove that a minor of a scrap is well-defined. From Section 4, we recall that a *scrap* is a triple $P = (V, L, B)$ of a finite set V , a totally isotropic subspace L of K^V , and an ordered basis B of L^\perp/L .

Lemma 6.1. *Let $P = (V, L, B)$ be a scrap and $v \in V$, $x \in K \setminus \{0\}$. If $\delta_x^v \notin L^\perp \setminus L$, then there is a sequence $b_1, b_2, \dots, b_m \in L^\perp$ such that $b_i(v) \in \{0, x\}$ and $B = \{b_1 + L, b_2 + L, \dots, b_m + L\}$.*

Proof. Let $B = \{a_1 + L, a_2 + L, \dots, a_m + L\}$ with $a_i \in L^\perp$. If $\delta_x^v \in L$, then $a_i(v) \in \{0, x\}$ for all i . Hence we may assume that $\delta_x^v \notin L$ and so $\delta_x^v \notin L^\perp$. Therefore there is $y \in L$ such that $\langle y, \delta_x^v \rangle = 1$. Thus, $y(v) \notin \{0, x\}$. Let

$$b_i = \begin{cases} a_i & \text{if } a_i(v) \in \{0, x\}, \\ a_i + y & \text{otherwise.} \end{cases}$$

Then, $b_i + L = a_i + L$ and $b_i(v) \in \{0, x\}$. \square

Proposition 6.2. *Let $P = (V, L, B)$ be a scrap. If $\delta_x^v \notin L^\perp \setminus L$, then $P|_x^v$ is well-defined and is a scrap.*

Proof. Let us first show that it is well-defined. Let $b_1, b_2, \dots, b_k \in L^\perp$ be such that $b_i(v) \in \{0, x\}$ and $B = \{b_i + L : i = 1, 2, \dots, k\}$. We claim that the choice of b_i does not change $P|_x^v$. Suppose $b_i - b'_i \in L$ and $b_i(v), b'_i(v) \in \{0, x\}$. Since $b_i - b'_i \in L$ and $(b_i - b'_i)(v) \in \{0, x\}$, we have $p_{V \setminus \{v\}}(b_i - b'_i) \in L|_x^v$. Therefore, $p_{V \setminus \{v\}}(b_i) + L|_x^v = p_{V \setminus \{v\}}(b'_i) + L|_x^v$.

Now, we claim that $P|_x^v$ is a scrap.

First, we show that $L|_x^v$ is a totally isotropic subspace of $K^{V \setminus \{v\}}$. For all $a, b \in L|_x^v$, there are $\bar{a}, \bar{b} \in L$ such that $\bar{a}(v), \bar{b}(v) \in \{0, x\}$, $p_{V \setminus \{v\}}(\bar{a}) = a$, $p_{V \setminus \{v\}}(\bar{b}) = b$, and $\bar{a}, \bar{b} \in L$. Hence $\langle a, b \rangle = \langle \bar{a}, \bar{b} \rangle = 0$.

Next, we show that $\{p_{V \setminus \{v\}}(b_i) + L|_x^v : i = 1, 2, \dots, k\}$ is a basis of $(L|_x^v)^\perp / (L|_x^v)$. Since $b_i(v) \in \{0, x\}$, we have $p_{V \setminus \{v\}}(b_i) \in (L|_x^v)^\perp = (L^\perp)|_x^v$. Suppose that there exists $C \neq \emptyset$ such that

$$\sum_{i \in C} (p_{V \setminus \{v\}}(b_i) + L|_x^v) = 0 + L|_x^v.$$

Since $\sum_{i \in C} p_{V \setminus \{v\}}(b_i) \in L|_x^v$, there exists $z \in L \subseteq L^\perp$ such that $z(v) \in \{0, x\}$ and $p_{V \setminus \{v\}}(z) = \sum_{i \in C} p_{V \setminus \{v\}}(b_i)$. By assumption, $\sum_{i \in C} b_i \notin L$. Since $p_{V \setminus \{v\}}(\sum_{i \in C} b_i - z) = 0$, we have $\sum_{i \in C} b_i - z = \delta_x^v \in L^\perp \setminus L$, a contradiction. Therefore, $\{p_{V \setminus \{v\}}(b_i) + L|_x^v : i = 1, 2, \dots, k\}$ is linearly independent. Moreover, $\dim((L|_x^v)^\perp / (L|_x^v)) = 2(|V| - 1 - \dim(L|_x^v)) = 2(|V| - \dim(L)) = \dim(L^\perp / L)$ because $\delta_x^v \notin L^\perp \setminus L$. \square

7. GENERALIZATION OF TUTTE'S LINKING THEOREM

In this section, we show a generalization of Tutte's linking theorem [16].

Lemma 7.1. *Let V be a finite set and $v \in V$. Let L be a totally isotropic subspace of K^V . Let $X_1, Y_1 \subseteq V \setminus \{v\}$. Let $x, y \in K \setminus \{0\}$, $x \neq y$. Then,*

$$\dim(L|_{\subseteq X_1 \cap Y_1}) + \dim(L|_{\subseteq X_1 \cup Y_1 \cup \{v\}}) \geq \dim(L|_x^v|_{\subseteq X_1}) + \dim(L|_y^v|_{\subseteq Y_1}).$$

In other words,

$$\lambda(L|_x^v|_{\subseteq X_1}) + \lambda(L|_y^v|_{\subseteq Y_1}) \geq \lambda(L|_{\subseteq X_1 \cap Y_1}) + \lambda(L|_{\subseteq X_1 \cup Y_1 \cup \{v\}}) - 1.$$

Proof. We may assume that $V = X_1 \cup Y_1 \cup \{v\}$ by taking $L' = L|_{\subseteq X \cup Y \cup \{v\}}$.

Let B be a minimum set of vectors in L such that $p_{X_1 \cap Y_1}(B)$ is a basis of $L|_{\subseteq X_1 \cap Y_1}$ and for every $z \in B$, $z(w) = 0$ for all $w \notin X_1 \cap Y_1$.

Let C be a minimum set of vectors in L such that $p_{X_1}(B \cup C)$ is a basis of $L|_x^v|_{\subseteq X_1}$ and for every $z \in C$, $z(w) = 0$ for all $w \notin X_1 \cup \{v\}$ and $z(v) \in \{0, x\}$. We may assume that at most one vector in C has x on v .

Let D be a minimum set of vectors in L such that $p_{Y_1}(B \cup D)$ is a basis of $L|_y^v|_{\subseteq Y_1}$ and for every $z \in D$, $z(w) = 0$ for all $w \notin Y_1 \cup \{v\}$ and $z(v) \in \{0, y\}$. We may assume that at most one vector in D has y on v .

We claim that $B \cup C \cup D$ is linearly independent. Suppose there are $B' \subseteq B$, $C' \subseteq C$, and $D' \subseteq D$ such that

$$\sum_{b \in B'} b + \sum_{c \in C'} c + \sum_{d \in D'} d = 0.$$

No element of C' has x on v , because all vectors in $B' \cup D'$ have 0 or y on v and at most one element of C' has x on v . Since $\sum_{c \in C'} c(w) = 0$ for all $w \in V \setminus (X_1 \cap Y_1)$, we have $p_{X_1 \cap Y_1}(\sum_{c \in C'} c) \in L|_{\subseteq X_1 \cap Y_1}$. Since $p_{X_1 \cap Y_1}(B)$ is a basis, there is $B'' \subseteq B$ such that $p_{X_1 \cap Y_1}(\sum_{c \in C'} c) = p_{X_1 \cap Y_1}(\sum_{b \in B''} b)$. So,

$$\sum_{c \in C'} c + \sum_{b \in B''} b = 0.$$

This means that $C' = \emptyset$ because $C \cup B$ is a basis. Similarly $D' = \emptyset$ and so $B' = \emptyset$. Hence $B \cup C \cup D$ is linearly independent indeed. So $\dim(L) \geq |B| + |C| + |D| = \dim(L|_x^v|_{\subseteq X_1}) + \dim(L|_y^v|_{\subseteq Y_1}) - \dim(L|_{\subseteq X_1 \cap Y_1})$. \square

Now, we translate Tutte's linking theorem into isotropic subspaces.

Theorem 7.2. *Let V be a finite set and X be a subset of V . Let L be a totally isotropic subspace of K^V . Let k be a constant. Let b be a complete vector of $K^{V \setminus X}$.*

Then $\lambda(L|_{\subseteq Z}) \geq k$ for all $Z \supseteq X$ if and only if there is a complete vector $a \in K^{V \setminus X}$ such that $\lambda(L|_a^{V \setminus X}) \geq k$ and $a(v) \neq b(v)$ for all $v \in V \setminus X$.

Proof. (\Leftarrow) Let Z be a subset of V such that $X \subseteq Z$. Let $a_1 = p_{V \setminus Z}(a)$, $a_2 = p_{Z \setminus X}(a)$. Since $L|_{\subseteq Z} \subseteq L|_{a_1}^{V \setminus Z}$, we have $\lambda(L|_{\subseteq Z}) \geq \lambda(L|_{a_1}^{V \setminus Z})$. So

$$\lambda(L|_{\subseteq Z}) \geq \lambda(L|_{a_1}^{V \setminus Z}) \geq \lambda(L|_{a_1}^{V \setminus Z}|_{a_2}^{Z \setminus X}) = \lambda(L|_a^{V \setminus X}) \geq k.$$

(\Rightarrow) Induction on $|V \setminus X|$. Suppose that there is no such complete vector $a \in K^{V \setminus X}$. We may assume that $|V \setminus X| \geq 1$.

Pick $v \in V \setminus X$. Let $K \setminus \{0, b(v)\} = \{x, y\}$. Since there is no complete vector $a' \in K^{V \setminus \{v\} \setminus X}$ such that $\lambda(L|_x^v|_{a'}^{V \setminus \{v\} \setminus X}) \geq k$, there exists X_1 such that $X \subseteq X_1 \subseteq V \setminus \{v\}$ and $\lambda(L|_x^v|_{\subseteq X_1}) < k$. Similarly, there exists Y_1 such that $X \subseteq Y_1 \subseteq V \setminus \{v\}$ and $\lambda(L|_y^v|_{\subseteq Y_1}) < k$. By Lemma 7.1, either $\lambda(L|_{\subseteq X_1 \cap Y_1}) < k$ or $\lambda(L|_{\subseteq X_1 \cup Y_1 \cup \{v\}}) < k$, a contradiction. \square

Corollary 7.3. *Let V be a finite set and X be a subset of V . Let L be a totally isotropic subspace of K^V . Let b be a complete vector of $K^{V \setminus X}$.*

If $\lambda(L|_{\subseteq Z}) \geq \lambda(L|_{\subseteq X})$ for all $Z \supseteq X$, then there is a complete vector $a \in K^{V \setminus X}$ such that $L|_a^{V \setminus X} = L|_{\subseteq X}$ and $a(v) \neq b(v)$ for all $v \in V \setminus X$.

Proof. By Theorem 7.2, there exists a complete vector $a \in K^{V \setminus X}$ such that

$$\lambda(L|_a^{V \setminus X}) = \lambda(L|_{\subseteq X}) \text{ and } a(v) \neq b(v) \text{ for all } v \in V \setminus X.$$

Since $L|_{\subseteq X} \subseteq L|_a^{V \setminus X}$ and $\dim(L|_{\subseteq X}) = \dim(L|_a^{V \setminus X})$, we have $L|_{\subseteq X} = L|_a^{V \setminus X}$. \square

Corollary 7.4. *Let $P = (V, L, B)$ be a scrap and $X \subseteq V$. If*

$$\lambda(P) = \lambda(L|_{\subseteq X}) = \min_{X \subseteq Z \subseteq V} \lambda(L|_{\subseteq Z}),$$

then there is an ordered set B' such that $Q = (X, L|_{\subseteq X}, B')$ is a scrap and an $\alpha\beta$ -minor of P .

Proof. By applying Corollary 7.3 with $b(v) = \gamma$ for all $v \in V \setminus X$, there is a complete vector $a \in K^{V \setminus X}$ such that $L|_a^{V \setminus X} = L|_{\subseteq X}$ and $a(v) \in \{\alpha, \beta\}$ for all $v \in V \setminus X$. Let $V \setminus X = \{y_1, y_2, \dots, y_m\}$ and $a_i = a(y_i)$. Then, $L|_{\subseteq X} = L|_{a_1|_{a_2}^{y_1}|_{a_2}^{y_2} \dots |_{a_m}^{y_m}}$. Let $L_0 = L$ and $L_i = L_{i-1}|_{a_i}^{y_i}$. By Proposition 5.3, $\lambda(L|_{\subseteq X}) = \lambda(L) = \lambda(L_i)$ implies $\delta_{a_{i+1}}^{y_{i+1}} \notin L_i^\perp \setminus L_i$. So, $P|_{a_1|_{a_2}^{y_1}|_{a_2}^{y_2} \dots |_{a_m}^{y_m}} = (X, L|_{\subseteq X}, B')$ is well-defined and is an $\alpha\beta$ -minor of P . \square

8. SUM

A scrap $P = (V, L, B)$ is called a *sum* of two disjoint scraps $P_1 = (V_1, L_1, B_1)$ and $P_2 = (V_2, L_2, B_2)$ if

$$V = V_1 \cup V_2, \quad L_1 = L|_{\subseteq V_1}, \quad \text{and} \quad L_2 = L|_{\subseteq V_2}.$$

For given two disjoint scraps, there could be many scraps that are sums of those. In this section, we define the *connection type*, which determines a sum uniquely.

Let $[n]$ denote the set $\{1, 2, 3, \dots, n\}$.

Definition 8.1. Let $P = (V, L, B)$ be a sum of two disjoint scraps $P_1 = (V_1, L_1, B_1)$ and $P_2 = (V_2, L_2, B_2)$ where $B = \{b_1 + L, b_2 + L, \dots, b_n + L\}$, $B_1 = \{b_1^1 + L_1, b_2^1 + L_1, \dots, b_m^1 + L_1\}$, and $B_2 = \{b_1^2 + L_2, b_2^2 + L_2, \dots, b_l^2 + L_2\}$. For $x_1 \in K^{V_1}$ and $x_2 \in K^{V_2}$, let $x_1 \oplus x_2$ denote the vector in K^V such that $p_{V_i}(x_1 \oplus x_2) = x_i$ for $i = 1, 2$. Let

$$C_0 = \left\{ (X, Y) : X \subseteq [m], Y \subseteq [l], \left(\sum_{i \in X} b_i^1 \right) \oplus \left(\sum_{j \in Y} b_j^2 \right) \in L \right\},$$

$$C_s = \left\{ (X, Y) : X \subseteq [m], Y \subseteq [l], \left(\sum_{i \in X} b_i^1 \right) \oplus \left(\sum_{j \in Y} b_j^2 \right) - b_s \in L \right\} \quad s = 1, \dots, n$$

The sequence $C(P, P_1, P_2) = (C_0, C_1, C_2, \dots, C_n)$ is called the *connection type* of this sum.

It is easy to see that if $\lambda(P), \lambda(P_1), \lambda(P_2) \leq k$, then the number of distinct connection types is bounded by a function of k , because $|B| = 2\lambda(P) \leq 2k$ and $|B_i| = 2\lambda(P_i) \leq 2k$ for $i = 1$ and 2 .

Proposition 8.2. *The connection type is well-defined.*

Proof. It is enough to show that the choice of b_i, b_i^1 , and b_i^2 does not affect C_i . Suppose $b_i + L = d_i + L, b_i^1 + L_1 = d_i^1 + L_1$, and $b_i^2 + L_2 = d_i^2 + L_2$. For any (X, Y) such that $X \subseteq [m], Y \subseteq [l]$, we have $\sum_{i \in X} (b_i^1 - d_i^1) \oplus \sum_{j \in Y} (b_j^2 - d_j^2) \in L$ and $b_s - d_s \in L$, and therefore C_0 and C_s are well-defined. \square

Proposition 8.3. *The connection type uniquely determines the sum of two disjoint scraps P_1 and P_2 .*

Proof. Suppose not. Let $P = (V, L, B), Q = (V, L', B')$ be two distinct sums of $P_1 = (V_1, L_1, B_1)$ and $P_2 = (V_2, L_2, B_2)$ by the same connection type (C_0, C_1, \dots, C_n) . Let $B_1 = \{b_1^1 + L_1, b_2^1 + L_1, \dots, b_m^1 + L_1\}$, and $B_2 = \{b_1^2 + L_2, b_2^2 + L_2, \dots, b_k^2 + L_2\}$.

We claim that $L = L'$. To show this, it is enough to show that $L \subseteq L'$. For any $a \in L, p_{V_1}(a) \in (L|_{\subseteq V_1})^\perp$ and $p_{V_2}(a) \in (L|_{\subseteq V_2})^\perp$. Therefore there is (X, Y) such that

$$x_1 = \sum_{i \in X} b_i^1 - p_{V_1}(a) \in L_1 \quad \text{and} \quad x_2 = \sum_{i \in Y} b_i^2 - p_{V_2}(a) \in L_2.$$

Since $x_1 \oplus 0, 0 \oplus x_2 \in L$, we have $x_1 \oplus x_2 \in L$. We deduce that $\sum_{i \in X} b_i^1 \oplus \sum_{i \in Y} b_i^2 = a + (x_1 \oplus x_2) \in L$. Therefore, $(X, Y) \in C_0$ and $a + (x_1 \oplus x_2) \in L'$. Since $x_1 \oplus 0, 0 \oplus x_2 \in L'$, we have $x_1 \oplus x_2 \in L'$, and so $a \in L'$.

Now, we show that $B = B'$. Let $b_j + L$ be the j -th element of B with $b_j \in L^\perp$. Let $b'_j + L$ be the j -th element of B' with $b'_j \in L^\perp$. Since $p_{V_i}(b_j) \in (L|_{\subseteq V_i})^\perp$, there is (X, Y) such that

$$x_1 = \sum_{i \in X} b_i^1 - p_{V_1}(b_j) \in L_1 \quad \text{and} \quad x_2 = \sum_{i \in Y} b_i^2 - p_{V_2}(b_j) \in L_2.$$

Since $x_1 \oplus 0, 0 \oplus x_2 \in L$, we have $x_1 \oplus x_2 \in L$, and therefore $\sum_{i \in X} b_i^1 \oplus \sum_{i \in Y} b_i^2 - b_j \in L$. Thus, $(X, Y) \in C_s$, and so

$$\sum_{i \in X} b_i^1 \oplus \sum_{i \in Y} b_i^2 - b_j \in L' = L.$$

Thus, $b_j + L = b'_j + L = b'_j + L'$. \square

Proposition 8.4. *Let $P_1 = (V_1, L_1, B_1), P_2 = (V_2, L_2, B_2)$ be two disjoint scraps. Let P be the sum of P_1 and P_2 by connection type $C(P, P_1, P_2)$. If $v \in V_1$ and $\delta_x^v \notin L_1^\perp \setminus L_1$, then $\delta_x^v \notin L^\perp \setminus L$ and $P|_x^v$ is the sum of $P_1|_x^v$ and P_2 by connection type $C(P, P_1, P_2)$.*

Proof. If $\delta_x^v \in L^\perp \setminus L$, then $\delta_x^v \in (L^\perp)|_{V_1} = (L|_{\subseteq V_1})^\perp = L_1^\perp$ and $\delta_x^v \notin L|_{\subseteq V_1}$. This contradicts $\delta_x^v \notin L_1^\perp \setminus L_1$. So, $\delta_x^v \notin L^\perp \setminus L$.

First, we claim that $P|_x^v$ is a sum of $P_1|_x^v$ and P_2 . This is equivalent with

$$L|_x^v|_{\subseteq V_1 \setminus \{v\}} = L|_{\subseteq V_1}|_x^v \quad \text{and} \quad L|_x^v|_{\subseteq V_2} = L|_{\subseteq V_2}.$$

So, as clearly $L|_x^v|_{\subseteq V_1 \setminus \{v\}} = L|_{\subseteq V_1}|_x^v$ and $L|_{\subseteq V_2} \subseteq L|_x^v|_{\subseteq V_2}$, it suffices to show that

$$L|_x^v|_{\subseteq V_2} \subseteq L|_{\subseteq V_2}.$$

Suppose $z \in L|_{\subseteq V_2}^v$. Let $\bar{z} \in K^V$ such that $p_{V_2}(\bar{z}) = z$, $\bar{z}(v) \in \{0, x\}$, and $p_{V_1 \setminus \{v\}}(\bar{z}) = 0$. If $\bar{z}(v) = 0$, then $z \in L|_{\subseteq V_2}$. If $\bar{z}(v) = x$, then $p_{V_1}(z) = \delta_x^v \in L^\perp|_{V_1} = L_1^\perp$, and therefore $\delta_x^v \in L_1$. So, $\delta_x^v \in L$ and $z + \delta_x^v \in L$. Since $(z + \delta_x^v)(v) = 0$, we have $z \in L|_{\subseteq V_2}$. This proves that $P|_x^v$ is a sum of $P_1|_x^v$ and P_2 .

Now, let us show that $C(P, P_1, P_2) = C(P|_x^v, P_1|_x^v, P_2)$. Let $B_1 = \{b_1^1 + L_1, b_2^1 + L_1, \dots, b_m^1 + L_1\}$, and $B_2 = \{b_1^2 + L_2, b_2^2 + L_2, \dots, b_k^2 + L_2\}$. For $s \in K^{V_1}$ and $t \in K^{V_2}$, let $s \oplus t$ denote the vector in K^V such that $p_{V_1}(s \oplus t) = s$ and $p_{V_2}(s \oplus t) = t$. We may assume that $b_i^1(v) \in \{0, x\}$ for all i by Lemma 6.1. Let $b \in L^\perp$ be such that $b(v) \in \{0, x\}$. Let $a(X, Y) = \left(\sum_{i \in X} b_i^1\right) \oplus \left(\sum_{j \in Y} b_j^2\right) - b$. Suppose we have (X, Y) such that $X \subseteq [m]$, $Y \subseteq [k]$, and $a(X, Y) \in L$. Since $\left(\sum_{i \in X} b_i^1(v)\right) - b(v) \in \{0, x\}$, we have

$$p_{V \setminus \{v\}}(a(X, Y)) = \left(\sum_{i \in X} p_{V_1 \setminus \{v\}}(b_i^1)\right) \oplus \left(\sum_{j \in Y} b_j^2\right) - p_{V \setminus \{v\}}(b) \in L|_x^v.$$

Conversely, let us suppose that there is (X, Y) such that $X \subseteq [m]$, $Y \subseteq [k]$, and $\left(\sum_{i \in X} p_{V \setminus \{v\}}(b_i^1)\right) \oplus \left(\sum_{j \in Y} b_j^2\right) - p_{V \setminus \{v\}}(b) \in L|_x^v$. Then, either $a(X, Y) \in L$ or $a(X, Y) + \delta_x^v \in L$. If $\delta_x^v \in L$, then $a(X, Y) \in L$. If $\delta_x^v \notin L^\perp$, then $a(X, Y) + \delta_x^v \notin L^\perp$ because $a(X, Y) \in L^\perp$, and therefore $a(X, Y) \in L$. This proves that $C(P, P_1, P_2) = C(P|_x^v, P_1|_x^v, P_2)$. \square

Lemma 8.5. *Let P_1, P_2, Q_1, Q_2 be scraps. Let P be a sum of P_1 and P_2 and Q be a sum of Q_1 and Q_2 . If P_i is a minor of Q_i for each $i = 1, 2$ and the connection type of P_1 and P_2 is equal to the connection type of Q_1 and Q_2 , then P is a minor of Q .*

Moreover, if P_i is an $\alpha\beta$ -minor of Q_i for each $i \in \{1, 2\}$ and the connection type of P_1 and P_2 is equal to the connection type of Q_1 and Q_2 , then P is an $\alpha\beta$ -minor of Q .

Proof. Induction on $|V(Q_1) \setminus V(P_1)| + |V(Q_2) \setminus V(P_2)|$. We may assume $|V(Q_1) \setminus V(P_1)| + |V(Q_2) \setminus V(P_2)| > 0$ and $V(Q_1) \neq V(P_1)$ by symmetry. There are $v \in V(Q_1) \setminus V(P_1)$, $x \in K \setminus \{0\}$, $X = V(Q_1) \setminus V(P_1) \setminus \{v\}$, and a complete vector $a \in K^X$ such that $P_1 = Q_1|_{x|a}^v$. If P_1 is an $\alpha\beta$ -minor of Q_1 , then we may assume $x \in \{\alpha, \beta\}$ and $a(w) \in \{\alpha, \beta\}$ for all $w \in X$.

By Proposition 8.4, $Q|_x^v$ is the sum of $Q_1|_x^v$ and Q_2 with the connection type $C(Q|_x^v, Q_1|_x^v, Q_2) = C(Q, Q_1, Q_2) = C(P, P_1, P_2)$. So, P is a minor of $Q|_x^v$ by induction and therefore P is a minor of Q .

Similarly if P_1 is an $\alpha\beta$ -minor of Q_1 and P_2 is an $\alpha\beta$ -minor of Q_2 , then by induction P is an $\alpha\beta$ -minor of Q . \square

9. WELL-QUASI-ORDERING

Proposition 9.1. *Let k be a constant. If $\{S_1, S_2, S_3, \dots\}$ is an infinite sequence of isotropic systems of branch-width at most k , then there exist $i < j$ such that S_i is simply isomorphic to an $\alpha\beta$ -minor of S_j .*

Proof. By Corollary 4.4, there is a linked branch-decomposition (T_i, \mathcal{L}_i) of $S_i = (V_i, L_i)$ of width at most k for each i . In T_i , we pick an edge and attach a root and direct every edge so that each leaf has a directed path from the root. Let F be the forest such that the i -th component is T_i .

For each edge e of T_i , let X_e be the set of leaves of T_i having a directed path from e . Let $A_e = \mathcal{L}_i^{-1}(X_e)$. We associate e with a scrap $P_e = (A_e, L_i|_{\subseteq A_e}, B_e)$ and $\lambda(e) = \lambda(L_i|_{\subseteq A_e}) \leq k$ where B_e is chosen to satisfy the following:

If f is λ -linked to e , then P_e is an $\alpha\beta$ -minor of P_f .

We claim that we can choose B_e satisfying the above property. We prove it by induction on the length of the directed path from a root edge to e . If no other edge is λ -linked to e , let B_e be a basis of $(L_i|_{\subseteq A_e})^\perp / (L_i|_{\subseteq A_e})$ in an arbitrary order. If f , other than e , is λ -linked to e , choose f such that the distance between e and f is minimal. By induction, there exists B_f so that whenever g is λ -linked to f , P_f is an $\alpha\beta$ -minor of P_g . By Corollary 7.4, there is an ordered basis B_e such that P_e is an $\alpha\beta$ -minor of P_f . Suppose that g is λ -linked to e . Then f is on the path from g to e and therefore g is λ -linked to f . Thus, P_f is an $\alpha\beta$ -minor of P_g and therefore P_e is an $\alpha\beta$ -minor of P_g . We conclude that there is B_e satisfying the property.

For $e, f \in E(F)$, let $e \leq f$ denote that a scrap P_e is isomorphic to an $\alpha\beta$ -minor of a scrap P_f . Clearly, \leq has no infinite strictly descending sequences, because there are finitely many scraps of bounded number of elements up to isomorphisms. By construction if f is λ -linked to e , then $e \leq f$.

The leaf edges of F are well-quasi-ordered, because there are only finitely many distinct scraps of one element up to isomorphisms.

Suppose the root edges are not well-quasi-ordered. By Lemma 4.5, F contains an infinite sequence (e_0, e_1, \dots) of non-leaf edges such that

- (i) $\{e_0, e_1, \dots\}$ is an antichain with respect to \leq ,
- (ii) $l(e_0) \leq l(e_1) \leq \dots$,
- (iii) $r(e_0) \leq r(e_1) \leq \dots$.

Since $\lambda(e_i) \leq k$ for all i , we may assume that $\lambda(e_i)$ is a constant for all i , by taking a subsequence.

Since the number of distinct connection types $C(P_{e_i}, P_{l(e_i)}, P_{r(e_i)})$ is finite, we may assume that the connection types are same for all i , also by taking a subsequence.

Then, by Lemma 8.5, P_{e_0} is isomorphic to an $\alpha\beta$ -minor of P_{e_1} , that is $e_0 \leq e_1$. This contradicts that $\{e_0, e_1, \dots\}$ is an antichain with respect to \leq .

Therefore, root edges are well-quasi-ordered, and there exist $i < j$ such that a scrap (V_i, L_i, \emptyset) is isomorphic to an $\alpha\beta$ -minor of a scrap (V_j, L_j, \emptyset) . Thus, S_i is simply isomorphic to an $\alpha\beta$ -minor of S_j . \square

10. PIVOT-MINOR AND $\alpha\beta$ -MINOR

In this section, we shall show a relation between pivot-minors of graphs and $\alpha\beta$ -minors of isotropic systems.

Proposition 10.1. *For $i \in \{1, 2\}$, let S_i be an isotropic system with a graphic presentation (G_i, a_i, b_i) such that $a_i(v), b_i(v) \in \{\alpha, \beta\}$ for all $v \in V(G_i)$.*

If $S_1 = S_2$, then G_1 can be obtained from G_2 by applying a sequence of pivots.

Proof. Let $V = V(G_1) = V(G_2)$ and let $S = S_1 = S_2 = (V, L)$ be an isotropic system. We show Proposition 10.1 by induction on $N(a_1, a_2) = |\{v \in V : a_1(v) \neq a_2(v)\}|$.

If $N(a_1, a_2) = 0$, then $b_1 = b_2$, so $G_1 = G_2$. Hence we may assume that $N(a_1, a_2) \geq 1$. Let $u \in V$ with $a_1(u) \neq a_2(u)$. This implies that $a_1(u) = b_2(u)$ because $a_i(u), b_i(u) \in \{\alpha, \beta\}$. By Proposition 3.5, there exists a vector $\hat{b}_u \in L$ such that $\hat{b}_u(u) = b_2(u) = a_1(u)$ and

$\hat{b}_u(w) \in \{0, a_2(w)\}$ for all $w \neq u$. As a_1 is Eulerian and as $\hat{b}_u \in L$, there exists $v \in V$ such that $\hat{b}_u(v) \notin \{0, a_1(v)\}$. Hence $\hat{b}_u(v) = a_2(v)$ and $a_1(v) \neq a_2(v)$.

Since $\hat{b}_u(u) \neq a_2(u)$, this means that $v \neq u$. Moreover, since $\hat{b}_u(v) \neq 0$, the vertices u and v are adjacent in G_2 . Pivoting (G_2, a_2, b_2) on uv yields $(G_2 \wedge uv, a_2[V \setminus \{u, v\}] + b_2[\{u, v\}], b_2[V \setminus \{u, v\}] + a_2[\{u, v\}])$. As $N(a_1, a_2[V \setminus \{u, v\}] + b_2[\{u, v\}]) = N(a_1, a_2) - 2$, it follows by induction that $G_2 \wedge uv$ can be obtained from G_1 by a sequence of pivots, hence so can G_2 . \square

Lemma 10.2. *For $i \in \{1, 2\}$, let S_i be the isotropic system with a graphic presentation (G_i, a_i, b_i) such that $a_i(v), b_i(v) \in \{\alpha, \beta\}$ for all $v \in V(G_i)$. If S_1 is an $\alpha\beta$ -minor of S_2 , then G_1 is a pivot-minor of G_2 .*

Proof. This lemma is a straightforward consequence of Proposition 10.1 and Proposition 3.7. \square

11. BINARY MATROIDS

We would like to show that Theorem 4.1 implies the well-quasi-ordering theorem of Geelen, Gerards, and Whittle [8] for binary matroids. The proof uses the following lemma.

Lemma 11.1 (Higman's lemma). *Let \leq be a well-quasi-order on X . For finite subsets $A, B \subseteq X$, we write $A \leq B$ if there is an injective mapping $f : A \rightarrow B$ such that $a \leq f(a)$ for all $a \in A$. Then \leq is a well-quasi-ordering on the set of all finite subsets of X . (For proof, see Diestel's book [7, Lemma 12.1.3].)*

For a binary matroid M with a fixed base B , the *fundamental graph* of M with respect to B is a bipartite graph $Bip(M, B)$ such that $V(Bip(M, B)) = E(M)$ and $v \in E(M) \setminus B$ is adjacent to $w \in B$ if and only if w is in the fundamental circuit of v with respect to B . For a bipartite graph $G = (V, E)$ with a bipartition $V = A \cup B$, $Bin(G, A, B)$ is a binary matroid on V , represented by a $A \times V$ matrix $(I_A \quad A_G[A, B])$, where I_A is a $A \times A$ identity matrix.

Lemma 11.2. *Let M_1, M_2 be binary matroids and let B_i be a fixed base of M_i . If M_1 is connected and $Bip(M_1, B_1)$ is a pivot-minor of $Bip(M_2, B_2)$, then M_1 is a minor of either M_2 or M_2^* .*

Proof. Let $H = Bip(M_1, B_1)$ and $G = Bip(M_2, B_2)$. If H is a pivot-minor of a bipartite graph G , then there is a bipartition (A', B') of H such that a binary matroid $M_3 = Bin(H, A', B')$ is a minor of $M_2 = Bin(G, B_2, V(G) \setminus B_2)$. Since M_1 is connected, H is connected and therefore H has a unique bipartition. So, $M_1 = M_3$ if $A = A'$ or $M_1 = M_3^*$ if $A = B'$. \square

Corollary 11.3. *Let k be a constant. If $\{M_1, M_2, M_3, \dots\}$ is an infinite sequence of binary matroids of branch-width at most k , then there exist $i < j$ such that M_i is isomorphic to a minor of M_j .*

Proof. First, we claim that if M_i is connected for all i , then the statement is true. Let B_i be a fixed base of M_i and $G_i = Bip(M_i, B_i)$ for all i . The rank-width of G_i is at most $k - 1$, since rank-width of G_i is equal to (branch-width of M_i) - 1, shown in [11]. By Theorem 4.1, there is an infinite subsequence $G_{a_1}, G_{a_2}, G_{a_3}, \dots$ such that G_{a_i} is isomorphic to a pivot-minor of $G_{a_{i+1}}$ for all i . By Lemma 11.2, M_{a_1} is isomorphic to a minor of either M_{a_2} or $M_{a_2}^*$, and M_{a_2}

is isomorphic to a minor of either M_{a_3} or $M_{a_3}^*$. It follows that M_{a_1} is isomorphic to a minor of M_{a_2} , or M_{a_2} is isomorphic to a minor of M_{a_3} , or M_{a_1} is isomorphic to a minor of M_{a_3} . This proves the above claim.

Now, we prove the main statement. We may consider each M_i as a set of disjoint connected matroids and then M_i is isomorphic to a minor of M_j if and only if there is an injective function f from components of M_i to components of M_j such that a is isomorphic to a minor of $f(a)$ for every component a of M_i . By Higman's lemma, there exist $i < j$ such that M_i is isomorphic to a minor of M_j . \square

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