Graphs of Bounded Rank-width

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Abstract

We define *rank-width* of graphs to investigate *clique-width*. Rank-width is a complexity measure of decomposing a graph in a kind of tree-structure, called a *rank-decomposition*. We show that graphs have bounded rank-width if and only if they have bounded clique-width.

It is unknown how to recognize graphs of clique-width at most k for fixed $k > 3$ in polynomial time. However, we find an algorithm recognizing graphs of rank-width at most k, by combining following three ingredients.

First, we construct a polynomial-time algorithm, for fixed k, that confirms rank-width is larger than k or outputs a rank-decomposition of width at most $f(k)$ for some function f. It was known that many hard graph problems have polynomial-time algorithms for graphs of bounded clique-width, however, requiring a given decomposition corresponding to clique-width (k-*expression*); we remove this requirement.

Second, we define graph *vertex-minors* which generalizes matroid minors, and prove that if $\{G_1, G_2, \ldots\}$ is an infinite sequence of graphs of bounded rank-width, then there exist $i < j$ such that G_i is isomorphic to a *vertex-minor* of G_j. Consequently there is a finite list $\mathcal{C}_k$ of graphs such that a graph has rank-width at most k if and only if none of its vertex-minors are isomorphic to a graph in $\mathcal{C}_k$.

Finally we construct, for fixed graph H, a modulo-2 counting monadic second-order logic formula expressing a graph contains a vertex-minor isomorphic to H. It is known that such logic formulas are solvable in linear time on graphs of bounded clique-width if the k-expression is given as an input.

Another open problem in the area of clique-width is Seese's conjecture; if a set of graphs have an algorithm to answer whether a given monadic second-order logic formula is true for all graphs in the set, then it has bounded rank-width. We prove a weaker statement; if the algorithm answers for all modulo-2 counting monadic second-order logic formulas, then the set has bounded rank-width.

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To Laehee and Seyoung.

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Chapter 1

Introduction

Some algorithmic problems, NP-hard on general graphs, are known to be solvable in polynomial time when the input graph admits a decomposition into trivial pieces by means of a tree-structure of cutsets of bounded order. However, it makes a difference whether the input graph is presented together with the corresponding tree-structure of cutsets or not. We have in mind two kinds of decompositions, "tree-width" and "clique-width" decompositions. These are similar graph invariants, and while the results of this paper concern clique-width, we begin with tree-width for purposes of comparison.

Having bounded clique-width is more general than having bounded tree-width, in the following sense. Every graph G of tree-width at most k has clique-width at most $O(2^k)$ (Corneil and Rotics [11], Courcelle and Olariu [19]), and for such graphs (for k fixed) the clique-width of G can be determined in linear time (Espelage et al. [24]). No bound in the reverse direction holds, for there are graphs of arbitrary large tree-width with clique-width at most k. (But, for fixed t, if G does not contain $K_{t,t}$ as a subgraph, then the tree-width is at most $3k(t-1)-1$, shown by Gurski and Wanke [31].)

The algorithmic situation with tree-width is as follows:

- Numerous problems have been shown to be solvable in polynomial time when the input graph is presented together with a decomposition of bounded tree-width. Indeed, every graph property expressible by monadic second order logic formulas with quantifications over vertices, vertex sets, edges, and edge sets (MS_2 logic formula) can be solved in polynomial time (see Courcelle [15]).

- For fixed k there is a polynomial time algorithm that either decides that an input graph has tree-width at least $k+1$, or outputs a decomposition of tree-width at most $4k$ (this is an easy modification of the algorithm to estimate graph branchwidth presented by Robertson and Seymour [49]).

- Consequently, by combining these algorithms, it follows that the same class of problems mentioned above can be solved on inputs of bounded tree-width; the input does not need to come equipped with the corresponding decomposition.

- In particular, one of these problems is the problem of deciding whether a graph has tree-width at most k. Consequently, for fixed k there is a polynomial (indeed, linear) time algorithm by Bodlaender [3] to test whether an input graph has tree-width at most k, and if so to output the corresponding decomposition.

For inputs of bounded clique-width, less progress has so far been made. (We will define clique-width properly later.)

- Some problems have been shown to be solvable in polynomial time when the input graph is presented together with a decomposition of bounded clique-width. This class of problems is smaller than the corresponding set for tree-width, but still of interest. For instance, deciding whether the graph is Hamiltonian (Wanke [58]), finding the chromatic number (Kobler and Rotics [39]), and various partition problems (Espelage et al. [23]) are solvable in polynomial time; and so is any problem that can be expressed in monadic second order logic with quantifications over vertices and vertex sets (MS logic; see Courcelle et al. [18] and Courcelle [15]).

- For fixed (general) k there was so far no known polynomial time algorithm that either decides that an input graph has clique-width at least $k+1$, or outputs a decomposition of clique-width bounded by any function of k. The best hitherto was an algorithm of Johansson [38], that with input an n-vertex graph G, either decides that G has clique-width at least $k+1$ or outputs a decomposition of clique-width at most $2k \log n$. Our main result fills this gap.

- Consequently, it follows that the same class of problems mentioned above can be solved on inputs of bounded clique-width; the input does not need to come equipped with the corresponding decomposition.

- However, the problem of deciding whether a graph has clique-width at most k is not known to belong to this class. There is still no polynomial time algorithm to test whether G has clique-width at most k, for fixed general $k > 3$.

Rank-width

In order to study graphs of bounded clique-width, we define another graph parameter, called *rank-width*, in Section 3.2. Rank-width is based on the notion of *branch-width* defined on symmetric submodular functions by Robertson and Seymour [48]. A tree-like decomposition for branch-width is called a *branch-decomposition*, and we measure its *width*, and the branch-width is the minimum possible width of all branch-decompositions. We define certain symmetric submodular functions on graphs, called *cut-rank* functions, by using a matrix rank over GF(2). By using *cut-rank* functions, we define rank-width and *rank-decompositions* of graphs as branch-width and branch-decompositions of their cut-rank functions. It turns out that a set of graphs has bounded rank-width if and only if it has bounded clique-width. More precisely, we obtain the following inequality

$$\text{rank-width} \leq \text{clique-width} \leq 2^{1+\text{rank-width}} - 1.$$

Basically we will show results based on rank-width, but they can be formulated in terms of clique-width as well by this inequality.

Approximation algorithms

A big open problem in the area of clique-width was how to remove the need of a decomposition of bounded clique-width as an input. Since there were no known methods to find a decomposition, most algorithms just assume that it is given as an input. To solve this problem, ideally we would like to have an algorithm, for fixed k, that constructs a decomposition of clique-width at most k, called a *k-expression* if an input graph has clique-width at most k and is given by its adjacency list. But we do not have such an algorithm yet. Instead, we construct a polynomial-time algorithm that constructs a decomposition of clique-width at most $f(k)$ ($f(k)$-expression) or confirms that the input graph has clique-width at least $k+1$, for a fixed function f. In fact, this is enough to remove the need of k-expressions as an input to many algorithms requiring them, because we can provide $f(k)$-expressions instead of k-expressions and we still obtain polynomial-time algorithms.

To obtain this "approximating" algorithm, we show that branch-width of certain symmetric submodular functions can be in fact "approximated" in the following sense: there is an algorithm that outputs a branch-decomposition of width at most $O(3k)$ or confirms that it has branch-width larger than k. As an easy corollary, we obtain an approximating algorithm for rank-width. In Section 7.1 and 7.2, we show two quicker algorithms approximating rank-width. We have a $O(n^4)$-time algorithm with $f(k) = 3k+1$ in Section 7.1, and a $O(n^3)$-time algorithm with $f(k) = 24k$ in Section 7.2 where n is the number of vertices in the input graph.

We also apply this algorithm to matroids, and obtain an algorithm to approximate the branch-width of matroids, which was known before only for representable matroids by Hliněný [32]. We prove:

Theorem 1.1. *For fixed k there is an algorithm which, with input an n-element matroid $\mathcal{M}$ in terms of its rank oracle, either decides that $\mathcal{M}$ has branch-width at least $k+1$, or outputs a branch-decomposition for $\mathcal{M}$ of width at most $3k-1$. Its running time and number of oracle calls is at most $O(n^4)$.*

Vertex-minors and well-quasi-ordering

Tree-width of graphs is interesting when considered together with the *graph minor* relation. Contraction of an edge e is the operation that deletes e and identifies the ends of e. A graph H is a *minor* of a graph G if H can be obtained from G by a sequence of contractions, vertex deletions, and edge deletions. If H is a minor of G, then the tree-width of H is at most that of G. This implies that for fixed k, the set of all graphs having tree-width at most k is closed under the graph minor relation.

To have similar statements for clique-width, we need an appropriate containment relation on graphs such that many theorems relating the graph minor relation to tree-width can be translated into theorems relating our containment relation to clique-width. Minor containment is not appropriate for clique-width because every graph G is a minor of the complete graph K_n on $n = |V(G)|$ vertices, and K_n has clique-width 2 if $n > 1$.

Courcelle and Olariu [19] showed that if H is an induced subgraph of a graph G, then the clique-width of H is at most that of G. But, induced subgraph containment is not rich enough; Corneil, Habib, Lanlignel, Reed, and Rotics wrote the following comment in their paper [9].

> Unfortunately, there does not seem to be a succinct forbidden subgraph characterization of graphs with clique-width at most 3, similar to the P_4-free characterization of graphs with clique-width at most 2. In fact every cycle C_n with $n \geq 7$ has clique-width 4, thereby showing an infinite set of minimal forbidden induced subgraphs for Clique-width≤ 3.

We have not yet found an appropriate containment relation for clique-width, but by generalizing the matroid minor relation, we define the *vertex-minor* relation of graphs. (It was orignally called *l-reduction* by Bouchet [8].) For a graph G and a vertex v of G, let $G * v$ be a graph, obtained by the local complementation at v, that is, replacing the graph induced on the set of neighbors of v by its complement. We say that G is *locally equivalent* to H if H can be obtained from G by applying a sequence of local complementations. A graph H is a *vertex-minor* of G if H can be obtained from G by applying a sequence of vertex deletions and local complementations. A simple fact is that if H is a vertex-minor of G, then the rank-width of H is at most that of G. For an edge uv of G, a pivoting uv is a composition of three local complementations, $G * u * v * u$. It is an easy exercise to show that $G * u * v * u = G * v * u * v$. We say that H is a pivot-minor of G if H can be obtained from G by applying a sequence of vertex deletions and pivotings. Every pivot-minor of G is a vertex-minor of G, but not vice versa.

In this paper, we prove the following.

Theorem 1.2. *Let k be a constant. If $\{G_1, G_2, G_3, \cdots\}$ is an infinite sequence of graphs of rank-width at most k, then there exist $i < j$ such that G_i is isomorphic to a pivot-minor of G_j, and therefore isomorphic to a vertex-minor of G_j.*

This implies that for each k, there is a finite list of graphs, such that a graph G has rank-width at most k if and only if no graph in the list is isomorphic to a vertex-minor of G.

This theorem was motivated by the following two theorems. The first one is for graphs of bounded tree-width, proved by Robertson and Seymour [47].

Theorem 1.3 (Robertson and Seymour [47]). *Let k be a constant.*

If $\{G_1, G_2, G_3, \cdots\}$ is an infinite sequence of graphs of tree-width at most k, then there exist $i < j$ such that G_i is isomorphic to a minor of G_j.

The next one, generalizing the previous one, was shown by Geelen, Gerards, and Whittle [27].

Theorem 1.4 (Geelen, Gerards, and Whittle [27]). *Let k be a constant. Let $\mathbb{F}$ be a finite field. If $\{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \cdots\}$ is an infinite sequence of $\mathbb{F}$-representable matroids of branch-width at most k, then there exist $i < j$ such that $\mathcal{M}_i$ is isomorphic to a minor of $\mathcal{M}_j$.*

If we set $\mathbb{F} = \mathrm{GF}(2)$, then Theorem 1.4 implies Theorem 1.2 for bipartite graphs. We will also show that Theorem 1.2 implies Theorem 1.4 if $\mathbb{F} = \mathrm{GF}(2)$. In fact, the main idea of proving Theorem 1.4 remains in our paper, although we have to go through a different technical notion.

In the original proof of Theorem 1.4, they used "configuration" to represent $\mathbb{F}$-representable matroids, and then convert the matroid problem into a vector space problem. We use a similar approach, but use a different notion. Research done by Bouchet [4, 7, 5] was very helpful. He developed the notion of isotropic systems, which generalize binary matroids. Informally speaking, an isotropic system can be considered as an equivalence class of graphs by *local equivalence*. A detailed definition will be reviewed in Section 4.1.

Seese's conjecture

We have seen that many NP-hard problems can be effectively solved for graphs of bounded tree-width or bounded clique-width. This fact is not only an observation but we have theorems stating this as follows.

- If a graph problem can be expressed by MS_2 logic formulas, then there is an algorithm that answers this problem in polynomial time if an input graph has bounded tree-width. (see [15])
- If a graph problem can be expressed by MS logic formulas, then there is an algorithm that answers this problem in polynomial time if an input graph has bounded clique-width [18].

Since there are many graph problems expressible by MS_2 logic formulas or MS logic formulas, the above two theorems prove usefulness of tree-width and clique-width.

We would like to ask another question related to logic formulas.

> Let $\mathcal{C}$ be a set of graphs. When does there exists an algorithm (not necessarily polynomial-time) that answers whether a given logic formula is satisfied for all graphs in $\mathcal{C}$?

The answer of this problem will depend on the set of logic formulas that will be given as an input. We are interested in two kinds of logic formulas on graphs, MS logic formulas and MS_2 logic formulas. If there is such an algorithm, then we call that $\mathcal{C}$ has a *decidable MS theory* or has a *decidable* MS_2 *theory* depending on the choice of logic formulas.

For MS_2 logic formulas, we have the following theorem called Seese's theorem [52]: if a set of graphs has a decidable MS_2 theory, then it has bounded tree-width. This answers the previous problem for MS_2 logic formulas. The proof uses the "grid theorem" by Robertson and Seymour [46] stating that if a set of graphs has bounded tree-width, then no graph in the set contains a minor isomorphic to a sufficiently large grid.

We are interested in answering the problem for MS logic formulas. The statement analogous to Seese's Theorem for MS formulas is a conjecture, also made by D. Seese in [52]. This conjecture says that if a set of graphs has a decidable MS theory, then it has bounded clique-width. Its hypothesis concerns less formulas, hence is weaker than that of Seese's Theorem. Since a set of graphs has bounded clique-width if it has bounded tree-width, Seese's Theorem is actually a weakening of the Conjecture.

In Chapter 5, we will actually prove a slight weakening of the Conjecture, by assuming that the considered sets of graphs has a decidable satisfiability problem for *C_2MS logic formulas*, in other words, for MS logic formulas that can be written with the set predicate $\mathsf{Card}_2(X)$, that we will write $\mathsf{Even}(X)$ for simplicity.

Recognizing rank-width

Our main objective was to find an exact algorithm that answers whether an input graph has clique-width at most k in polynomial time, but we were unable to find such an algorithm. This problem seems very hard because it is still unknown whether it is in co-NP to recognize graphs of clique-width at most k for fixed $k > 3$. Instead, we developed rank-width and may ask the same question but with rank-width.

In Section 2.3, we will show that, for a given symmetric submodular function that satisfies certain conditions and can be evaluated in polynomial time from the input, it is in NP∩co-NP to answer whether branch-width is at most k. This implies, in particular, that recognizing graphs of rank-width at most k is in NP∩co-NP.

We would like to have an algorithm that recognize graphs of rank-width at most k. Let us first see some analogous results for tree-width. To recognize graphs having tree-width at most k, we can use the following two theorems.

(1) For fixed k, there is a finite list of graphs such that a graph G has tree-width at most k if and only if no graph in the list is isomorphic to a minor of G (Robertson and Seymour [47]).

(2) For fixed graph H, there is a $O(|V(G)|^3)$-time algorithm that answers whether an input graph G contains a minor isomorphic to H (Robertson and Seymour [49]).

When we combine these two facts, we prove the existence of a polynomial-time algorithm to answer whether a given graph has tree-width at most k.

We now pay attention to rank-width. From the well-quasi-ordering theorem (Theorem 1.2), we have a theorem analogous to (1) in the above as follows: for fixed k, there is a finite list of graphs such that a graph G has rank-width at most k if and only if no graph in the list is isomorphic to a vertex-minor of G. But we do not have a polynomial-time algorithm to answer whether an input graph contains a vertex-minor isomorphic to a fixed graph.

Instead, we construct a C_2MS logic formula for a fixed graph H such that it is true if and only if an input graph contains a vertex-minor isomorphic to H. Since every C_2MS logic formula can be determined for graphs of bounded clique-width, we can recognize graphs of clique-width at most k by combining the following four statements.

- (Section 7.1 and 7.2) For fixed k, there is a polynomial-time algorithm that outputs a rank-decomposition of width $3k+1$ or confirms that the rank-width of the input graph is larger than k.
- (Section 6.8) For fixed k, there is a finite list of graphs such that a graph G has rank-width at most k if and only if no graph in the list is isomorphic to a vertex-minor of G.
- (Section 4.4) For fixed graph H, there is a C_2MS logic formula such that it is true on a graph G if and only if G contains a vertex-minor isomorphic to H.
- (Section 4.3) Every C_2MS logic formula on graphs can be decided in polynomial time if the input graph has bounded clique-width.

Conventions

In this thesis, we assume that graphs are simple undirected and finite.

Notes

Chapter 2 and Section 3.2 are joint work with P. Seymour [43]. Section 4.4, Chapter 5 (except Section 5.1), and Section 7.3 are joint work with B. Courcelle [20]: Section 2.1, 3.1, 4.1–4.3 are reviews of previous results. Other results without attribution are claimed to be original research. Section 3.3–3.7 come from the author's paper [42] that was accepted to Journal of Combinatorial Theory series B.

Chapter 2

Branch-width of Symmetric Submodular Functions

This chapter begins with the definition of *branch-width* of *symmetric submodular* functions. After defining branch-width, one natural question would be the following.

Problem 2.1. *Let k be a fixed constant and let V be a finite set. What is the time complexity of deciding whether the branch-width of a symmetric submodular function $f : 2^V \to \mathbb{Z}$ is at most k?*
(We assume that f is given by an oracle.*)*

We answer this question partially when f satisfies

$$f(\{v\}) - f(\emptyset) \leq 1 \text{ for all } v \in V. \tag{2.1}$$

In Section 2.3, we show that if the branch-width is larger than k, then there is a certificate of length polynomial in $|V|$ such that we can prove it using this certificate in a polynomial (of $|V|$) time, assuming that f satisfies (2.1) and is given by an oracle.

We were not yet able to find a polynomial-time algorithm to decide whether branch-width is at most k, but in Section 2.4, we show a polynomial-time "approximation" algorithm that, for fixed k, either confirms that branch-width is larger than k or obtains a branch-decomposition of width at most $3k + 1 - 2f(\emptyset)$, assuming that f satisfies (2.1).

There are some instances of f having better algorithmic properties. If V is the element set of a matroid $\mathcal{M}$ and f is the *connectivity function* of $\mathcal{M}$, then we obtain an approximation algorithm for the branch-width of matroids, and in Section 2.5 we show how to make the above algorithm faster by using properties of connectivity functions of matroids. In next chapter, we define the *rank-width* of graphs by using a certain symmetric submodular function on the set of vertices. In this case, the above approximation algorithm can run quickly, which will be discussed in Chapter 7.

2.1 Definition of branch-width

Let us write $\mathbb{Z}$ to denote the set of integers. Let V be a finite set and $f : 2^V \to \mathbb{Z}$ be a function. If

$$f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$$

for all $X, Y \subseteq V$, then f is said to be *submodular*. If f satisfies $f(X) = f(V \setminus X)$ for all $X \subseteq V$, then f is said to be *symmetric*.

A *subcubic tree* is a tree with at least two vertices such that every vertex is incident with at most three edges. A *leaf* of a tree is a vertex incident with exactly one edge. We call $(T, \mathcal{L})$ a *partial branch-decomposition* of a symmetric submodular function f if T is a subcubic tree and $\mathcal{L} : V \to \{t : t \text{ is a leaf of } T\}$ is a surjective function. (If $|V| \leq 1$ then f admits no partial branch-decomposition.) If in addition $\mathcal{L}$ is bijective, we call $(T, \mathcal{L})$ a *branch-decomposition* of f. If $\mathcal{L}(v) = t$, then we say t is *labeled by* v and v is *a label of* t.

For an edge e of T, the connected components of $T \setminus e$ induce a partition (X, Y) of the set of leaves of T. The *width* of an edge e of a partial branch-decomposition $(T, \mathcal{L})$ is $f(\mathcal{L}^{-1}(X))$. The *width* of $(T, \mathcal{L})$ is the maximum width of all edges of T. The *branch-width* $\text{bw}(f)$ of f is the minimum width of a branch-decomposition of f. (If $|V| \leq 1$, we define $\text{bw}(f) = f(\emptyset)$.)

We define a *linked* branch-decomposition. For a branch-decomposition $(T, \mathcal{L})$ of f, let e_1 and e_2 be two edges of T. Let E be the set of leaves of T in the component of $T \setminus e_1$ not containing e_2, and let F be the set of leaves of T in the component of $T \setminus e_2$ not containing e_1. Let P be the shortest path in T containing e_1 and e_2. We call e_1 and e_2 *linked* if

$$\min_{h \in E(P)} (\text{width of } h \text{ of } (T, \mathcal{L})) = \min_{\mathcal{L}^{-1}(E) \subseteq Z \subseteq V \setminus \mathcal{L}^{-1}(F)} f(Z).$$

We call a branch-decomposition $(T, \mathcal{L})$ *linked* if each pair of edges of T is linked.

2.2 Interpolation of a submodular function

In this section, we define an *interpolation* of a certain submodular function. Later we will use it to prove other theorems.

For a finite set V, we define (with a slight abuse of terminology) 3^V to be $\{(X, Y) : X, Y \subseteq V, X \cap Y = \emptyset\}$.

Definition 2.2. *Let $f : 2^V \to \mathbb{Z}$ be a submodular function such that $f(\emptyset) \leq f(X)$ for all $X \subseteq V$. We call $f^* : 3^V \to \mathbb{Z}$ an* interpolation *of f if*

i) *$f^*(X, V \setminus X) = f(X)$ for all $X \subseteq V$,*

ii) *(*uniform*) if $C \cap D = \emptyset$, $A \subseteq C$, and $B \subseteq D$, then $f^*(A, B) \leq f^*(C, D)$,*

iii) *(*submodular*) $f^*(A, B) + f^*(C, D) \geq f^*(A \cap C, B \cup D) + f^*(A \cup C, B \cap D)$ for all $(A, B), (C, D) \in 3^V$.*

iv) $f^*(\emptyset, \emptyset) = f(\emptyset)$.

Assuming that $\emptyset$ is a minimizer of f is not a serious restriction, because first of all it is true for all symmetric submodular functions, and secondly if we let

$$g(X) = \begin{cases} f(X) & \text{if } X \neq \emptyset \\ \min_Z f(Z) & \text{otherwise,} \end{cases}$$

then g is also submodular.

Proposition 2.3. *Let $f : 2^V \to \mathbb{Z}$ be a submodular function such that $f(\emptyset) \leq f(X)$ for all $X \subseteq V$, and let $f^* : 3^V \to \mathbb{Z}$ be an interpolation of f. Then:*

(1) *$f^*(X, Y) \leq \min_{X \subseteq Z \subseteq V \setminus Y} f(Z)$ for all $(X, Y) \in 3^V$,*

(2) *$f^*(\emptyset, Y) = f(\emptyset)$ for all $Y \subseteq V$.*

(3) *If $f(\{v\}) - f(\emptyset) \leq 1$ for every $v \in V$, then for every fixed $B \subseteq V$, $f^*(X, B) - f(\emptyset)$ is the rank function of a matroid on $V \setminus B$.*

We note that the matroid theory is reviewed in Section 2.5.

Proof.

(1) If $X \subseteq Z \subseteq V \setminus Y$, then $f^*(X, Y) \leq f^*(Z, V \setminus Z) = f(Z)$.

(2) $f(\emptyset) = f^*(\emptyset, \emptyset) \leq f^*(\emptyset, Y) \leq f^*(\emptyset, V) = f(\emptyset)$.

(3) Let $r(X) = f^*(X, B) - f(\emptyset)$. It is trivial that r is submodular and nondecreasing. Moreover,

$$0 \leq r(X) = f^*(X, B) - f(\emptyset) \leq f(X) - f(\emptyset) \leq |X|,$$

and therefore r is the rank function of a matroid on $V \setminus B$. □

We define $f_{\min}(X, Y) = \min f(Z)$, the minimum being taken over all Z satisfying $X \subseteq Z \subseteq V \setminus Y$.

Proposition 2.4. *Let $f : 2^V \to \mathbb{Z}$ be a submodular function such that $f(\emptyset) \leq f(X)$ for all $X \subseteq V$. Then $f_{\min}$ is an interpolation of f.*

Proof. The first, second, and last conditions are trivial. Let us prove submodularity. Let X, Y be subsets of V such that $A \subseteq X \subseteq V \setminus B$, $C \subseteq Y \subseteq V \setminus D$, $f_{\min}(A, B) = f(X)$, and $f_{\min}(C, D) = f(Y)$. Then

$$\begin{aligned} f(X) + f(Y) &\geq f(X \cap Y) + f(X \cup Y) \\ &\geq f_{\min}(A \cap C, B \cup D) + f_{\min}(A \cup C, B \cap D). \end{aligned}$$

Thus, $f_{\min}$ is an interpolation. □

In general $f_{\min}$ is not the only interpolation of a function f, and sometimes it is better for us to work with other interpolations that can be evaluated more quickly.

We remark that if $f^* : 3^V \to \mathbb{Z}$ is a uniform submodular function satisfying $f^*(\emptyset, \emptyset) = f^*(\emptyset, V)$, then there is a submodular function $f : 2^V \to \mathbb{Z}$ such that $f(\emptyset) \leq f(X)$ for all $X \subseteq V$ and f^* is an interpolation of f.

2.3 Comparing branch-width with a fixed number

Let V be a finite set and $f : 2^V \to \mathbb{Z}$ be a symmetric submodular function such that

$$f(\{v\}) - f(\emptyset) \leq 1 \text{ for all } v \in V.$$

In this section, we show that a statement, "branch-width of f is at most k", for fixed k, can be disproved in polynomial time (of $|V|$) by using a certificate of polynomial size (of $|V|$), when f is given by an oracle. To prove the statement, we have a natural certificate, a branch-decomposition of width at most k. However it is nontrivial to disprove the statement. We use the notion called *tangles*, which is dual to the notion of branch-width and was introduced by Robertson and Seymour [48].

A class $\mathcal{T}$ of subsets of V is called a *tangle* of f of order k if it satisfies the following four axioms.

(T1) For all $A \in \mathcal{T}$, we have $f(A) < k$.

(T2) If $f(A) < k$, then either $A \in \mathcal{T}$ or $V \setminus A \in \mathcal{T}$.

(T3) If $A, B, C \in \mathcal{T}$, then $A \cup B \cup C \neq V$.

(T4) For all $v \in V$, we have $V \setminus \{v\} \notin \mathcal{T}$.

We call that A is *small* if A is contained in a tangle. Informally speaking, the following proposition shows that a subset of a small set is small.

Proposition 2.5. *Let $\mathcal{T}$ be a tangle of f of order k. If $A \in \mathcal{T}$, $B \subseteq A$, and $f(B) < k$, then $B \in \mathcal{T}$.*

Proof. By (T2), either $B \in \mathcal{T}$ or $V \setminus B \in \mathcal{T}$. Since $(V \setminus B) \cup A \cup A = V$, the tangle $\mathcal{T}$ can not contain $V \setminus B$ by (T3). Hence $B \in \mathcal{T}$. □

Robertson and Seymour [48] showed that tangles are related to branch-width.

Theorem 2.6 (Robertson and Seymour [48, (3.5)]). *The following are equivalent:*

(i) *there is no tangle of f of order $k + 1$,*

(ii) *the branch-width of f is at most k.*

Therefore to show that the branch-width of f is larger than k for fixed k, it is enough to provide a tangle $\mathcal{T}$ of f of order $k + 1$. However, $\mathcal{T}$ might contain exponentially many subsets of V. So, we need to devise a way to encode a tangle into a certificate of polynomial size. If f satisfies that $f(\{v\}) - f(\emptyset) \leq 1$ for all $v \in V$, then there is a method to encode a tangle into a certificate of polynomial size as follows.

Theorem 2.7. *Let V be a finite set and $f : 2^V \to \mathbb{Z}$ be a symmetric submodular function such that $f(\{v\}) - f(\emptyset) \leq 1$ for all $v \in V$. For fixed k, there is a certificate of size at most a polynomial in $|V|$, that can be used to prove, in time polynomial in $|V|$, that f has branch-width larger than k, assuming that f is given by an oracle.*

Proof. Let $n = |V|$. We may assume that $n > 1$ because branch-width of f is $f(\emptyset)$ if $n \leq 1$. We may assume that $f(\emptyset) = 0$. Let $\mathcal{T}$ be a tangle of f of order $k+1$. Let $f_{\min}(X, Y) = \min_{X \subseteq Z \subseteq V \setminus Y} f(Z)$ for disjoint subsets X, Y of V. Let

$$P = \{(X, Y) : X \cap Y = \emptyset, |X| = |Y| = f_{\min}(X, Y) \leq k\}.$$

We claim that for each $(X, Y) \in P$, there is a unique maximal set $Z \in \mathcal{T}$, denoted by $Z = \mu(X, Y)$, such that $X \subseteq Z \subseteq V \setminus Y$ and $f(Z) = f_{\min}(X, Y)$. Suppose that Z_1 and Z_2 are contained in $\mathcal{T}$ and $X \subseteq Z_1 \subseteq V \setminus Y$, $X \subseteq Z_2 \subseteq V \setminus Y$, and $f(Z_1) = f(Z_2) = f_{\min}(X, Y)$. By submodularity,

$$f(Z_1 \cup Z_2) + f(Z_1 \cap Z_2) \leq f(Z_1) + f(Z_2) = 2f_{\min}(X, Y).$$

Since both $f(Z_1 \cup Z_2)$ and $f(Z_1 \cap Z_2)$ are bigger than or equal to $f_{\min}(X, Y)$, they are equal to $f_{\min}(X, Y)$. Since $Z_1 \cup Z_2 \cup (V \setminus (Z_1 \cup Z_2)) = V$, we obtain that $Z_1 \cup Z_2 \in \mathcal{T}$. Thus $\mu : P \to 2^V$ is well-defined.

We provide (P, μ) to our algorithm as a certificate showing that branch-width of f is larger than k. Since $|P| \leq \binom{n}{k}^2$, a description of (P, μ) has polynomial size in n.

Now we describe a polynomial-time algorithm that decides whether there is a tangle giving (P, μ).

By using submodular function minimization algorithms like [51] or [37], we can calculate $f_{\min}$ in polynomial time, and therefore we can check whether P is correct.

To ensure that (P, μ) is obtained by a tangle, our algorithm tests the following:

(1) $\mu(X_1, Y_1) \cup \mu(X_2, Y_2) \cup \mu(X_3, Y_3) \neq V$ for all $(X_i, Y_i) \in P$ for $i \in \{1, 2, 3\}$.

(2) for all $(A, B) \in P$, there exists no Z such that $A \subseteq Z \subseteq V \setminus B$, $f(Z) = k$, and $Z \not\subseteq \mu(A, B)$ and $V \setminus Z \not\subseteq \mu(B, A)$.

Equivalently for all $x \in V \setminus (\mu(A, B) \cup B)$ and $y \in V \setminus (\mu(B, A) \cup A)$, if $x \neq y$, then $f_{\min}(A \cup \{x\}, B \cup \{y\}) > k$.

(3) $|\mu(X, Y)| \neq |V| - 1$ for all $(X, Y) \in P$.

These can be done in polynomial time. We claim that if (P, μ) is obtained from a tangle $\mathcal{T}$, then (P, μ) will satisfy those tests. The first test is trivially true from the axiom of tangles. Now let us consider the second test. Suppose $A \subseteq Z \subseteq V \setminus B$, $f(Z) = k$. Then, either $Z \in \mathcal{T}$ or $V \setminus Z \in \mathcal{T}$. In either case, we obtain $Z \subseteq \mu(A, B)$ or $V \setminus Z \subseteq \mu(B, A)$. The third test is true because $V \setminus \{v\} \notin \mathcal{T}$ for all $v \in T$. Therefore if at least one of them fails, then (P, μ) is not obtained from a tangle. We now assume that (P, μ) passed those tests.

We claim that we can construct a tangle $\mathcal{T}$ of f of order $k+1$ from (P, μ) uniquely as follows:

For all Z such that $f(Z) \leq k$, we choose $(A, B) \in P$ such that

$$|A| = |B| = f(Z) \text{ and } A \subseteq Z \subseteq V \setminus B.$$

If $Z \subseteq \mu(A, B)$, then $Z \in \mathcal{T}$. Otherwise, $V \setminus Z \in \mathcal{T}$.

Let us first show that this is well-defined. Let Z be a subset of V such that $f(Z) \leq k$. By Proposition 2.4 and (3) of Proposition 2.3, we may choose $A \subseteq Z$ such that $f_{\min}(A, V \setminus Z) = |A| = f(Z)$, and then choose $B \subseteq V \setminus Z$ such that $f_{\min}(A, B) = |A| = |B| = f_{\min}(A, V \setminus Z) = f(Z) \leq k$. Thus there exists a wanted pair $(A, B) \in P$. Suppose that there are two wanted pairs $(A_1, B_1), (A_2, B_2) \in P$ such that $Z \subseteq \mu(A_1, B_1)$ but $Z \not\subseteq \mu(A_2, B_2)$. We obtain that $\mu(B_2, A_2) \cup \mu(A_1, B_1) = V$, because $V \setminus Z \subseteq \mu(B_2, A_2)$ by the second test. This contradicts to the first test.

We now claim that the axioms of tangles are satisfied by $\mathcal{T}$. Axioms (T1) and (T2) are true by construction. To show (T3), assume that $A_i \in \mathcal{T}$ for all $i \in 1, 2, 3$. There exist $(X_i, Y_i) \in P$ for each i such that $A_i \subseteq \mu(X_i, Y_i)$, and therefore $A_1 \cup A_2 \cup A_3 \subseteq \mu(X_1, Y_1) \cup \mu(X_2, Y_2) \cup \mu(X_3, Y_3) \neq V$. To obtain (T4), suppose that $V \setminus \{v\} \in \mathcal{T}$. Then, there exists $(X, Y) \in P$ such that $V \setminus \{v\} \subseteq \mu(X, Y)$. Hence $\mu(X, Y) = V$ or $\mu(X, Y) = V \setminus \{v\}$, but we obtain a contradiction because of (1) and (3). □

Suppose that we can calculate f by using an input of size in polynomial of $|V|$ in polynomial time. By the previous theorem, we conclude that deciding whether the branch-width is at most k for fixed k is in NP∩co-NP. But, it is still open whether it is in P. But it is known to be in P for some symmetric submodular functions. One example will be discussed in Chapter 7.

2.4 Approximating branch-width

In this section, we would like to show a polynomial-time algorithm that, for fixed k, outputs a branch-decomposition of bounded width or confirms that the branch-width is larger than k.

Definition 2.8. *Let V be a finite set and let $f : 2^V \to \mathbb{Z}$ be a symmetric submodular function satisfying $f(\emptyset) = 0$. We say that $W \subseteq V$ is* well-linked *with respect to f if for every partition (X, Y) of W and every Z with $X \subseteq Z \subseteq V \setminus Y$, we have*

$$f(Z) \geq \min(|X|, |Y|).$$

This notion is analogous to the notion of well-linkedness [45] related to tree-width of graphs.

Theorem 2.9. *Let V be a finite set with $|V| \geq 2$, and let $f : 2^V \to \mathbb{Z}$ be a symmetric submodular function such that $f(\emptyset) = 0$. If with respect to f there is a well-linked set of size k, then $\mathrm{bw}(f) \geq k/3$.*

Proof. Let W be a well-linked set of size k, and suppose that $(T, \mathcal{L})$ is a branch decomposition of f. We will show that $(T, \mathcal{L})$ has width at least $k/3$. We may assume that T does not have a vertex of degree 2, by suppressing any such vertices. For each edge $e = uv$ of T, let A_{uv} be the set of elements of V that are mapped by L into the connected component of $T \setminus e$ containing u, and let $B_{uv} = V \setminus A_{uv}$.

We may assume that $W \neq \emptyset$; choose $w \in W$. Since W is well-linked with respect to f, $f(\{w\}) \geq 1$, and therefore the width of $(T, \mathcal{L})$ is at least 1. Consequently we may assume that $k > 3$.

Suppose first that $\min(|A_{uv} \cap W|, |B_{uv} \cap W|) < k/3$ for every edge uv of T. Direct every edge uv from u to v if $|A_{uv} \cap W| < k/3$ and $|B_{uv} \cap W| \geq k/3$. By the assumption, each edge is given a unique direction. Since the number of vertices is more than the number of edges in T, there is a vertex $t \in V(T)$ such that every edge incident with t has head t.

If t is a leaf of T, let s be the neighbor of t. Since ts has head t, it follows that $|B_{st} \cap W| \geq k/3$. But $|B_{st}| = 1 < k/3$, a contradiction.

So, t has three neighbours x, y, z in T such that $|A_{xt} \cap W| < k/3$, $|A_{yt} \cap W| < k/3$, and $|A_{zt} \cap W| < k/3$. But $|W| = |A_{xt} \cap W| + |A_{yt} \cap W| + |A_{zt} \cap W| < k = |W|$, a contradiction.

We deduce that there exists $uv \in E(T)$ such that $|A_{uv} \cap W| \geq k/3$ and $|B_{uv} \cap W| \geq k/3$. Hence $f(A_{uv}) \geq \min(|A_{uv} \cap W|, |B_{uv} \cap W|) \geq k/3$, and the width of $(T, \mathcal{L})$ is at least $k/3$. □

Theorem 2.10. *Let V be a finite set, let $f : 2^V \to \mathbb{Z}$ be a symmetric submodular function such that $f(\{v\}) \leq 1$ for all $v \in V$ and $f(\emptyset) = 0$, and let $k \geq 0$ be an integer. If with respect to f, there is no well-linked set of size k, then* $\mathrm{bw}(f) \leq k$.

Proof. We may assume that $\mathrm{bw}(f) > 0$, and so $|V| \geq 2$. We may assume that $k > 0$. For two partial branch-decompositions $(T, \mathcal{L})$ and $(T', \mathcal{L}')$ of f, we say that $(T, \mathcal{L})$ *extends* $(T', \mathcal{L}')$ if T' is obtained by contracting some edges of T and for every $v \in V$, $\mathcal{L}'(v)$ is the vertex of T' that corresponds to $\mathcal{L}(v)$ under the contraction.

We will prove that, if there is no well-linked set of size k with respect to f, then for every partial branch-decomposition $(T_s, \mathcal{L}_s)$ of f with width at most k, there is a branch-decomposition of f of width at most k extending $(T_s, \mathcal{L}_s)$. Since $k \geq 1$ and f trivially admits a partial branch-decomposition of width 1 (using the two-vertex tree with vertices u, v, and mapping all vertices of V except one to u, and the last to v), this implies the statement of the theorem.

Pick a partial branch-decomposition $(T, \mathcal{L})$ of f extending $(T_s, \mathcal{L}_s)$ such that the width of $(T, \mathcal{L})$ is at most k and the number of leaves of T is maximum.

We claim that $(T, \mathcal{L})$ is a branch-decomposition of f. It is enough to show that $\mathcal{L}$ is a bijection. Suppose therefore that there is a leaf t of T such that $B = \mathcal{L}^{-1}(\{t\})$ has more than one element.

We claim that $f(B) = k$. Suppose that $f(B) < k$. Let $v \in B$. Construct a subcubic tree T' by adding two vertices t_1 and t_2 and edges t_1t, t_2t to T. Let $\mathcal{L}'(v) = t_1$ and $\mathcal{L}'(w) = t_2$ for all $w \in B \setminus \{v\}$ and $\mathcal{L}'(x) = \mathcal{L}(x)$ for all $x \in V \setminus B$. Then $(T', \mathcal{L}')$ is a partial branch-decomposition extending $(T, \mathcal{L})$. Moreover $f(\{v\}) \leq 1 \leq k$ and $f(B \setminus \{v\}) \leq f(B) + f(\{v\}) \leq k$, and so the width of $(T', \mathcal{L}')$ is at most k. But the number of leaves of T' is greater than that of T, a contradiction.

Let f^* be an interpolation of f. By Proposition 2.3, $f^*(X, B)$ is the rank function of a matroid on $V \setminus B$. Let X be a base of this matroid. Then $|X| = f^*(V \setminus B, B) = f(B) = k$.

Since X is not well-linked, there exists $Z \subseteq V$ such that

$$f(Z) < \min(|Z \cap X|, |(V \setminus Z) \cap X|).$$

Since $f(Z \setminus B) = f^*(Z \setminus B, B \cup (V \setminus Z)) \geq f^*(Z \cap X, B) = |Z \cap X| > f(Z)$, it follows that $Z \cap B \neq \emptyset$. Similarly $B \setminus Z = (V \setminus Z) \cap B \neq \emptyset$.

Construct a subcubic tree T' by adding two vertices t_1 and t_2 and edges t_1t, t_2t to T. Let $\mathcal{L}'(x) = t_1$ if $x \in B \cap Z$, $\mathcal{L}'(x) = t_2$ if $x \in B \setminus Z$ and $\mathcal{L}'(x) = \mathcal{L}(x)$ otherwise.

By submodularity,

$$\begin{aligned}|(V \setminus Z) \cap X| + f(B) > f(Z) + f(B) &\geq f(Z \cup B) + f(Z \cap B)\\ &= f((V \setminus Z) \setminus B) + f(Z \cap B)\\ &\geq f^*((V \setminus Z) \cap X, B) + f(Z \cap B)\\ &= |(V \setminus Z) \cap X| + f(Z \cap B),\end{aligned}$$

and so $f(Z \cap B) < f(B) \leq k$ and similarly $f(B \setminus Z) < f(B) \leq k$. Therefore $(T', \mathcal{L}')$ is a partial branch-decomposition extending $(T, \mathcal{L})$ of width at most k. But the number of leaves of T' is greater than that of T, a contradiction. □

Corollary 2.11. *For all $k \geq 0$, there is a polynomial-time algorithm that, with input a set V with $|V| \geq 2$ and a symmetric submodular function $f : 2^V \to \mathbb{Z}$ with $f(\{v\}) \leq 1$ for all $v \in V$ and $f(\emptyset) = 0$, outputs either a well-linked set of size k or a branch-decomposition of width at most k.*

The proof of Theorem 2.10 provides an algorithm that either finds a well-linked set of size k, or constructs a branch-decomposition of f of width at most k. By combining with Theorem 2.9, we get an algorithm that either concludes that $\mathrm{bw}(f) > k$ or finds a branch-decomposition of width at most $3k + 1$.

Let us analyze the running time of the algorithm of Theorem 2.10. To do so, we must be more precise about how the input function f and f^* are accessed. We consider two different situations, as follows:

- In the first case, we assume that only f is given as input, and in the sense that we can compute $f(X)$ for a set X; and we need to compute values of f^* from this input.
- In the second case, we assume that an interpolation f^* of f is given as input (in the same sense, that for any pair (X, Y) we can compute $f^*(X, Y)$), and we need to compute f from f^*.

For the first analysis, let γ be the time to compute $f(X)$ for any set X. In this case we shall use $f^* = f_{\min}$. To calculate $f_{\min}$, we use the submodular function minimization algorithm [37], whose running time is $O(n^5 \gamma \log M)$ where M is the maximum value of f and $n = |V|$. Thus, we can calculate $f_{\min}$ in $O(n^5 \gamma \log n)$ time.

Finding a base X can be done by calculating f^* at most $O(n)$ times, and therefore takes time $O(n^6\gamma \log n)$. To check whether X is well-linked, we try all partitions of X; 2^{k-1} tries (a constant). And finding the set Z for a given partition of X can be done in time $O(n^5\gamma \log n)$ by submodular function minimization algorithms. Since the process is cycled through at most $O(n)$ times (because if $(T, \mathcal{L})$ is a partial branch-decomposition then $|V(T)| \leq 2n-2$), it follows that in this case the time complexity is $O(n^7\gamma \log n)$.

For the second analysis, let δ be the time to compute $f^*(X)$ for any set X. Finding a base X can be done in time $O(n\delta)$. Finding Z to show that X is not well-linked can be done in time $O(n^5\delta \log n)$. Thus, the time complexity in this case is $O(n^6\delta \log n)$.

In summary, then, we have shown the following two statements.

Corollary 2.12. *For given k, there is an algorithm as follows. It takes as input a finite set V with $|V| \geq 2$ and a symmetric submodular function $f : 2^V \to \mathbb{Z}$, such that $f(\{v\}) \leq 1$ for all $v \in V$ and $f(\emptyset) = 0$. It either concludes that $\mathrm{bw}(f) > k$ or outputs a branch-decomposition of f of width at most $3k+1$; and its running time (excluding evaluating f) and number of evaluations of f are both $O(|V|^7 \log |V|)$.*

Corollary 2.13. *For given k, there is an algorithm as follows. It takes as input a finite set V with $|V| \geq 2$ and a function f^* which is an interpolation of some symmetric submodular function $f : 2^V \to \mathbb{Z}$, such that $f(\{v\}) \leq 1$ for all $v \in V$ and $f(\emptyset) = 0$. It either concludes that $\mathrm{bw}(f) > k$ or outputs a branch-decomposition of f of width at most $3k+1$; and its running time is $O(|V|^6\delta \log |V|)$, where δ is the time for each evaluation of f^*.*

2.5 Application to matroid branch-width

The connectivity function of a matroid is a special kind of symmetric submodular function, and we have been able to modify our general algorithm so that it runs much more quickly for functions of this type. There are two separate modifications. First, there is an interpolation of the connectivity function λ of a matroid that can be evaluated faster than $\lambda_{\min}$. Second, we can apply the matroid intersection algorithm instead of the general submodular function minimization algorithms.

Let us review matroid theory first. For general matroid theory, we refer to Oxley's book [44]. We call $\mathcal{M} = (E, \mathcal{I})$ a *matroid* if E is a finite set and $\mathcal{I}$ is a collection of subsets of E, satisfying

(i) $\emptyset \in \mathcal{I}$

(ii) If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.

(iii) For every $Z \subseteq E$, maximal subsets of Z in $\mathcal{I}$ all have the same size $r(Z)$. We call $r(Z)$ the *rank* of Z.

An element of $\mathcal{I}$ is called *independent* in $\mathcal{M}$. We let $E(\mathcal{M}) = E$. We call $B \subseteq E$ a *base* if it is maximally independent. A matroid may also be defined by axioms on the set of bases. We call $B' \subseteq E$ a *cobase* if $E \setminus B'$ is a base. The *dual matroid* $\mathcal{M}^*$ of $\mathcal{M}$ is the matroid on $E(\mathcal{M})$ such that the set of cobases of $\mathcal{M}$ is equal to the set of bases of $\mathcal{M}^*$.

A matroid $\mathcal{M} = (E, \mathcal{I})$ is *binary* if there exists a matrix N over GF(2) such that E is a set of column vectors of N and $\mathcal{I} = \{X \subseteq E : X \text{ is linearly independent}\}$.

For $e \in E(\mathcal{M})$, $\mathcal{M} \setminus e$ is the matroid $(E \setminus \{e\}, \mathcal{I}')$ such that

$$\mathcal{I}' = \{X \subseteq E(\mathcal{M}) \setminus \{e\} : X \in \mathcal{I}\}.$$

This operation is called *deletion* of e. For $e \in E(\mathcal{M})$, $\mathcal{M}/e = (\mathcal{M}^* \setminus e)^*$ and this operation is called *contraction* of e. A matroid $\mathcal{N}$ is called a *minor* of $\mathcal{M}$ if $\mathcal{N}$ can be obtained from $\mathcal{M}$ by applying a sequence of deletions and contractions.

The *connectivity* function $\lambda_{\mathcal{M}}$ of $\mathcal{M}$ is

$$\lambda_{\mathcal{M}}(X) = r(X) + r(E \setminus X) - r(E) + 1.$$

Note that $\lambda_{\mathcal{M}}$ is a symmetric submodular function. A branch-decomposition $(T, \mathcal{L})$ of $\lambda_{\mathcal{M}}$ is called a *branch-decomposition* of $\mathcal{M}$. The *branch-width* $\mathrm{bw}(\mathcal{M})$ of $\mathcal{M}$ is the branch-width of λ_M.

The following proposition is due to Jim Geelen (private communication).

Proposition 2.14. *Let $\mathcal{M}$ be a matroid with rank function r, with connectivity function*

$$\lambda(X) = r(X) + r(E(\mathcal{M}) \setminus X) - r(E(\mathcal{M})) + 1.$$

Let B be a base of $\mathcal{M}$. Then

$$\lambda_B(X, Y) = r(X \cup (B \setminus Y)) + r(Y \cup (B \setminus X)) - |B \setminus X| - |B \setminus Y| + 1$$

is an interpolation of λ.

Proof. We verify the three conditions of the definition of an interpolation.
1) If $Y = E(\mathcal{M}) \setminus X$, then

$$\lambda_B(X, Y) = r(X) + r(Y) - r(B \cap X) - r(B \cap Y) + 1 = r(X) + r(Y) - |B| + 1 = \lambda(X).$$

2) Let $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$. Then

$$r(X_2 \cup (B \setminus Y_2)) \geq r(X_1 \cup (B \setminus Y_2)) \geq r(X_1 \cup (B \setminus Y_1)) - (|B \setminus Y_1| - |B \setminus Y_2|).$$

Therefore,

$$r(X_2 \cup (B \setminus Y_2)) - |B \setminus Y_2| \geq r(X_1 \cup (B \setminus Y_1)) - |B \setminus Y_1|.$$

Similarly,

$$r(Y_2 \cup (B \setminus X_2)) - |B \setminus X_2| \geq r(Y_1 \cup (B \setminus X_1)) - |B \setminus X_1|.$$

By adding both inequalities, we deduce that $\lambda_B(X_2, Y_2) \geq \lambda_B(X_1, Y_1)$.

3) Let $X_1 \cap Y_1 = \emptyset$ and $X_2 \cap Y_2 = \emptyset$. It is easy to show that

$$(P \cap R) \cup (Q \cap S) \subseteq (P \cup Q) \cap (R \cup S)$$

for any choice of sets P, Q, R, S. Since r is submodular and increasing,

$$\begin{aligned} & r(X_1 \cup (B \setminus Y_1)) + r(X_2 \cup (B \setminus Y_2)) \\ & \geq r((X_1 \cup (B \setminus Y_1)) \cup (X_2 \cup (B \setminus Y_2))) + r((X_1 \cup (B \setminus Y_1)) \cap (X_2 \cup (B \setminus Y_2))) \\ & \geq r((X_1 \cup X_2) \cup (B \setminus (Y_1 \cap Y_2))) + r((X_1 \cap X_2) \cup (B \setminus (Y_1 \cup Y_2))). \end{aligned}$$

Similarly

$$\begin{aligned} & r(Y_1 \cup (B \setminus X_1)) + r(Y_2 \cup (B \setminus X_2)) \\ & \qquad \geq r((Y_1 \cup Y_2) \cup (B \setminus (X_1 \cap X_2))) + r((Y_1 \cap Y_2) \cup (B \setminus (X_1 \cup X_2))). \end{aligned}$$

But also

$$|B \setminus X_1| + |B \setminus X_2| = |B \setminus (X_1 \cap X_2)| + |B \setminus (X_1 \cup X_2)|.$$

By adding, we deduce that

$$\lambda_B(X_1, Y_1) + \lambda_B(X_2, Y_2) \geq \lambda_B(X_1 \cap X_2, Y_1 \cup Y_2) + \lambda(X_1 \cup X_2, Y_1 \cap Y_2). \qquad \square$$

Now, we discuss a method to avoid the general submodular function minimization algorithm. To apply Corollary 2.13 to matroid branch-width, we needed a submodular function minimization algorithm that, given a matroid $\mathcal{M}$ and two disjoint subsets X and Y, will output $Z \subseteq E(\mathcal{M})$ such that $X \subseteq Z \subseteq E(\mathcal{M}) \setminus Y$ and $\lambda(Z)$ is minimum. We claim that that this can be done by the matroid intersection algorithm. Let $\mathcal{M}_1 = \mathcal{M}/X \setminus Y$ and $\mathcal{M}_2 = \mathcal{M} \setminus X/Y$, with rank functions r_1, r_2 respectively. Then by the matroid intersection algorithm, we can find $U \subseteq E(\mathcal{M}) \setminus X \setminus Y$ minimizing $r_1(U) + r_2(E(\mathcal{M}) \setminus X \setminus Y \setminus U)$. Using the fact $r_1(U) = r(U \cup X) - r(X)$, $r_2(U) = r(U \cup Y) - r(Y)$, we construct a set Z with $X \subseteq Z \subseteq E(\mathcal{M}) \setminus Y$ that minimizes $\lambda(Z)$. And this can be done in $O(n^3)$ time (if $\mathcal{M}$ is input in terms of its rank oracle), where $n = |E(\mathcal{M})|$.

We deduce:

Corollary 2.15. *For given k, there is an algorithm that, with input an n-element matroid $\mathcal{M}$, given by its rank oracle, either concludes that* $\mathrm{bw}(\mathcal{M}) > k$ *or outputs a branch-decomposition of $\mathcal{M}$ of width at most $3k-1$. Its running time and number of oracle calls is at most $O(n^4)$.*

Proof. Pick a base B of $\mathcal{M}$ arbitrarily. We use λ_B as an interpolation of λ. For a given partition (A, B), finding a base X can be done in time $O(n)$. Finding Z to prove that X is not well-linked can be done in $O(2^{3k-2}n^3)$. Therefore, the time complexity is $O(n + n(n + 2^{3k-2}n^3)) = O(8^k n^4)$. $\square$

We note that previous algorithm by P. Hliněný [32] to approximate matroid branch-width was only for matroids representable over a finite field.

Chapter 3

Rank-width and Vertex-minors

3.1 Clique-width

The notion of clique-width was first introduced by Courcelle and Olariu [19]. Let k be a positive integer. We call (G, lab) a *k-graph* if G is a graph and lab is a mapping from its vertex set to $\{1, 2, \dots, k\}$. (In this paper, all graphs are finite and have no loops or parallel edges.) We call $lab(v)$ the *label* of a vertex v.

We need the following definitions of operations on k-graphs.

(1) For $i \in \{1, \dots, k\}$, let $\cdot_i$ denote a k-graph with a single vertex labeled by i.

(2) For $i, j \in \{1, 2, \dots, k\}$ with $i \neq j$, we define a unary operator $\eta_{i,j}$ such that

$$\eta_{i,j}(G, lab) = (G', lab)$$

where $V(G') = V(G)$, and $E(G') = E(G) \cup \{vw : v, w \in V, lab(v) = i, lab(w) = j\}$. This adds edges between vertices of label i and vertices of label j.

(3) We let $\rho_{i \to j}$ be the unary operator such that

$$\rho_{i \to j}(G, lab) = (G, lab')$$

where

$$lab'(v) = \begin{cases} j & \text{if } lab(v) = i, \\ lab(v) & \text{otherwise.} \end{cases}$$

This mapping relabels every vertex labeled by i into j.

(4) Finally, $\oplus$ is a binary operation that makes the disjoint union. Note that $G \oplus G \neq G$.

A well-formed expression t in these symbols is called a *k-expression*. The k-graph produced by performing these operations in order therefore has vertex set the set of occurrences of the constant symbols $(\cdot_i)$ in t; and this k-graph (and any k-graph isomorphic to it) is called the *value* of t, denoted by $val(t)$. If a k-expression t has value (G, lab), we say that t is a *k-expression of G*. The *clique-width* of a graph G, denoted by $\mathrm{cwd}(G)$, is the minimum k such that there is a k-expression of G.

For instance, K_4 (the complete graph with four vertices) can be constructed by

$$\rho_{2\to 1}(\eta_{1,2}(\rho_{2\to 1}(\eta_{1,2}(\rho_{2\to 1}(\eta_{1,2}(\cdot_1 \oplus \cdot_2)) \oplus \cdot_2)) \oplus \cdot_2)).$$

Therefore, K_4 has a 2-expression, and $\mathrm{cwd}(K_4) \le 2$. It is easy to see that $\mathrm{cwd}(K_4) > 1$, and therefore $\mathrm{cwd}(K_4) = 2$.

Some other examples: cographs, which are graphs with no induced path of length 3, are exactly the graphs of clique-width at most 2; the complete graph K_n $(n > 1)$ has clique-width 2; and trees have clique-width at most 3 [19].

For some classes of graphs, it is known that clique-width is bounded and algorithms to construct a k-expression have been found. For example, cographs [10], graphs of clique-width at most 3 [9], and P_4-sparse graphs (every five vertices have at most one induced subgraph isomorphic to a path of length 3) [18] have such algorithms.

3.2 Rank-width and clique-width

In this section, we define the *rank-width* of a graph and show that a set of graphs has bounded rank-width if and only if it has bounded clique-width.

For a matrix $M = (m_{ij} : i \in C, j \in R)$ over a field F, if $X \subseteq R$ and $Y \subseteq C$, let $M[X,Y]$ denote the submatrix $(m_{ij} : i \in X, j \in Y)$. For a graph G, let $A(G)$ be its adjacency matrix over GF(2).

Definition 3.1. *Let G be a graph. For two disjoint subsets $X, Y \subseteq V(G)$, we define $\rho^*_G(X,Y) = \mathrm{rk}(A(G)[X,Y])$ where* rk *is the matrix rank function; and we define the* cut-rank *function ρ_G of G by letting $\rho_G(X) = \rho^*_G(X, V(G)\setminus X)$ for $X \subseteq V(G)$.*

We will show that ρ_G is symmetric submodular and ρ^*_G is an interpolation of ρ_G.

Proposition 3.2. *Let $M = (m_{ij} : i \in C, j \in R)$ be a matrix over a field F. Then for all $X_1, X_2 \subseteq R$ and $Y_1, Y_2 \subseteq C$, we have*

$$\mathrm{rk}(M[X_1,Y_1]) + \mathrm{rk}(M[X_2,Y_2]) \ge \mathrm{rk}(M[X_1 \cup X_2, Y_1 \cap Y_2]) + \mathrm{rk}(M[X_1 \cap X_2, Y_1 \cup Y_2]).$$

Proof. See [41, Proposition 2.1.9], [56, Lemma 2.3.11], or [55]. □

Corollary 3.3. *Let G be a graph. If $(X_1,Y_1),(X_2,Y_2) \in 3^{V(G)}$ then*

$$\rho^*_G(X_1,Y_1) + \rho^*_G(X_2,Y_2) \ge \rho^*_G(X_1 \cap X_2, Y_1 \cup Y_2) + \rho^*_G(X_1 \cup X_2, Y_1 \cap Y_2).$$

Moreover, if $X_1, X_2 \subseteq V(G)$, then

$$\rho_G(X_1) + \rho_G(X_2) \ge \rho_G(X_1 \cap X_2) + \rho_G(X_1 \cup X_2).$$

Proof. Let M be the adjacency matrix of G over GF(2). Then

$$\rho_G(X) = \mathrm{rk}(M[X, V(G)\setminus X]).$$

Apply Proposition 3.2. □

A *rank-decomposition* of G is a branch-decomposition of ρ_G, and the *rank-width* of G, denoted by $\mathrm{rwd}(G)$, is the branch-width of ρ_G.

The following proposition provides a link between clique-width and rank-width.

Proposition 3.4. *For a graph G,* $\mathrm{rwd}(G) \leq \mathrm{cwd}(G) \leq 2^{\mathrm{rwd}(G)+1} - 1$.

Proof. We may assume that $|V(G)| \geq 2$, because if $|V(G)| \leq 1$, then $\mathrm{rwd}(G) = 0$ and $\mathrm{cwd}(G) \leq 1$.

A *rooted binary tree* is a subcubic tree with a specified vertex, called the *root*, such that every non-root vertex has one, two or three incident edges and the root has at most two incident edges. A vertex u of a rooted binary tree is called a *descendant* of a vertex v if v belongs to the path from the root to u; and u is called a *child* of v if u, v are adjacent in T and u is a descendant of v.

First we show that $\mathrm{rwd}(G) \leq \mathrm{cwd}(G)$. Let $k = \mathrm{cwd}(G)$. Let t be a k-expression with value (G, lab) for some choice of lab. We recall that a k-expression is a well-formed expression with four types of symbols; the constants, two unary operators, and the binary operator forming disjoint union. The parentheses of the expression form a tree structure. Thus there is a rooted binary tree T, each vertex v of which corresponds to a k-expression, say $N(v)$; and letting V_0, V_1, V_2 denote the sets of vertices in T with zero, one and two children respectively, we have for each vertex $v \in V(T)$:

- if $v \in V_0$ then $N(v)$ is a 1-term expression consisting just of a constant term,
- if $v \in V_1$ with child u, then $N(v)$ is obtained from $N(u)$ by applying one of the two unary operators,
- if $v \in V_2$ with children u_1, u_2, then $N(v)$ is obtained from $N(u_1), N(u_2)$ by applying $\oplus$,
- if v is the root then $N(v) = (G, lab)$.

In particular, each vertex $v \in V_0$ gives rise to a unique vertex w of G; let us write this $\mathcal{L}(w) = v$. Then $\mathcal{L}$ is a bijection between $V(G)$ and the set of leaves of T. Consequently $(T, \mathcal{L})$ is a branch-decomposition of ρ_G. Let us study its width. Let $u, v \in V(T)$, where u is a child of v, and let T_1, T_2 be the components of $T \setminus e$, where e is the edge uv and $u \in V(T_1)$. Let $X_i = \{\mathcal{L}^{-1}(t) : t \in V_0 \cap V(T_i)\}$ for $i = 1, 2$. Thus (X_1, X_2) is a partition of $V(G)$, and we need to investigate $\rho_G(X_1)$. Let $N(u) = (G_1, lab_1)$. Thus $V(G_1) = X_1$. If $x, y \in X_1$ and $lab_1(x) = lab_1(y)$, then x, y are adjacent in G to the same members of X_2, from the properties of the iterative construction of (G, lab); and since the function lab_1 has at most k different values, it follows that X_1 can be partitioned into k subsets so that the members of each subset have the same neighbors in X_2. Consequently $\rho_G(X_1) \leq k$. Since this applies for every edge of T, we deduce that $(T, \mathcal{L})$ is a branch-decomposition of ρ_G with width at most k. Hence $\mathrm{rwd}(G) \leq k = \mathrm{cwd}(G)$.

Now we show the second statement of the theorem, that $\mathrm{cwd}(G) \le 2^{\mathrm{rwd}(G)+1} - 1$. Let $k = \mathrm{rwd}(G)$ and $(T, \mathcal{L})$ be a rank-decomposition of G of width k. By subdividing one edge of T, and suppressing all other vertices of T with degree 2, we may assume that T is a rooted binary tree; its root has degree 2, and all other vertices have degree 1 or 3.

For $v \in V(T)$, let $D_v = \{x \in V(G) : \mathcal{L}(x)$ is a descendant of v in $T\}$, and let G_v denote the subgraph of G induced on D_v. We claim that for every $v \in V(T)$, there is a map lab_v and a $(2^{k+1} - 1)$-expression t_v with value (G_v, lab_v), such that

(i) if $lab_v(x) = 1$ then $x \in D_v$ is nonadjacent to every vertex of $G \setminus D_v$,

(ii) if $x, y \in D_v$ and there exists $z \in V(G) \setminus D_v$ such that x is adjacent to z but y is not, then $lab_v(x) \ne lab_v(y)$,

(iii) for each $x \in D_v$, $lab_v(x) \in \{1, 2, \dots, 2^k\}$.

We prove this by induction on the number of vertices of T that are descendants of v. If v is a leaf, let $t_v = \cdot_1$. Then t_v satisfies the above conditions. Thus we may assume that v has exactly two children v_1, v_2.

By the inductive hypothesis, there are $(2^{k+1} - 1)$-expressions t_1, t_2 with values (G_{v_i}, lab_{v_i}) for $i = 1, 2$, satisfying the statements above. Let F be the set of pairs (i, j) with $i, j \in \{1, 2, \dots, 2^k\}$, such that there is an edge xy of G, with $x \in D_{v_1}$, $lab_{v_1}(x) = i$, $y \in D_{v_2}$ and $lab_{v_2}(y) = j$. It follows from the second condition above that if $(i, j) \in F$ then every vertex $x \in D_{v_1}$ with $lab_{v_1}(x) = i$ is adjacent in G to every vertex $y \in D_{v_2}$ with $lab_{v_2}(y) = j$. Let

$$t^* = \left(\mathop{\circ}_{(i,j)\in F} \eta_{i,j+2^k-1} \right) \left(t_{v_1} \oplus \left(\mathop{\circ}_{i=2}^{2^k} \rho_{i \to i+2^k-1} \right) (t_{v_2}) \right).$$

Then t^* is a $(2^{k+1} - 1)$-expression with value (G_v, lab^*) say, and it satisfies the first two displayed conditions above. However, it need not yet satisfy the third. Let us choose a $(2^{k+1} - 1)$-expression t_v with value (G_v, lab_v) say, satisfying the first two conditions above, and satisfying the following:

- $\{lab_v(x) : x \in D_v\}$ is minimal,
- subject to this condition, $\max_{x \in D_v} lab_v(x)$ ($= r$ say) is as small as possible.

(We call these the "first and second optimizations".) For $i = 1, \dots, r$, let $X_i = \{x \in D_v : lab_v(x) = i\}$. The definition of r implies that $X_r \ne \emptyset$. If there exists i with $2 \le i < r$ such that $X_i = \emptyset$, then applying the operation $\rho_{r \to i}$ to t_v produces a k-expression contradicting the second optimization. Thus, $X_2, \dots, X_r$ are all nonempty. For $1 \le i \le r$, let Y_i be the set of vertices of $V(G) \setminus D_v$ with a neighbor in X_i. From the first condition (i), $Y_1 = \emptyset$. From the second condition (ii), every vertex in X_i is adjacent to every member of Y_i for all i with $1 \le i \le r$. If there exist i, j with $1 \le i < j \le r$ such that $Y_i = Y_j$, then applying $\rho_{j \to i}$ to t_v produces a k-expression contradicting the first optimization. Thus $Y_1, \dots, Y_r$ are all distinct.

Let M be the matrix $A(G)[D_v, V(G) \setminus D_v]$. Then M has $r-1$ distinct nonzero rows. Since $(T, \mathcal{L})$ has width k, it follows that M has rank at most k, and therefore M has at most $2^k - 1$ distinct nonzero rows (this is an easy fact about any matrix over GF(2)). We deduce that $r \leq 2^k$, and therefore t_v satisfies the third condition above.

This completes the proof that the k-expressions t_v exist as described above. In particular, if v is the root of T then $G_v = G$, and so t_v is a $(2^{k+1}-1)$-expression of G. We deduce that $\mathrm{cwd}(G) \leq 2^{k+1} - 1$. □

The above proof gives an algorithm that converts a rank-decomposition of width k into a $(2^{k+1}-1)$-expression. Let $n = |V(G)|$, and let $(T, \mathcal{L})$ be the input rank-decomposition. At each non-leaf vertex v of T, we first construct F, in $O((2^k)^2) = O(1)$ time. Then merging sets with the same neighbors outside D_v will take time $O\left(2^{2k} n\right) = O(n)$. The number of non-leaf vertices v of T is $O(n)$. Therefore, the time complexity is $O(n^2)$. Note that we may assume that checking the adjacency of two vertices can be done in constant time, because we preprocess the input to construct an adjacency matrix in time $O(n^2)$.

3.3 Graphs having rank-width at most 1

We call a graph G *distance-hereditary* if and only if for every connected induced subgraph H of G, the distance between every pair of vertices in H is the same as in G. Howorka [36] defined distance-hereditary graphs, and Bandelt and Mulder [2] found a recursive characterization of distance-hereditary graphs, which we will use here. In this section, we show that a graph is distance-hereditary if and only if it has rank-width at most 1.

Two distinct vertices v, w are called *twins* of G if for every $x \in V(G) \setminus \{v, w\}$, v is adjacent to x if and only if w is adjacent to x. We call v a *pendant vertex* of G if it has only one incident edge in G.

Proposition 3.5. *Let G be a graph. If $v, w \in V(G)$ are twins of G and $G \setminus v$ has at least one edge different from vw, then* $\mathrm{rwd}(G \setminus v) = \mathrm{rwd}(G)$. *Note that we do not require that $vw \in E(G)$.*

Proof. It is enough to show that $\mathrm{rwd}(G \setminus v) \geq \mathrm{rwd}(G)$. Since $|V(G \setminus v)| \geq 2$, there is a rank-decomposition $(T, \mathcal{L})$ of $G \setminus v$ of width $\mathrm{rwd}(G \setminus v)$. Let $x = \mathcal{L}(w)$ and let $y \in V(T)$ be such that $xy \in E(T)$.

Let T' be a tree obtained from T by deleting xy, adding two new vertices x', z, and adding three new edges xz, zx', zy. Let $\mathcal{L}'(x') = v$ and $\mathcal{L}'(u) = \mathcal{L}(u)$ for all $u \neq x'$.

So, $(T', \mathcal{L}')$ is a rank-decomposition of G. For every edge e except zx' and zx in T', the width of e in $(T', \mathcal{L}')$ is equal to the width of e in $(T, \mathcal{L})$, because v and w are twins. Both the width of zx and the width of zx' are at most 1. Since G has at least one edge $e \neq vw$ and v, w are twins, $G \setminus v$ has at least one edge and $\mathrm{rwd}(G \setminus v) \geq 1$, and therefore the width of $(T', \mathcal{L}')$ is $rwd(G \setminus v)$. Therefore, $\mathrm{rwd}(G \setminus v) \geq \mathrm{rwd}(G)$. □

Proposition 3.6. *If G has rank-width at most 1 and $|V(G)| \geq 2$, then G has a pair of vertices v and w such that either they are twins or w has no neighbor different from v.*

Proof. If $|V(G)| = 2$, then the claim is trivial, and so we may assume that $|V(G)| \geq 3$.

Let $(T, \mathcal{L})$ be a rank-decomposition of G of width at most 1. Since T has at least three leaves, there exists a vertex x of T that is adjacent to two leaves $\mathcal{L}(v)$, $\mathcal{L}(w)$ of T. Let y be the vertex of T adjacent to x different from $\mathcal{L}(v)$ and $\mathcal{L}(w)$. The partition of $V(G)$ induced by xy is $(\{v, w\}, V(G) \setminus \{v, w\})$. So, the width of xy is $\rho_G(\{v, w\}) \leq 1$. That means either v, w are twins or v has no neighbor different from w or w has no neighbor different from v. □

Proposition 3.7. *G is distance-hereditary if and only if the rank-width of G is at most* 1*.*

Proof. Bandelt and Mulder [2] showed that every distance-hereditary graph can be obtained by creating twins, adding an isolated vertex, or adding a pendant vertex to a distance-hereditary graph or is a graph with one vertex. So, the rank-width of every distance-hereditary graphs is at most 1 by Proposition 3.5. Conversely, if a graph has rank-width at most 1, then by Proposition 3.6, it is distance-hereditary. □

Golumbic and Rotics [30] proved that distance-hereditary graphs have clique-width at most 3, and this can be proved as a corollary of Proposition 3.7.

Corollary 3.8. *Distance-hereditary graphs have clique-width at most* 3*.*

Proof. By Proposition 3.4, clique-width of a graph G is at most $2^{\mathrm{rwd}(G)+1} - 1$. □

3.4 Local complementations and vertex-minors

We define *local complementation*, *pivoting*, *vertex-minors*, and *pivot-minors*. In fact, vertex-minor containment was called *l-reduction* by Bouchet [8], but the author thinks "vertex-minor" is a better name, because of the many analogies with matroid minors discussed in Section 3.5. For two sets A and B, let $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

Definition 3.9. *Let $G = (V, E)$ be a graph and $v \in V$. The graph obtained by applying* local complementation *at v to G is*

$$G * v = (V, E \Delta \{xy : xv, yv \in E, x \neq y\}).$$

For an edge $uv \in E$, the graph obtained by pivoting *uv is defined by $G \wedge uv = G * u * v * u$. We call H* locally equivalent *to G if G can be obtained by applying a sequence of local complementations to G. We call H a* vertex-minor *of G if H can be obtained by applying a sequence of vertex deletions and local complementations to G. We call H a* pivot-minor *of G if H can be obtained by applying a sequence of vertex deletions and pivotings. A vertex-minor H of G is called a* proper *vertex-minor if H has fewer vertices than G and similarly a pivot-minor H of G is called a* proper *pivot-minor if H has fewer vertices than G.*

Figure 3.1: Local complementation

A pivoting is well-defined because $G*u*v*u = G*v*u*v$ if u and v are adjacent. To prove this, we prove the following proposition that describes pivoting directly.

Proposition 3.10. *For a graph H and $u, v \in V(H)$, let H_{uv} be a graph obtained by exchanging u and v in H. For $X, Y \subseteq V(H)$, let $H*(X,Y)$ be the graph $(V(H), E')$ where $E' = E(H)\Delta\{xy : x \in X, y \in Y, x \neq y\}$. Let $G = (V,E)$ be a graph. For $x \in V$, let $N(x)$ be the set of neighbors of x in G. For $uv \in E$, let $V_1 = N(u) \cap N(v)$, $V_2 = N(u) \setminus N(v) \setminus \{v\}$, and $V_3 = N(v) \setminus N(u) \setminus \{v\}$. (See Figure 3.2.) Then*

$$G \wedge uv = (G*(V_1,V_2)*(V_2,V_3)*(V_3,V_1))_{uv}.$$

In other words, pivoting uv is an operation that,

(1) *for each $(x,y) \in (V_1 \times V_2) \cup (V_2 \times V_3) \cup (V_3 \times V_1)$, adds a new edge xy if $xy \notin E(G)$ or deletes it otherwise,*

(2) *and then, exchanges u and v.*

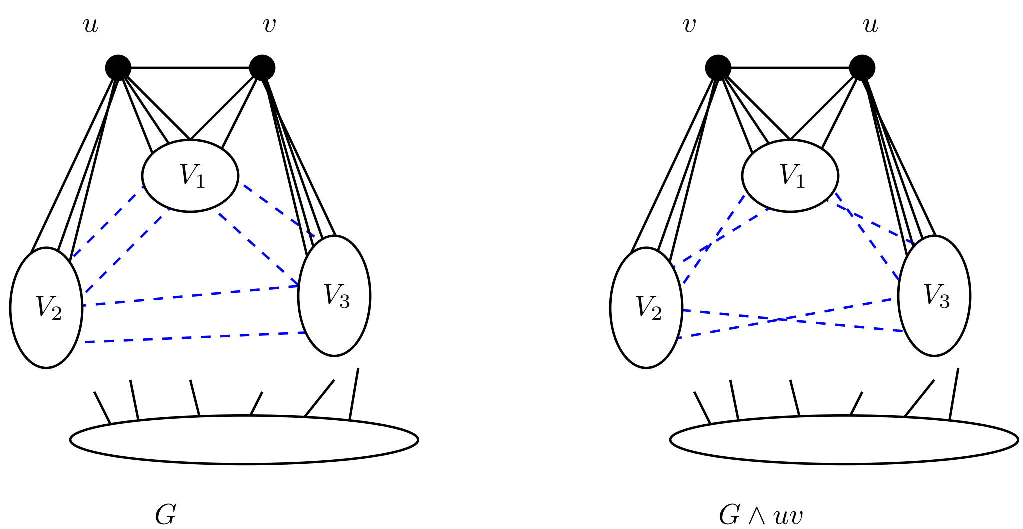

Figure 3.2: Pivoting

Proof. Note that V_1, V_2, V_3 are disjoint subsets of $V(G)$. For a graph H and $X \subseteq V(H)$, let $H * (X)^2 = H * (X, X)$.

Let us first consider the neighbors of u and v in $G*u*v*u$. The set of neighbors of u in G is $N(u) = V_1 \cup V_2 \cup \{v\}$. The set of neighbors of v in $G*u$ is $N(v)\Delta(N(u)\setminus\{v\}) = V_2 \cup V_3 \cup \{u\}$. The set of neighbors of u in $G*u*v$ is $N(u)\Delta(V_2 \cup V_3) = V_1 \cup V_3 \cup \{v\}$. Therefore, $G*u*v*u = G*(V_1 \cup V_2 \cup \{v\})^2 * (V_2 \cup V_3 \cup \{u\})^2 * (V_1 \cup V_3 \cup \{v\})^2$.

Now, we use the simple facts that $G*(X \cup Y)^2 = G*(X)^2*(Y)^2*(X,Y)$ for $X \cap Y = \emptyset$, $G*(X,Y)*(Z,W) = G*(Z,W)*(X,Y)$, $G*(X,Y)*(X,Y) = G$, and $G*(\{x\})^2 = G$. So, $G*(V_1 \cup V_2 \cup \{v\})^2 = G*(V_1)^2*(V_2)^2*(V_1,V_2)*(V_1,\{v\})*(V_2,\{v\})$.

By applying these, we obtain the following.

$$
\begin{aligned}
&G*u*v*u \\
&= G*(V_1,V_2)*(V_2,V_3)*(V_3,V_1) \\
&\quad *(V_1,\{v\})*(V_2,\{v\})*(V_2,\{u\})*(V_3,\{u\})*(V_1,\{v\})*(V_3,\{v\}) \\
&= G*(V_1,V_2)*(V_2,V_3)*(V_3,V_1)*(V_2,\{v\})*(V_2,\{u\})*(V_3,\{v\})*(V_3,\{u\}) \\
&= (G*(V_1,V_2)*(V_2,V_3)*(V_3,V_1))_{uv}
\end{aligned}
$$
□

Corollary 3.11. *If G is a graph and $uv \in E(G)$, then $G*u*v*u = G*v*u*v$.*

Proof. This is immediate from Proposition 3.10. □

Corollary 3.12. *If a graph G is bipartite and $uv \in E(G)$, $G \wedge uv$ is also bipartite.*

Proof. Let V_1, V_2, and V_3 be sets defined in Proposition 3.10. Since G is bipartite, $V_1 = \emptyset$. It does not break bipartiteness to add edges between V_2 and V_3. □

For a graph H, let $x \simeq_H y$ denote that either $x = y$ or they are adjacent in H. Let $a \oplus b$ denote $(a \wedge \neg b) \vee (\neg a \wedge b)$. This operation is usually called the logical "exclusive or" operation. (Note that we use the $\wedge$ symbol with two meanings: one for pivoting and another for the logical "and" operation.)

The next corollary is a reformulation of the above proposition.

Corollary 3.13. *Let G be a graph and let $uv \in E(G)$. For all $x, y \in V(G)$, $x \simeq_{G \wedge uv} y$ if and only if $(x \simeq_G y) \oplus (x \simeq_G u \wedge y \simeq_G v) \oplus (x \simeq_G v \wedge y \simeq_G u)$.*

Proof. If $x = y$, then it is clear.

Suppose $\{x, y\} \cap \{u, v\} = \emptyset$ and $x \neq y$. Let V_1, V_2, and V_3 be sets defined in Proposition 3.10. We add or remove an edge xy if and only if there exist $i, j \in \{1, 2, 3\}$ such that $x \in V_i$, $y \in V_j$, and $i \neq j$. It is equivalent to say that $(x \simeq_G u \wedge y \simeq_G v) \oplus (x \simeq_G v \wedge y \simeq_G u)$ is true.

Now, consider when one of x or y is u or v. We may assume that $x = u$ without loss of generality. Then

$$\begin{aligned}
&(x \simeq_G y) \oplus (x \simeq_G u \wedge y \simeq_G v) \oplus (x \simeq_G v \wedge y \simeq_G u) \\
&= (u \simeq_G y) \oplus (y \simeq_G v) \oplus (y \simeq_G u) && \text{because } u \text{ is adjacent to } v. \\
&= y \simeq_G v \\
&= y \simeq_{G \wedge uv} u && \text{because we exchanged } u \text{ and } v. \\
&= x \simeq_{G \wedge uv} y
\end{aligned}$$

□

Equivalent formulations of the following proposition were independently shown by Arratia, Bollabás, and Sorkin [1, Lemma 10] and Genest [29, Proposition 1.3.5]. But our proof does not require much case checking.

Proposition 3.14. *If $vv_1, vv_2 \in E(G)$ are two distinct edges incident with v, then*

$$G \wedge vv_1 \wedge v_1v_2 = G \wedge vv_2.$$

Proof. First of all, $G \wedge vv_1 \wedge v_1v_2$ is well-defined because v_1 and v_2 are adjacent in $G \wedge vv_1$. Let $G' = G \wedge vv_1$. Corollary 3.13 implies that $x \simeq_{G \wedge uv} y$ if and only if

$$(x \simeq_G y) \oplus (x \simeq_G u \wedge y \simeq_G v) \oplus (x \simeq_G v \wedge y \simeq_G u).$$

For simplicity, we write $\simeq$ instead of $\simeq_G$.

$$x \simeq_{G' \wedge v_1v_2} y = (x \simeq_{G'} y) \oplus (x \simeq_{G'} v_1 \wedge y \simeq_{G'} v_2) \oplus (x \simeq_{G'} v_2 \wedge y \simeq_{G'} v_1) \tag{3.1}$$
$$x \simeq_{G'} y = (x \simeq y) \oplus (x \simeq v \wedge y \simeq v_1) \oplus (x \simeq v_1 \wedge y \simeq v) \tag{3.2}$$
$$x \simeq_{G'} v_1 = x \simeq v \tag{3.3}$$
$$y \simeq_{G'} v_2 = (y \simeq v_2) \oplus (y \simeq v_1) \oplus (y \simeq v \wedge v_2 \simeq v_1) \tag{3.4}$$
$$x \simeq_{G'} v_2 = (x \simeq v_2) \oplus (x \simeq v_1) \oplus (x \simeq v \wedge v_2 \simeq v_1) \tag{3.5}$$
$$y \simeq_{G'} v_1 = y \simeq v \tag{3.6}$$

Now, let us apply (3.2) — (3.6) to (3.1). We use the fact that $a \wedge (b \oplus c) = (a \wedge b) \oplus (a \wedge c)$.

$$\begin{aligned}
x \simeq_{G' \wedge v_1v_2} y &= (x \simeq_{G'} y) \oplus (x \simeq_{G'} v_1 \wedge y \simeq_{G'} v_2) \oplus (x \simeq_{G'} v_2 \wedge y \simeq_{G'} v_1) \\
&= (x \simeq y) \oplus (x \simeq v \wedge y \simeq v_1) \oplus (x \simeq v_1 \wedge y \simeq v) \\
&\quad \oplus (x \simeq v \wedge y \simeq v_2) \oplus (x \simeq v \wedge y \simeq v_1) \oplus (x \simeq v \wedge y \simeq v \wedge v_2 \simeq v_1) \\
&\quad \oplus (x \simeq v_2 \wedge y \simeq v) \oplus (x \simeq v_1 \wedge y \simeq v) \oplus (x \simeq v \wedge y \simeq v \wedge v_2 \simeq v_1) \\
&= (x \simeq y) \oplus (x \simeq v \wedge y \simeq v_2) \oplus (x \simeq v_2 \wedge y \simeq v) \\
&= x \simeq_{G \wedge vv_2} y
\end{aligned}$$

Therefore, $x \simeq_{G \wedge vv_1 \wedge v_1v_2} y$ if and only if $x \simeq_{G \wedge vv_2} y$. □

The following observation is fundamental.

Proposition 3.15. *Let $G' = G * v$. Then for every $X \subseteq V(G)$,*

$$\rho_G(X) = \rho_{G'}(X).$$

Proof. We may assume that $v \in X$ by the symmetry of cut-rank.

Let $M = A(G)[X, V(G) \setminus X]$ and $M' = A(G')[X, V(G) \setminus X]$. It is easy to see that M' is obtained from M by adding the row of v to the rows of its neighbors in X. Therefore, $\rho_G(X) = \text{rk}(M) = \text{rk}(M') = \rho_{G'}(X)$. □

Corollary 3.16. *If H is locally equivalent to G, then the rank-width of H is equal to the rank-width of G. If H is a vertex-minor of G, then the rank-width of H is at most the rank-width of G.*

Proof. The first statement is obvious. Since vertex deletion does not increase cut-rank, it does not increase rank-width, and therefore the second statement is true. □

3.5 Bipartite graphs and binary matroids

In this section, we discuss the relation between branch-width of binary matroids and rank-width of bipartite graphs. We will also discuss further properties relating binary matroids and bipartite graphs. As an example, we will show the implication of the grid theorem for binary matroids by Geelen, Gerards, and Whittle [28]. The notion of matroids was reviewed in Section 2.5.

Let $G = (V, E)$ be a bipartite graph with a bipartition $V = A \cup B$. Let $Bin(G, A, B)$ be the binary matroid on V, represented by the $A \times V$ matrix

$$\begin{pmatrix} I_A & A(G)[A, B] \end{pmatrix},$$

where I_A is the $A \times A$ identity matrix. If $\mathcal{M} = Bin(G, A, B)$, then G is called a *fundamental graph* of $\mathcal{M}$.

Here is a major observation, which gives a relation between connectivity of binary matroids and cut-rank of bipartite graphs.

Proposition 3.17. *Let $G = (V, E)$ be a bipartite graph with a bipartition $V = A \cup B$ and let $\mathcal{M} = Bin(G, A, B)$. Then for every $X \subseteq V$, $\lambda_{\mathcal{M}}(X) = \rho_G(X) + 1$.*

Proof. Let $M = A(G)$. First note that

$$M[X, V \setminus X] = \begin{pmatrix} 0 & M[X \cap A, (V \setminus X) \cap B] \\ M[X \cap B, (V \setminus X) \cap A] & 0 \end{pmatrix}.$$

Therefore, $\rho_G(X) = \text{rk}(M[X, V \setminus X]) = \text{rk}(M[X \cap B, (V \setminus X) \cap A]) + \text{rk}(M[X \cap A, (V \setminus$

$X) \cap B]$). Consequently,

$$\begin{aligned}
\lambda_{\mathcal{M}}(X) &= r(X) + r(V \setminus X) - r(V) + 1 \\
&= \operatorname{rk}\begin{pmatrix} 0 & M[(V \setminus X) \cap A, X \cap B] \\ I_{X \cap A} & M[X \cap A, X \cap B] \end{pmatrix} \\
&\quad + \operatorname{rk}\begin{pmatrix} 0 & M[X \cap A, (V \setminus X) \cap B] \\ I_{(V \setminus X) \cap A} & M[(V \setminus X) \cap A, (V \setminus X) \cap B] \end{pmatrix} - |A| + 1 \\
&= \operatorname{rk}(M[(V \setminus X) \cap A, X \cap B]) + \operatorname{rk}(M[X \cap A, (V \setminus X) \cap B]) + 1 \\
&= \rho_G(X) + 1.
\end{aligned}$$

□

An easy corollary of Proposition 3.17 is the following.

Corollary 3.18. *Let $G = (V, E)$ be a bipartite graph with a bipartition $V = A \cup B$ and let $\mathcal{M} = Bin(G, A, B)$. Then the branch-width of $\mathcal{M}$ is one more than the rank-width of G.*

Proof. This is trivial because $(T, \mathcal{L})$ is a branch-decomposition of $\mathcal{M}$ of width $k + 1$ if and only if it is a rank-decomposition of G of width k. □

Now, let us discuss the relation between matroid minors and graph vertex-minors.

Proposition 3.19. *Let $G = (V, E)$ be a bipartite graph with a bipartition $V = A \cup B$ and let $\mathcal{M} = Bin(G, A, B)$. Then*

(1) *$Bin(G, B, A) = \mathcal{M}^*$,*

(2) *For $uv \in E(G)$, $Bin(G \wedge uv, A\Delta\{u, v\}, B\Delta\{u, v\}) = \mathcal{M}$.*

(3) $Bin(G \setminus v, A \setminus \{v\}, B \setminus \{v\}) = \begin{cases} \mathcal{M}/v & \text{if } v \in A, \\ \mathcal{M} \setminus v & \text{if } v \in B. \end{cases}$

Proof. Let M be the adjacency matrix of G. Then, $\mathcal{M}$ is represented by a matrix $\begin{pmatrix} I & M[A, B] \end{pmatrix}$.

(1): It is known that $\mathcal{M}^*$ is represented by a matrix $\begin{pmatrix} M[B, A] & I \end{pmatrix}$. Therefore, $\mathcal{M}^* = Bin(G, B, A)$

(2): We may assume that $u \in A$, $v \in B$. Let $R = (r_{ij} : i \in A, j \in V) = \begin{pmatrix} I & M[A, B] \end{pmatrix}$ be a matrix over GF(2). (So, $r_{ij} = 1$ if $j \in B$ and $ij \in E(G)$ or $i = j$, and $r_{ij} = 0$ otherwise.) We know that elementary row operations on R do not change the associated matroid $\mathcal{M}$.

By adding the row vector of u, that is $(r_{uj} : j \in V)$, to the rows of neighbors of u in A, we obtain another matrix $R' = (r'_{ij} : i \in A, j \in V)$ representing the same matroid. We first observe that $R'[A, (A \setminus \{u\}) \cup \{v\}]$ is an identity matrix, because $r_{uv} = 1$ and when we obtain R', we changed all 1's into 0's in the column of v. We also observe that the column vector of u, v in R' is equal to the column vector of v, u in R respectively. Moreover for $i \neq u$ and $j \in B \setminus \{v\}$, $r'_{ij} \neq r_{ij}$ if and only if $r_{uj} = 1$ and $r_{iv} = 1$, or equivalently $iv, ju \in E(G)$. By Proposition 3.10, we know that for

$i \in A \setminus \{u\}$ and $j \in B \setminus \{v\}$, ij belongs to exactly one of $E(G)$ and $E(G \wedge uv)$ if and only if $iv, ju \in E(G)$. (Because G is bipartite, $iu, jv \notin E(G)$.) Moreover the set of neighbors of u, v in $G \wedge uv$ is equal to the set of neighbors of v, u in G respectively. Therefore, we conclude that $\mathcal{M} = Bin(G \wedge uv, A\Delta\{v,w\}, B\Delta\{v,w\})$.

(3): If $v \in B$, by deleting the column of v in $\begin{pmatrix} I & M[A,B] \end{pmatrix}$, we obtain a matrix representation of $\mathcal{M} \setminus v$ and therefore $\mathcal{M} \setminus v = Bin(G \setminus v, A, B \setminus \{v\})$.

If $v \in A$, then $\mathcal{M}^* = Bin(G, B, A)$, and therefore $\mathcal{M}^* \setminus v = Bin(G, B, A \setminus \{v\})$ and $\mathcal{M}/v = Bin(G, A \setminus \{v\}, B)$. □

Corollary 3.20. *Let $\mathcal{M}$ be a binary matroid and G be the fundamental graph of $\mathcal{M}$ with a bipartition $V(G) = A \cup B$ such that $\mathcal{M} = Bin(G, A, B)$. If v has no neighbor in G, then*

$$\mathcal{M} \setminus v = \mathcal{M}/v = Bin(G \setminus v, A \setminus \{v\}, B \setminus \{v\}).$$

Otherwise let w be a neighbor of v.

$$\text{(1)}\ \ \mathcal{M} \setminus v = \begin{cases} Bin(G \wedge vw \setminus v, A\Delta\{v,w\}, B\Delta\{v,w\} \setminus \{v\}) & \text{if } v \in A, \\ Bin(G \setminus v, A \setminus \{v\}, B \setminus \{v\}) & \text{otherwise.} \end{cases}$$

$$\text{(2)}\ \ \mathcal{M}/v = \begin{cases} Bin(G \wedge vw \setminus v, A\Delta\{v,w\} \setminus \{v\}, B\Delta\{v,w\}) & \text{if } v \in B, \\ Bin(G \setminus v, A \setminus \{v\}, B \setminus \{v\}) & \text{otherwise.} \end{cases}$$

Note that the matroid $Bin(G \wedge vw \setminus v, A\Delta\{v,w\} \setminus \{v\}, B\Delta\{v,w\} \setminus \{v\})$ is independent of the choice of w by Proposition 3.14 and (2) of Proposition 3.19.

Proof. If v has no neighbor in G, then v is a loop or a coloop of $\mathcal{M}$, and therefore $\mathcal{M}\setminus v = \mathcal{M}/v$. By (3) of Proposition 3.19, we deduce that $Bin(G\setminus v, A\setminus\{v\}, B\setminus\{v\}) = \mathcal{M} \setminus v = \mathcal{M}/v$.

Now we assume that w is a neighbor of v. By (1) of Proposition 3.19, it is enough to show (1). If $v \in B$, then by (3) of Proposition 3.19, we obtain that $\mathcal{M} \setminus v = Bin(G \setminus v, A, B \setminus \{v\})$. If $v \in A$, then $\mathcal{M} = Bin(G \wedge vw, A\Delta\{v,w\}, B\Delta\{v,w\})$, and therefore $\mathcal{M} \setminus v = Bin(G \wedge vw, A\Delta\{v,w\}, B\Delta\{v,w\} \setminus \{v\})$. □

Corollary 3.21. *If G, H are bipartite graphs with bipartitions $A \cup B = V(G)$ and $A' \cup B' = V(H)$ and $Bin(H, A', B') = Bin(G, A, B)$, then H can be obtained by applying a sequence of pivotings to G, and therefore H is locally equivalent to G.*

Proof. We proceed by induction on $|A'\Delta A|$.

Let $\mathcal{M} = Bin(G, A, B) = Bin(H, A', B')$. If $A' = A$, then $G = H$ because $\mathcal{M}$ determines every fundamental circuit with respect to A.

Now, we may assume that $A' \neq A$. Since A and A' are bases of $\mathcal{M}$, we may pick $w \in A' \setminus A$ and $v \in A \setminus A'$ such that w is in the fundamental circuit of v with respect to A', and therefore $vw \in E(H)$. Let $H' = H \wedge vw$. By (2) of Proposition 3.19, $\mathcal{M} = Bin(H', A'\Delta\{v,w\}, B'\Delta\{v,w\})$. By induction, H' can be obtained by applying a sequence of pivotings to G. Since $H = H' \wedge vw$, H can be obtained by applying a sequence of pivotings to G. □

Corollary 3.22.

(1) *Let $\mathcal{N}$, $\mathcal{M}$ be binary matroids, and H, G be fundamental graphs of $\mathcal{N}$, $\mathcal{M}$ respectively. If $\mathcal{N}$ is a minor of $\mathcal{M}$, then H is a pivot-minor of G, and therefore H is a vertex-minor of G.*

(2) *Let G be a bipartite graph with a bipartition $A \cup B = V(G)$. If H is a pivot-minor of G, then there is a bipartition $A' \cup B' = V(H)$ of H such that $Bin(H, A', B')$ is a minor of $Bin(G, A, B)$.*

Proof. (1) We proceed by induction on $|E(\mathcal{M}) \setminus E(\mathcal{N})|$. By Corollary 3.21, we may assume that $\mathcal{M} \neq \mathcal{N}$. By induction, it is enough to show it when $\mathcal{N} = \mathcal{M} \setminus v$ or $\mathcal{N} = \mathcal{M}/v$ for $v \in V(G)$. By Corollary 3.20, either $G \wedge vw \setminus v$ for some $w \in V(G)$ or $G \setminus v$ is a fundamental graph of $\mathcal{N}$. By Corollary 3.21, H can be obtained from either $G \wedge vw \setminus v$ or $G \setminus v$ by applying a sequence of pivotings.

(2): By (2) and (3) of Proposition 3.19, we obtain a bipartition (A', B') of H such that $Bin(H, A', B')$ is a minor of $Bin(G, A, B)$. □

By Proposition 3.19, theorems about branch-width of binary matroids give corollaries about rank-width of bipartite graphs. One of the recent theorems about branch-width of binary matroids was proved by Geelen, Gerards, and Whittle. Let us recall their theorem in the context of binary matroids. The $n \times n$ *grid* is a graph on the vertex set $\{1, 2, \dots, n\} \times \{1, 2, \dots, n\}$ such that (x_1, y_1) and (x_2, y_2) are adjacent if and only if $|x_1 - x_2| + |y_1 - y_2| = 1$.

Theorem 3.23 (Grid theorem for binary matroids [28]). *For every positive integer k, there is an integer l such that if $\mathcal{M}$ is a binary matroid with branch-width at least l, then $\mathcal{M}$ contains a minor isomorphic to the cycle matroid of the $k \times k$ grid.*

To make corollaries about rank-width from this theorem, it is helpful to replace the $k \times k$ grid by a planar graph whose cycle matroid has a simpler fundamental graph. We define a planar graph $R_k = (V, E)$ (Figure 3.3) as following:

$$V = \{v_1, v_2, \cdots, v_{k^2}\},$$
$$E = \{v_i v_{i+1} : 1 \leq i \leq k^2 - 1\} \cup \{v_i v_{i+k} : 1 \leq i \leq k^2 - k\}.$$

We can obtain a minor of R_k isomorphic to the $k \times k$ grid by deleting edges $v_{ik} v_{ik+1}$ for all $1 \leq i \leq k - 1$. To show that R_k is isomorphic to a minor of the $l \times l$ grid for a big l, let us cite a useful lemma by Robertson, Seymour, and Thomas.

Lemma 3.24 (Robertson, Seymour, and Thomas [50, (1.5)]). *If H is a planar graph with $|V(H)| + 2|E(H)| \leq n$, then H is isomorphic to a minor of the $2n \times 2n$ grid.*

By this lemma, R_k is isomorphic to a minor of the $6k^2 \times 6k^2$ grid. Therefore, Theorem 3.23 is still true if R_k is used instead of the $k \times k$ grid.

Now, let us construct a fundamental graph S_k of the cycle matroid of R_k. Since edges of R_k represent elements of the cycle matroid of R_k, they are vertices of S_k. Let

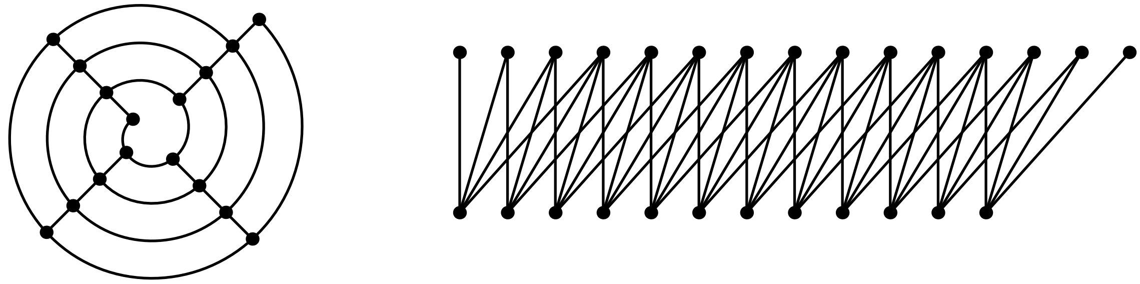

Figure 3.3: R_4 and S_4

$a_i = v_i v_{i+1}$ and $b_i = v_i v_{i+k}$. Let $A = \{a_i : 1 \le i \le k^2-1\}$ and $B = \{b_i : 1 \le i \le k^2-k\}$ so that A is the set of edges of a spanning tree of R_k. For each $b_j \in B$, $a_i b_j \in E(S_k)$ if and only if a_i is in the fundamental cycle of b_j with respect to the spanning tree of R_k with the edge set A. In summary, S_k is a bipartite graph with $V(S_k) = A \cup B$ such that $a_i b_j \in E(S_k)$ if and only if $i \le j < i + k$ (Figure 3.3). By Corollary 3.22, we obtain the following.

Corollary 3.25. *For every positive integer k, there is an integer l such that if a bipartite graph G has rank-width at least l, then it contains a vertex-minor isomorphic to S_k.*

This corollary will be used in Chapter 5 to prove a weaker version of Seese's conjecture.

3.6 Inequalities on cut-rank and vertex-minors

Submodularity plays an important role in many places of combinatorics. In this section, we prove inequalities concerning the cut-rank function.

Proposition 3.26. *Let $G = (V, E)$ be a graph and let $v \in V$ and $Y_1 \subseteq V$. Let $M = A(G)$ be the adjacency matrix of G over* GF(2)*. Then*

$$\rho_{G*v\setminus v}(Y_1) = \mathrm{rk}\begin{pmatrix} 1 & M[\{v\}, V \setminus Y_1 \setminus \{v\}] \\ M[Y_1, \{v\}] & M[Y_1, V \setminus Y_1 \setminus \{v\}] \end{pmatrix} - 1.$$

Moreover, if w is a neighbor of v, then

$$\rho_{G\wedge vw\setminus v}(Y_1) = \mathrm{rk}\begin{pmatrix} 0 & M[\{v\}, V \setminus Y_1 \setminus \{v\}] \\ M[Y_1, \{v\}] & M[Y_1, V \setminus Y_1 \setminus \{v\}] \end{pmatrix} - 1.$$

Proof. We will use elementary row operations on matrices to prove the claim. Let N be the set of neighbors of v in G. Let J_A^B be a matrix $(1)_{i\in A, j\in B}$. We will write J instead of J_A^B if it is not confusing. Let $V = V(G)$. Let $Y_2 = V \setminus Y_1 \setminus \{v\}$. Let $L_{11} = M[Y_1 \cap N, Y_2 \cap N]$, $L_{12} = M[Y_1 \cap N, Y_2 \setminus N]$, $L_{21} = M[Y_1 \setminus N, Y_2 \cap N]$, and

$L_{22} = M[Y_1 \setminus N, Y_2 \setminus N]$. Then

$$\begin{aligned}
\rho_{G*v\setminus v}(Y_1) &= \operatorname{rk}(A(G*v)[Y_1,Y_2]) \\
&= \operatorname{rk}\begin{pmatrix} L_{11}+J & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \\
&= \operatorname{rk}\begin{pmatrix} 1 & 1\ 1\ 1\ \cdots 1 & 0\ 0\ 0 \cdots 0 \\ 0 & L_{11}+J & L_{12} \\ 0 & L_{21} & L_{22} \end{pmatrix} - 1 \\
&= \operatorname{rk}\begin{pmatrix} 1 & 1\ 1\ 1\ \cdots 1 & 0\ 0\ 0 \cdots 0 \\ J & L_{11} & L_{12} \\ 0 & L_{21} & L_{22} \end{pmatrix} - 1 \\
&= \operatorname{rk}\begin{pmatrix} 1 & M[\{v\},Y_2] \\ M[Y_1,\{v\}] & M[Y_1,Y_2] \end{pmatrix} - 1.
\end{aligned}$$

Let W be the set of neighbors of w. We may assume that $w \in Y_1$ by symmetry. Consequently $w \in Y_1 \cap (N \setminus W)$. Let $N_1 = N \setminus W \setminus \{w\}$, $N_2 = N \cap W$, $N_3 = W \setminus N$, $N_4 = V \setminus N \setminus W \setminus \{w\}$. Let $M_{ij} = M[Y_1 \cap N_i, Y_2 \cap N_j]$ for all $i, j \in \{1,2,3,4\}$. Then

$$\begin{aligned}
\rho_{G\wedge vw\setminus v}(Y_1) &= \operatorname{rk}(A(G\wedge vw)[Y_1,Y_2]) \\
&= \operatorname{rk}\begin{pmatrix} 1\ 1\ 1 \cdots 1 & 1\ 1\ 1 \cdots 1 & 0\ 0\ 0 \cdots 0 & 0\ 0\ 0 \cdots 0 \\ M_{11} & M_{12}+J & M_{13}+J & M_{14} \\ M_{21}+J & M_{22} & M_{23}+J & M_{24} \\ M_{31}+J & M_{32}+J & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix} \\
&= \operatorname{rk}\begin{pmatrix} 1 & 0\ 0\ 0 \cdots 0 & 1\ 1\ 1 \cdots 1 & 1\ 1\ 1 \cdots 1 & 0\ 0\ 0 \cdots 0 \\ 0 & 1\ 1\ 1 \cdots 1 & 1\ 1\ 1 \cdots 1 & 0\ 0\ 0 \cdots 0 & 0\ 0\ 0 \cdots 0 \\ 0 & M_{11} & M_{12}+J & M_{13}+J & M_{14} \\ 0 & M_{21}+J & M_{22} & M_{23}+J & M_{24} \\ 0 & M_{31}+J & M_{32}+J & M_{33} & M_{34} \\ 0 & M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix} - 1 \\
&= \operatorname{rk}\begin{pmatrix} 1 & 0\ 0\ 0 \cdots 0 & 1\ 1\ 1 \cdots 1 & 1\ 1\ 1 \cdots 1 & 0\ 0\ 0 \cdots 0 \\ 0 & 1\ 1\ 1 \cdots 1 & 1\ 1\ 1 \cdots 1 & 0\ 0\ 0 \cdots 0 & 0\ 0\ 0 \cdots 0 \\ J & M_{11} & M_{12} & M_{13} & M_{14} \\ J & M_{21}+J & M_{22}+J & M_{23} & M_{24} \\ 0 & M_{31}+J & M_{32}+J & M_{33} & M_{34} \\ 0 & M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix} - 1 \\
&= \operatorname{rk}\begin{pmatrix} 0 & 1\ 1\ 1 \cdots 1 & 1\ 1\ 1 \cdots 1 & 0\ 0\ 0 \cdots 0 & 0\ 0\ 0 \cdots 0 \\ 1 & 0\ 0\ 0 \cdots 0 & 1\ 1\ 1 \cdots 1 & 1\ 1\ 1 \cdots 1 & 0\ 0\ 0 \cdots 0 \\ J & M_{11} & M_{12} & M_{13} & M_{14} \\ J & M_{21} & M_{22} & M_{23} & M_{24} \\ 0 & M_{31} & M_{32} & M_{33} & M_{34} \\ 0 & M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix} - 1
\end{aligned}$$

$$= \mathrm{rk}\begin{pmatrix} 0 & M[\{v\}, Y_2] \\ M[Y_1, \{v\}] & M[Y_1, Y_2] \end{pmatrix} - 1. \qquad \square$$

The following lemma is analogous to an inequality on connectivity functions of matroids [27, (5.2)]. Later we will show an equivalent statement in Lemma 6.11 with another proof.

Lemma 3.27. *Let G be a graph and $v \in V(G)$. Suppose that (X_1, X_2) and (Y_1, Y_2) are partitions of $V(G) \setminus \{v\}$. Then*

$$\rho_{G\setminus v}(X_1) + \rho_{G*v\setminus v}(Y_1) \geq \rho_G(X_1 \cap Y_1) + \rho_G(X_2 \cap Y_2) - 1.$$

If w is a neighbor of v, then

$$\rho_{G\setminus v}(X_1) + \rho_{G\wedge vw\setminus v}(Y_1) \geq \rho_G(X_1 \cap Y_1) + \rho_G(X_2 \cap Y_2) - 1,$$
$$\rho_{G*v\setminus v}(X_1) + \rho_{G\wedge vw\setminus v}(Y_1) \geq \rho_G(X_1 \cap Y_1) + \rho_G(X_2 \cap Y_2) - 1.$$

Proof. We use Proposition 3.26 and apply Proposition 3.2. Let $M = A(G)$ be the adjacency matrix of G over GF(2). Then

$$\begin{aligned}
&\rho_{G\setminus v}(X_1) + \rho_{G\wedge vw\setminus v}(Y_1) \\
&= \mathrm{rk}(M[X_1, X_2] + \mathrm{rk}(M[Y_1 \cup \{v\}, Y_2 \cup \{v\}]) - 1 \\
&\geq \mathrm{rk}(M[X_1 \cap Y_1, X_2 \cup \{v\} \cup Y_2] + \mathrm{rk}(M[X_1 \cup \{v\} \cup Y_1, Y_2 \cap X_2]) - 1 \\
&= \rho_G(X_1 \cap Y_1) + \rho_G(X_2 \cap Y_2) - 1.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\rho_{G\setminus v}(X_1) + \rho_{G*v\setminus v}(Y_1) \\
&= \mathrm{rk}(M[X_1, X_2] + \mathrm{rk}\begin{pmatrix} 1 & M[\{v\}, Y_2] \\ M[Y_1, \{v\}] & M[Y_1, Y_2] \end{pmatrix} - 1 \\
&\geq \mathrm{rk}(M[X_1 \cap Y_1, X_2 \cup \{v\} \cup Y_2]) + \mathrm{rk}(M[X_1 \cup \{v\} \cup Y_1, Y_2 \cap X_2]) - 1 \\
&= \rho_G(X_1 \cap Y_1) + \rho_G(X_2 \cap Y_2) - 1.
\end{aligned}$$

Since $G * v \wedge vw = G \wedge vw * w$, we obtain that

$$G * v \wedge vw \setminus v = G \wedge vw \setminus v * w.$$

Let $H = G * v$. We deduce that

$$\rho_{H\setminus v}(X_1) + \rho_{H\wedge vw\setminus v}(Y_1) \geq \rho_H(X_1 \cap Y_1) + \rho_H(X_2 \cap Y_2) - 1.$$

Therefore $\rho_{G*v\setminus v}(X_1) + \rho_{G\wedge vw\setminus v*w}(Y_1) \geq \rho_{G*v}(X_1 \cap Y_1) + \rho_{G*v}(X_2 \cap Y_2) - 1$. We note that $\rho_{H*x}(Z) = \rho_H(Z)$ for every graph H, $x \in V(H)$, and $Z \subseteq V(H)$. $\square$

3.7 Tutte's linking theorem

In this section, we prove a theorem analogous to Tutte's linking theorem [57]. In the following theorem, we show that the minimum cut-rank of cuts separating two disjoint sets X, Y of vertices of a graph G is equal to the maximum cut-rank of X in all vertex-minors of G having $X \cup Y$ as the set of vertices.

Theorem 3.28. *Let G be a graph and X, Y be disjoint subsets of $V(G)$. The following are equivalent.*

(1) $\min_{X \subseteq Z \subseteq V(G) \setminus Y} \rho_G(Z) \geq k$.

(2) *There exists a vertex-minor G' of G such that $V(G') = X \cup Y$ and $\rho_{G'}(X) \geq k$.*

(3) *There exists a pivot-minor G' of G such that $V(G') = X \cup Y$ and $\rho_{G'}(X) \geq k$.*

Proof. (2)⇒(1): We may assume that G' is an induced subgraph of G by applying local complementations to G. For all Z satisfying $X \subseteq Z \subseteq V(G) \setminus Y$, we have

$$k \leq \rho_{G'}(X) = \rho^*_G(X, Y) \leq \rho^*_G(Z, V(G) \setminus Z) = \rho_G(Z).$$

(3)⇒(2): Trivial.

(1)⇒(3): We proceed by induction on $|V(G) \setminus (X \cup Y)|$. Suppose there is no such graph G'. If $X \cup Y = V(G)$, then it is trivial. Let $x \in V(G) \setminus (X \cup Y)$. If x has no neighbor, then $\rho_{G \setminus x}(Z) = \rho_G(Z)$ for all $Z \subseteq V(G) \setminus \{x\}$. Therefore, $\min_{X \subseteq Z \subseteq V(G) \setminus Y} \rho_G(Z) = \min_{X \subseteq Z \subseteq V(G) \setminus \{x\} \setminus Y} \rho_{G \setminus x}(Z)$.

So, we may assume that x has a neighbor y. By induction, there exists $A \subseteq V(G) \setminus \{x\}$ such that $\rho_{G \setminus x}(A) \leq k-1$. Also, there exists $B \subseteq V(G) \setminus \{x\}$ such that $\rho_{G \wedge xy \setminus x}(B) \leq k-1$. By Lemma 3.27, either $\rho_G(A \cap B) \leq k-1$ or $\rho_G(A \cup B) \leq k-1$. Consequently, $\min_{X \subseteq Z \subseteq V(G) \setminus Y} \rho_G(Z) \leq k-1$. □

We can deduce Tutte's linking theorem for binary matroids from the above theorem. Here is the statement of Tutte's linking theorem for binary matroids.

Corollary 3.29. *Let $\mathcal{M} = (E, \mathcal{I})$ be a binary matroid and let X, Y be disjoint subsets of E. Then*

$$\min_{X \subseteq Z \subseteq E \setminus Y} \lambda_{\mathcal{M}}(Z) \geq k$$

if and only if there is a minor $\mathcal{M}'$ of $\mathcal{M}$ such that $E(\mathcal{M}') = X \cup Y$ and $\lambda_{\mathcal{M}'}(X) \geq k$.

Proof. Let G be a bipartite graph with a bipartition $A \cup B = V(G)$ such that $Bin(G, A, B) = \mathcal{M}$. There exists a minor $\mathcal{M}'$ of $\mathcal{M}$ such that $E(\mathcal{M}') = X \cup Y$ and $\lambda_{\mathcal{M}'}(X) \geq k$ if and only if there exists a pivot-minor H of G such that $V(H) = X \cup Y$ and $\rho_H(X) \geq k-1$ by Corollary 3.22. The remaining proof is routine by Proposition 3.17 and Proposition 3.28. □

Chapter 4

Testing Vertex-minors

For fixed graph H, Robertson and Seymour gave a $O(|V(G)|^3)$-time algorithm to test whether the input graph G contains H as a minor in [49]. We may ask the same question for vertex-minors, but are not yet able to answer this question completely. However, we show a polynomial-time algorithm that works only for graphs of bounded rank-width, by using a logic formula describing vertex-minors. To construct these logic formulas, we use the notion of *isotropic systems* and their *minors*. Informally speaking, isotropic systems are equivalence classes of graphs by local equivalence. Therefore, it enables us to describe vertex-minors in terms of minors of isotropic systems. In Section 4.1, we review the notion of isotropic systems. In Section 4.2, we review monadic second-order logic formulas. In Section 4.3, we discuss an algorithm evaluating monadic second-order logic formulas. By combining these sections, we will build monadic-second order logic formulas describing vertex-minors in Section 4.4.

4.1 Review on isotropic systems

In this section, the notion of *isotropic systems* and a few useful theorems will be reviewed. All materials are from Bouchet's papers [4, 5, 7]. We change a little notation for readability; in particular, Bouchet used capital letters to denote vectors, and we use small letters.

4.1.1 Definition of isotropic systems

Let us begin with a definition for vector spaces. For a vector space W with a bilinear form $\langle\, , \rangle$, a subspace L of W is called *totally isotropic* if and only if $\langle x, y\rangle = 0$ for all $x, y \in L$.

Let $K = \{0, \alpha, \beta, \gamma\}$ be the two-dimensional vector space over GF(2) with the bilinear form $\langle\, , \rangle$ such that $\alpha + \beta + \gamma = 0$ and

$$\langle x, y\rangle = \begin{cases} 1 & \text{if } x \neq y \text{ and } x, y \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let V be a finite set. Let K^V be the set of functions from V to K, and so K^V is a vector space over GF(2). We attach the following bilinear form to K^V:

$$\text{for } x, y \in K^V, \quad \langle x, y \rangle = \sum_{v \in V} \langle x(v), y(v) \rangle \in \text{GF}(2).$$

Definition 4.1 (Bouchet [4]). *We call $S = (V, L)$ an* isotropic system *if V is a finite set and L is a totally isotropic subspace of K^V with* $\dim(L) = |V|$*. We call V the* element set *of S.*

Let us define some notation. For $X \subseteq V$, let $p_X : K^V \to K^X$ be the *canonical projection* such that

$$(p_X(a))(v) = a(v) \quad \text{for all } v \in X \text{ and } a \in K^V.$$

For $a \in K^V$ and $X \subseteq V$, $a[X]$ is a vector in K^V such that

$$a[X](v) = \begin{cases} a(v) & \text{if } v \in X, \\ 0 & \text{otherwise.} \end{cases}$$

We note that $p_X(a)$ should not be confused with $a[X]$. While $p_X(a)$ is a vector in K^X, $a[X]$ is a vector in K^V. Let L be a subspace of K^V and $v \in V$. Let $x \in K \setminus \{0\} = \{\alpha, \beta, \gamma\}$.

- Let $L^\perp$ be the subspace of K^V such that

$$L^\perp = \{z \in K^V : \langle z, y \rangle = 0 \text{ for all } y \in L\}.$$

- Let $L|_x^v$ be the subspace of $K^{V \setminus \{v\}}$ such that

$$L|_x^v = \{p_{V \setminus \{v\}}(a) : a \in L, a(v) = 0 \text{ or } x\}.$$

- Let $L|_{\subseteq X}$, $L|_X$ be the subspaces of K^X such that

$$\begin{aligned} L|_{\subseteq X} &= \{p_X(a) : a \in L, a(v) = 0 \text{ for all } v \in V \setminus X\} \\ L|_X &= \{p_X(a) : a \in L\} \end{aligned}$$

We remark that every totally isotropic subspace L of K^V has dimension at most $|V|$ because

$$2\dim(L) \leq \dim(L) + \dim(L^\perp) = 2|V|.$$

Therefore $|V|$ is the maximum possible dimension that totally isotropic subspaces can achieve.

Two vectors $a, b \in K^V$ are called *supplementary* if $\langle a(v), b(v) \rangle = 1$ for all $v \in V$. We call $a \in K^V$ *complete* if $a(v) \neq 0$ for all $v \in V$. For $X \subseteq V$ and a complete vector

a of K^X, $L|_a^X$ is the subspace of $K^{V\setminus X}$ such that

$$L|_a^X = \{p_{V\setminus X}(b) : b \in L, b(v) \in \{a(v), 0\} \text{ for all } v \in X\}.$$

Note that $L|_{x_1}^{v_1}|_{x_2}^{v_2}|_{x_3}^{v_3} \cdots |_{x_k}^{v_k} = L|_x^{\{v_1,v_2,\ldots,v_k\}}$ where $x \in K^{\{x_1,x_2,\ldots,x_k\}}$ such that $x(v_i) = x_i$.

Definition 4.2 (Bouchet [4, (8.1)]). *Let $S = (V, L)$ be an isotropic system and $v \in V$. For $x \in K \setminus \{0\}$, $S|_x^v = (V \setminus \{v\}, L|_x^v)$ is called an* elementary minor *of S. An isotropic system S' is called a* minor *of S if S' can be obtained from S by applying a sequence of elementary minor operations; in other words,*

$$S' = S|_{x_1}^{v_1}|_{x_2}^{v_2}|_{x_3}^{v_3} \cdots |_{x_k}^{v_k}$$

for $x_1, x_2, \ldots x_k \in K \setminus \{0\}$ and distinct $v_1, v_2, \ldots, v_k \in V$.

Bouchet proved that an elementary minor of an isotropic system is again an isotropic system. We show the proof for the completeness of this thesis.

Proposition 4.3 (Bouchet [4, (8.1)]). *Let $S = (V, L)$ be an isotropic system and $v \in V$. For each $x \in K \setminus \{0\}$, $S|_x^v$ is an isotropic system.*

Proof. It is easy to see that $L|_x^v$ is a subspace of $K^{V\setminus\{v\}}$, because $a + b \in L|_x^v$ for all $a, b \in L|_x^v$. Moreover $L|_x^v$ is totally isotropic, because if $\langle a(v), b(v)\rangle = 0$, then

$$\langle p_{V\setminus\{v\}}(a), p_{V\setminus\{v\}}(b)\rangle = \langle a, b\rangle$$

for all $a, b \in K^V$.

We claim that $\dim(L|_x^v) = |V| - 1$. We have $\dim(L|_x^v) \le |V| - 1$, because $L|_x^v$ is a totally isotropic subspace of $K^{V\setminus\{v\}}$. Let B be a basis of L. Since $\dim(L) = |V|$, B should contain at least one vector with a nonzero value at v. However we may assume that at most two vectors in B have nonzero values at v because $\alpha + \alpha = \beta + \beta = \gamma + \gamma = \alpha + \beta + \gamma = 0$.

If B has only one vector a with $a(v) \neq 0$, then $\{p_{V\setminus\{v\}}(b) : b(v) = 0, b \in B\}$ is independent in $L|_x^v$ and we deduce that $\dim(L|_x^v) \ge |V| - 1$.

Now let us assume that B has exactly two vectors a_1, a_2 with $a_1(v), a_2(v) \neq 0$. Let $B \setminus \{a_1, a_2\} = \{a_3, a_4, \ldots, a_{|V|}\}$. We may assume that $a_1(v) \neq a_2(v)$ because we can exchange a_2 by $a_2 + a_1$. We may assume that $a_1(v) = x$ or $a_2(v) = x$ because otherwise $a_1(v) + a_2(v) = x$. We may assume that $a_2(v) = x$. We claim that $\{p_{V\setminus\{v\}}(a_i) : 2 \le i \le |V|\}$ is independent in $L|_x^v$. Suppose not. There exists W such that $\emptyset \neq W \subseteq \{2, 3, \ldots, |V|\}$ and $\sum_{i\in W} p_{V\setminus\{v\}}(a_i) = 0$. It is clear that $\{p_{V\setminus\{v\}}(a_i) : 3 \le i \le |V|\}$ is independent, and therefore $2 \in W$. Since B is a basis of L,

$$\left(\sum_{i\in W} a_i\right)(w) = \begin{cases} x & \text{if } w = v, \\ 0 & \text{otherwise.} \end{cases}$$

Then we obtain $\langle a_1, \sum_{i\in W} a_i\rangle = 1$, which is a contradiction because L is totally isotropic and $\sum_{i\in W} a_i \in L$. Therefore, $\dim(L|_x^v) \ge |V| - 1$. □

4.1.2 Fundamental basis and fundamental graphs

The connection between isotropic systems and graphs was also studied by Bouchet [5].

Definition 4.4 (Bouchet [5]). *We call $x \in K^V$ an* Eulerian vector *of an isotropic system $S = (V, L)$ if*

(i) *x is complete and*

(ii) *$\emptyset \neq P \subseteq V$ implies $x[P] \notin L$.*

Proposition 4.5 (Bouchet [5, (4.1)]). *For every complete vector c of K^V, there is an Eulerian vector a of S, supplementary to c.*

Proof. Let $S = (V, L)$ be an isotropic system. We proceed by induction on $|V|$. Let $v \in V$. By symmetry, we may assume that $c(v) = \gamma$. If $|V| \leq 1$, then it is trivial. Suppose that S does not have an Eulerian vector. For $x \in K^{V\setminus\{v\}}$ and $y \in K$, we let $x \oplus y \in K^V$ be a vector such that $p_{V\setminus\{v\}}(x \oplus y) = x$ and $(x \oplus y)(v) = y$.

Let a be an Eulerian vector of $S|_\gamma^v$. Since $a \oplus \alpha$ is not Eulerian, there exists a nonempty set $X \subseteq V$ such that $(a \oplus \alpha)[X] \in L$. Since a is an Eulerian vector of $S|_\gamma^v$, we conclude that $v \in X$. Similarly we have a nonempty set $Y \subseteq V$ such that $(a \oplus \beta)[Y] \in L$ and $v \in Y$. By adding two vectors, we obtain

$$(a \oplus \alpha)[X] + (a \oplus \beta)[Y] = (a[X \Delta Y]) \oplus \gamma \in L,$$

and therefore $a[X\Delta Y] \in L|_\gamma^v$. Since a is an Eulerian vector of $S|_\gamma^v$, $X\Delta Y = \emptyset$ and therefore $X = Y$. But $\langle (a \oplus \alpha)[X], (a \oplus \beta)[Y] \rangle = \langle \alpha, \beta \rangle = 1$, contrary to the fact that L is totally isotropic. □

Proposition 4.6 (Bouchet [5, (4.3)]). *Let a be an Eulerian vector of an isotropic system $S = (V, L)$. For every $v \in V$, there exists a unique vector $b_v \in L$ such that*

(i) *$b_v(v) \neq 0$,*

(ii) *$b_v(w) \in \{0, a(w)\}$ for $w \neq v$.*

Moreover, the set $\{b_v : v \in V\}$ is a basis of L. We call $\{b_v : v \in V\}$ the fundamental basis *of L with respect to a.*

Proof. Existence: Let δ_x^v denote a vector in K^V such that $\delta_x^v(w) = 0$ if $w \neq v$ and $\delta_x^v(v) = x$. Let W be a vector space spanned by $\{\delta_{a(w)}^w : w \in V\}$. It is clear that $\dim(W) = |V|$. Let $L + W = \{x + y : x \in L, y \in W\}$. Since a is Eulerian, $L \cap W = \emptyset$ and therefore

$$\dim(L + W) = \dim(L) + \dim(W) = 2|V|,$$

and so $K^V = L + W$. Let $z \in K \setminus \{0, a(v)\}$. We can express $\delta_z^v = x + y$ for some $x \in L$ and $y \in W$. For all $w \neq v$, $0 = \langle \delta_z^v, \delta_{a(w)}^w \rangle = \langle x, \delta_{a(w)}^v \rangle$ and therefore $x(w) \in \{0, a(w)\}$. Moreover $1 = \langle \delta_z^v, \delta_{a(v)}^v \rangle = \langle x, \delta_{a(v)}^v \rangle$ implies that $x(v) \neq 0$. We let $b_v = x$.

Uniqueness: Suppose b_v, b'_v satisfy two conditions. Then,

$$0 = \langle b_v, b'_v \rangle = \langle b_v(v), b'_v(v) \rangle.$$

So, $b_v(v) = b'_v(v)$ and therefore $b_v - b'_v = a[P]$ for some $P \subseteq V$. Since a is Eulerian, $P = \emptyset$ and $b_v = b'_v$.

Independence: Suppose it is dependent. There exists $\emptyset \neq I \subseteq V$ such that $\sum_{v \in I} b_v = 0$. Choose $w \in I$. $\sum_{v \in I} b_v(w) = b_w(w) + \sum_{v \in I, v \neq w} b_v(w) = b_w(w)$ or $b_w(w) + a(w)$. Both are non-zero, because $b_w(w) \neq a(v)$. A contradiction. □

It is straightforward to construct an isotropic system from every graph.

Proposition 4.7 (Bouchet [5, (3.1)]). *Let $G = (V, E)$ be a graph and a, b be a pair of supplementary vectors of K^V. Let $n_G(v)$ be the set of neighbors of v. Let L be the subspace of K^V spanned by $\{a[n_G(v)] + b[\{v\}] : v \in V\}$. Then $S = (V, L)$ is an isotropic system. We call (G, a, b) a* graphic presentation *of S.*

Proof. It is enough to show that L is totally isotropic and $\dim(L) = |V|$.

For distinct $v, w \in V$,

$$\langle a[n_G(v)] + b[\{v\}], a[n_G(w)] + b[\{w\}] \rangle = \langle a[n_G(v)], b[\{w\}] \rangle + \langle b[\{v\}], a[n_G(w)] \rangle = 0$$

because $a[n_G(v)](w) \neq 0$ if and only if $a[n_G(w)](v) \neq 0$. Therefore L is totally isotropic.

We claim that, for a subset W of V, if $s = \sum_{w \in W} (a[n_G(w)] + b[\{w\}]) = 0$, then $W = \emptyset$. Suppose $v \in W$. Then $s(v) \in \{b(v), b(v) + a(v)\}$. Since $b(v) \neq 0$ and $b(v) + a(v) \neq 0$, we conclude that $s \neq 0$.

So $\{a[n_G(v)] + b[\{v\}] : v \in V\}$ is independent and therefore $\dim(L) = |V|$. □

It is interesting that the reverse direction also holds. Suppose an isotropic system $S = (V, L)$ is given with an Eulerian vector a. Let $\{b_v : v \in V\}$ be the fundamental basis of $S = (V, L)$ with respect to a. Let $G = (V, E)$ be a graph such that $vw \in E$ if and only if $v \neq w$ and $b_v(w) \neq 0$. Since $\langle b_v, b_w \rangle = 0$, $b_v(w) \neq 0$ if and only if $b_w(v) \neq 0$, and therefore G is undirected. We call G a *fundamental graph* of S with respect to a. In fact, if S has a graphic presentation (G, a, b), then G is a fundamental graph of S with respect to a.

Bouchet [5, (7.6)] showed that if (G, a, b) is a graphic presentation of an isotropic system $S = (V, L)$ and $v \in V$, then

$$(G * v, a + b[\{v\}], a[n_G(v)] + b)$$

is also a graphic presentation of S. Thus, local complementations do not change the associated isotropic system. If G and H are locally equivalent, associated isotropic systems can be chosen to be same by an appropriate choice of supplementary vectors. He also showed that if $uv \in E(G)$, then

$$(G \wedge uv, a[V \setminus \{u, v\}] + b[\{u, v\}], b[V \setminus \{u, v\}] + a[\{u, v\}])$$

is a graphic presentation of S. This fact will be used in Section 6.6.

A minor of an isotropic system is closely related to a vertex-minor of its fundamental graph as follows.

Proposition 4.8 (Bouchet [5, (9.1)]). *Let G be a graph and $n_G(v)$ be the set of neighbors of v in G. If (G, a, b) is a graphic presentation of an isotropic system $S = (V, L)$, then one of the following is a graphic presentation of an elementary minor $S|^v_x$.*

(i) $(G \setminus v, p_{V\setminus\{v\}}(a), p_{V\setminus\{v\}}(b))$ *if either $x = a(v)$ or $x = b(v)$ and v is an isolated vertex of G,*

(ii) $\big(G \wedge vw \setminus v, p_{V\setminus\{v\}}(a[V \setminus \{v, w\}] + b[\{v, w\}]), p_{V\setminus\{v\}}(b[V \setminus \{v, w\}] + a[\{v, w\}])\big)$ *if $x = b(v)$ and there is a neighbor w of v in G,*

(iii) $(G * v \setminus v, p_{V\setminus\{v\}}(a), p_{V\setminus\{v\}}(b + a[n_G(v)]))$ *otherwise.*

Proof. We know that (G, a, b), $(G * v, a + b[\{v\}], a[n_G(v)] + b)$, and $(G \wedge vw, a[V \setminus \{v, w\}] + b[\{v, w\}], b[V \setminus \{v, w\}] + a[\{v, w\}])$ (if $vw \in E(G)$) are graphic presentations of S. Therefore it is enough to show (i).

If v is an isolated vertex of G and $x = b(v)$, then $b[\{v\}] \in L$. Since every vector $c \in L$ satisfies $\langle c, b[\{v\}]\rangle = 0$, $c(v) \in \{0, b(v)\}$ for all $c \in L$. Moreover if $c(v) = b(v)$ for $c \in L$, then $c - b[\{v\}] \in L$. Therefore $S|^v_{a(v)} = S|^v_{b(v)}$.

Now we may assume that $x = a(v)$. Let $a' = p_{V\setminus\{v\}}(a)$ and $b' = p_{V\setminus\{v\}}(b)$. For all $w \in V \setminus \{v\}$, since $a[n_G(w)] + b[\{w\}] \in L$, $(a[n_G(w)] + b[\{w\}])(v) \in \{0, x\}$, and $p_{V\setminus\{v\}}(a[n_G(w)] + b[\{w\}]) = a'[n_{G'}(w)] + b'[\{w\}]$, we have $a'[n_{G\setminus v}(w)] + b'[\{w\}] \in L|^v_x$. Therefore $(G \setminus v, a', b')$ is a graphic presentation of $S|^v_x$. □

Corollary 4.9. *If we have two isotropic systems S_1 and S_2 such that S_1 is a minor of S_2, then every fundamental graph of S_1 is a vertex-minor of each fundamental graph of S_2. Conversely, if G_1 is a vertex-minor of a fundamental graph of an isotropic system S_2, then there exists a minor of S_2 having G_1 as a fundamental graph.*

Note that the choice of w in Proposition 4.8 does not affect the isotropic system because of Proposition 3.14.

4.1.3 Connectivity

For a subspace L of K^V, let $\lambda(L) = |V| - \dim(L)$. We recall from Subsection 4.1.1 that for $X \subseteq V$, we define $L|_{\subseteq X} = \{p_X(a) : a \in L, a(v) = 0 \text{ for all } v \in V \setminus X\}$.

Definition 4.10 (Bouchet [7]). *For an isotropic system $S = (V, L)$, we call $c : V \to \mathbb{Z}$ a* connectivity *function if $c(X) = \lambda(L|_{\subseteq X}) = |X| - \dim(L|_{\subseteq X})$.*

If L is a totally isotropic subspace of K^V, then $L|_{\subseteq X}$ is also a totally isotropic subspace of K^X. Thus, $\dim(L|_{\subseteq X}) \leq |X|$, and therefore $c(X) \geq 0$.

Bouchet observed the following proposition stating that the connectivity function of an isotropic system is equal to the cut-rank function of its fundamental graph.

Proposition 4.11 (Bouchet [7, Theorem 6]). *Let a be an Eulerian vector of an isotropic system $S = (V, L)$ and let c be the connectivity function of S. Let G be the fundamental graph of S with respect to a. Then, $c(X) = \rho_G(X)$ for all $X \subseteq V$.*

Proof. Let M be the adjacency matrix of G over GF(2). Let $A = M[X, V \setminus X]$. We have

$$\mathrm{rk}(A) = |X| - \mathrm{nullity}(A),$$

where the *nullity* of A is the dimension of the *null space* $\{P \in 2^X : AP = 0\}$. (We consider 2^X as a vector space over GF(2).)

Let $\{b_v : v \in V\}$ be the fundamental basis of L with respect to a. Let $\varphi : 2^V \to L$ be a linear transformation with $\varphi(P) = \sum_{v \in P} b_v$. Then, φ is an isomorphism and therefore we have the following:

$$\begin{aligned}
\dim(L|_{\subseteq X}) &= \dim(\{x \in L : p_{V\setminus X}(x) = 0\}) \\
&= \dim(\varphi^{-1}(\{x \in L : p_{V\setminus X}(x) = 0\})) \\
&= \dim\left(\left\{P \subseteq V : \sum_{v \in P} p_{V\setminus X}(b_v) = 0\right\}\right) \\
&= \dim\left(\left\{P \subseteq X : \sum_{v \in P} (n_G(v) \setminus X) = \emptyset\right\}\right) \\
&= \dim(\{P \in 2^X : AP = 0\}) \\
&= \mathrm{nullity}(A).
\end{aligned}$$

Therefore, $c(X) = |X| - \dim(L|_{\subseteq X}) = |X| - \mathrm{nullity}(A) = \mathrm{rk}(A) = \rho_G(X)$. □

By this property, we notice that $c(X) = c(V \setminus X)$ and $c(X) + c(Y) \geq c(X \cap Y) + c(X \cup Y)$. Since c is symmetric submodular, it is straightforward to define *branch-decomposition* and *branch-width* of an isotropic system $S = (V, L)$. We call $(T, \mathcal{L})$ a *branch-decomposition* of S if it is a branch-decomposition of c. The *branch-width* $\mathrm{bw}(S)$ of S is the branch-width of c. It is easy to see that branch-width of an isotropic system is equal to rank-width of its fundamental graph by Proposition 4.11.

4.2 Monadic second-order logic formulas

In this section, we review basics of *monadic second-order logic formulas* (MS logic formulas), transformations of *relational structures* expressed in this language, and its extensions. We will also discuss its relation to clique-width. For the main definitions and results on MS logic formulas and some examples of formulas, the reader is referred to the book chapter [15] written by Courcelle. Since we are interested only in the application to rank-width, we will not review in full detail, and therefore definitions will be simplified.

4.2.1 Relational structures

Let D be a finite set. A function $A : D^m \to \{\mathsf{true}, \mathsf{false}\}$ is called a *relation symbol* on D with *arity* m. Similarly a function $A : (2^D)^m \to \{\mathsf{true}, \mathsf{false}\}$ is called a *set predicate* on D with arity m.

A pair $S = \langle D, \{A_1, A_2, \ldots, A_k\}\rangle$ is called a *relational structure* if

(i) D is a finite set,

(ii) A_i is either a set predicate on D or a relation symbol on D for each i.

We would write $S = \langle D, A_1, A_2, \ldots, A_k\rangle$ if it is not ambiguous.

In general, we are interested in logic formulas described on relational structures so that we can express properties of our objects. We give two examples in which we construct relational structures from objects so that we preserve all information about objects.

Example 4.12 (Graphs; Courcelle [14, Definition 1.7]). *Let $G = (V, E)$ be a graph. Let* edg *be a relation symbol on V with arity two such that* $\mathsf{edg}(v_1, v_2)$ *is true if and only if v_1 and v_2 are adjacent in G. Then, G is represented by a relational structure* $\langle V, \mathsf{edg}\rangle$.

Example 4.13 (Matroids; Hliněný [33, 34]). *Let $\mathcal{M} = (E, I)$ be a matroid. Let* Indep *be a set predicate on E with arity one such that* $\mathsf{Indep}(F)$ *is true if and only if F is independent in $\mathcal{M}$. Then, $\mathcal{M}$ is represented by a relational structure* $\langle E, \mathsf{Indep}\rangle$.

As you can see, there could be many ways to describe an object in terms of relational structures. For instance, we could introduce $\mathsf{Base}(F)$ to test whether F is a base of $\mathcal{M}$ for matroids so that we express $\mathcal{M}$ by a relational structure $\langle E, \mathsf{Base}\rangle$. Graphs also have many ways to be described as relational structures. In the next example, we describe another way of expressing graphs.

Example 4.14 (Graphs; Courcelle [14, Definition 1.7]). *Let $G = (V, E)$ be a graph. Let* inc *be a relation symbol on $V \cup E$ with arity three such that* $\mathsf{inc}(x, y, z)$ *is true if and only if x and z are the ends of y. Then, G is represented by a relational structure* $\langle V \cup E, \mathsf{inc}\rangle$.

To distinguish different relational structures on the same object, we sometimes write that a relational structure $\langle D, \{A_1, A_2, \ldots, A_k\}\rangle$ is a $\{A_1, A_2, \ldots, A_k\}$*-structure*. For instance, in Example 4.12, we describe graphs by $\{\mathsf{edg}\}$-structures, but in Example 4.14, graphs were described by $\{\mathsf{inc}\}$-structures; however, both keep all information on graphs.

We will discuss relational structures expressing isotropic systems in Section 4.4.

4.2.2 Monadic second-order logic formulas

Let $\langle D, \{A_1, A_2, \ldots, A_k\}\rangle$ be a relational structure. A variable is called a *first-order variable* if it denotes an element of D, and is called a *set variable* if it denotes a subset of D. *Monadic second-order logic formulas* (*MS logic formulas*) on this relational structure are logic formulas written by using $\exists$, $\forall$, $\wedge$, $\neg$, $\vee$, $\in$, true, and A_i with first-order variables and set variables. More formally, we may recursively define monadic second-order logic formulas on the relational structure $\langle D, \{A_1, A_2, \ldots, A_k\}\rangle$ as follows.

(i) true is an MS logic formula.

(ii) If x and y are first-order variables, then $x = y$ is an MS logic formula.

(iii) If x is a first-order variable and Y is a set variable, then $x \in Y$ is an MS logic formula.

(iv) If A_i is a relation symbol with arity m, then $A_i(x_1, x_2, \ldots, x_m)$ is an MS logic formula with m first-order variables $x_1, x_2, \ldots, x_m$.

(v) If A_i is a set predicate with arity m, then $A_i(X_1, X_2, \ldots, X_m)$ is an MS logic formula with m set variables $X_1, X_2, \ldots, X_m$.

(vi) If φ is an MS logic formula, then so is $\neg\varphi$.

(vii) If φ_1 and φ_2 are MS logic formulas, then so are $(\varphi_1 \wedge \varphi_2)$ and $(\varphi_1 \vee \varphi_2)$.

(viii) If x is a first-order variable and φ is an MS logic formula with no $\exists x$ and no $\forall x$, then $\exists x\, \varphi$ and $\forall x\, \varphi$ are MS logic formulas.

(ix) If X is a set variable and φ is an MS logic formula with no $\exists X$ and no $\forall X$, then $\exists X\, \varphi$, $\forall X\, \varphi$ are MS logic formulas.

We call a variable x a *free* variable of an MS logic formula φ if φ does not have $\exists x$ or $\forall x$ in its expression but it uses x. If an MS logic formula φ has no free variable, then we call φ a *closed* MS logic formula. By convention, uppercase alphabets denote set variables and lowercase alphabets denote first-order variables.

Example 4.15. *Let $\langle E, \mathsf{Indep}\rangle$ be a relational structure representing a matroid $\mathcal{M}$ as in Example 4.13. For a subset X of E, we can write an MS logic formula $\varphi(X)$ on this relational structure describing whether X is a base of $\mathcal{M}$. To make it short, we write $A \subseteq B$ for $\forall z((\neg z \in A) \vee (z \in B))$. Then,*

$$\varphi(X) = \mathsf{Indep}(X) \wedge \forall Y(\neg(\mathsf{Indep}(Y) \wedge X \subseteq Y) \vee Y \subseteq X).$$

In this formula, X is a free variable and Y is not. Since $\varphi(X)$ has a free variable, $\varphi(X)$ is not closed.

We now extend MS logic formulas. We define a set predicate Even such that $\mathsf{Even}(X)$ is true if and only $|X|$ is even. By allowing $\mathsf{Even}(X)$ to the definition of MS logic formulas, we obtain a definition of *modulo-2 counting monadic second-order logic formulas* (*C_2MS logic formulas*). Similarly for $p > 1$, let $\mathsf{Card}_p(X)$ be a set predicate meaning $|X| \equiv 0 \pmod p$. If we allow $\mathsf{Card}_p(X)$ in the definition of MS logic formulas, we obtain a definition of *counting monadic second-order logic formulas* (*CMS logic formulas*).

4.2.3 MS theory and MS satisfiability problem for graphs

Let $\mathcal{C}$ be a set of graphs. We may consider $\mathcal{C}$ as a set of $\{\mathsf{edg}\}$-structures (see Example 4.12). A *MS satisfiability problem* for $\mathcal{C}$ is the following decision problem:

Given a closed MS logic formula φ, is there a graph in $\mathcal{C}$ satisfying φ?

This problem is called *decidable* if there is an algorithm that answers the problem for all MS logic formulas. We may reformulate decidability of the above problem as decidability of the following problem:

Given a closed MS logic formula φ, do all graphs in $\mathcal{C}$ satisfy φ?

If this problem is decidable, then we say that $\mathcal{C}$ has a *decidable monadic second-order theory* (*decidable MS theory*). If φ is a closed MS logic formula, then so is $\neg\varphi$, and therefore $\mathcal{C}$ has a decidable MS theory if and only if it has a decidable satisfiability problem. Similarly we define *C_2MS satisfiability problem*, *CMS satisfiability problem*, *decidable C_2MS theory*, and *decidable CMS theory* by using appropriate logic formulas in definitions.

The above definitions use closed MS logic formulas on $\{\mathsf{edg}\}$-structures of graphs. If we use $\{\mathsf{inc}\}$-structures of graphs instead (see Example 4.14), then we obtain the definition of *MS_2 satisfiability problem* and *decidable MS_2 theory* of graphs.

In Chapter 5, we will discuss the following conjecture by D. Seese [52]: if a set of graphs has a decidable MS satisfiability problem, then it has bounded rank-width.

4.2.4 Transductions of relational structures

We now introduce *MS transductions*, transformations of relational structures that can be formalized in MS logic (or its extensions). We will only need its restricted form. For more about MS transductions, we refer the reader to surveys by B. Courcelle [13, 15].

Let $R = \{A_1, A_2, \ldots, A_k\}$ and $Q = \{B_1, B_2, \ldots, B_l\}$ be two finite sets of relation symbols or set predicates. A function

$$\tau : \{\text{all } R\text{-structures}\} \to 2^{\{\text{all } Q\text{-structures}\}}$$

with *parameters* $Y_1, Y_2, \ldots, Y_j$ is called a *monadic second-order transduction* (*MS transduction*) if there is a triple $\Delta = (\varphi, \psi, \{\theta_{B_1}, \theta_{B_2}, \cdots, \theta_{B_l}\})$ of MS logic formulas on R-structures such that the following two conditions are equivalent for every R-structure $S = \langle D_S, R\rangle$:

(1) A Q-structure $T = \langle D_T, Q\rangle$ is in $\tau(S)$.

(2) There exist $Y_1, Y_2, \ldots, Y_j \subseteq D_S$ satisfying the following four conditions. (If $Y_1, \ldots, Y_j$ satisfy these four conditions, we write $T = \mathrm{def}_\Delta(S, (Y_1, Y_2, \ldots, Y_j))$.)

- $\varphi(Y_1, Y_2, \ldots, Y_j)$ is true on S,
- $D_T = \{x \in D_S : \psi(Y_1, Y_2, \ldots, Y_j, x) \text{ is true on S}\}$,

- if B_i is a relation symbol with arity m, then θ_{B_i} is an MS logic formula on R-structures with arity $m+j$ such that

$$B_i(x_1, x_2, \ldots, x_m) = \theta_{B_i}(Y_1, Y_2, \ldots, Y_j, x_1, x_2, \ldots, x_m),$$

- if B_i is a set predicate with arity m, then θ_{B_i} is an MS logic formula on R-structures with arity $m+j$ such that

$$B_i(X_1, X_2, \ldots, X_m) = \theta_{B_i}(Y_1, Y_2, \ldots, Y_j, X_1, X_2, \ldots, X_m).$$

The triple $\Delta = (\varphi, \psi, \{\theta_{B_1}, \theta_{B_2}, \cdots, \theta_{B_l}\})$ is called a *definition scheme* of an MS transduction τ. If a definition scheme Δ defines an MS transduction τ, then we write $\tau = \mathrm{def}_\Delta$.

If we allow logic formulas in definition scheme to be C_2MS logic formulas or CMS logic formulas, then we obtain definitions of *C_2MS transductions*, *C_2MS definition schemes*, or *CMS transductions*, *CMS definition schemes* respectively. We note that every MS transduction is a C_2MS transduction and every C_2MS transduction is a CMS transduction.

Example 4.16 (Induced subgraph). *Let $G = (V, E)$ be a graph and Y be a subset of V. We write $G[Y]$ be a subgraph of G induced by V, which is a graph obtained by deleting vertices in $V \setminus Y$. In this example, we would like to show an MS transduction τ that maps a graph into the set of its induced subgraphs. We assume that G is given by its $\{\mathsf{edg}\}$-structure. We have one parameter Y to define induced subgraphs. We first show its definition scheme $\Delta = (\varphi, \psi, \theta_{\mathsf{edg}})$.*

(i) $\varphi(Y) = \mathsf{true}$, *(Every Y would induce an induced subgraph.)*

(ii) $\psi(Y, x) = (x \in Y)$, *(The set of vertices of $G[Y]$ is Y.)*

(iii) $\theta_{\mathsf{edg}}(Y, x, y) = \mathsf{edg}(x, y)$. *(Edges are preserved if $x, y \in Y$.)*

Let $\tau : \{\text{all } R\text{-structures}\} \to 2^{\{\text{all } Q\text{-structures}\}}$ be an MS transduction with parameters $Y_1, Y_2, \ldots, Y_j$. Let S be a R-structure and β be an MS logic formula on Q-structures with free variables $x_1, x_2, \ldots, x_k, X_1, X_2, \ldots, X_l$. Suppose that we want to evaluate β on a Q-structure $T \in \tau(S)$. Since the definition scheme of τ describes all set predicates and relational symbols of Q-structures in terms of MS logic formulas in R-structures, we obtain the following proposition.

Proposition 4.17 (Courcelle [13, 15]). *Let*

$$\tau : \{\textit{all } R\textit{-structures}\} \to 2^{\{\textit{all } Q\textit{-structures}\}}$$

be an MS transduction with parameters $Y_1, Y_2, \ldots, Y_j$, given by a definition scheme $\Delta = (\varphi, \psi, (\theta_B)_{B \in Q})$. Let S be a R-structure and β be an MS logic formula on Q-structures with free variables $x_1, x_2, \ldots, x_k, X_1, X_2, \ldots, X_l$.

Then there is an MS logic formula $\beta^\#$ on R-structures such that S satisfies $\beta^\#(Y_1, Y_2, \ldots, Y_j, x_1, x_2, \ldots, x_k, X_1, X_2, \ldots, X_l)$ if and only if

- $\varphi(Y_1, Y_2, \ldots, Y_j)$ *is true on* S, *(so that* $\mathrm{def}_\Delta(S, (Y_1, \ldots, Y_j))$ *is well-defined)*
- $\beta(x_1, x_2, \ldots, x_k, X_1, X_2, \ldots, X_l)$ *is true on* $T = \langle D_T, Q\rangle = \mathrm{def}_\Delta(S, (Y_1, \ldots, Y_j))$,
- $x_i \in D_T$ *for all* i *(or* $\psi(x_i)$ *is true on* S*), and*
- $X_i \subseteq D_T$ *for all* i.

We call $\beta^{\#}$ the *backwards translation* of β relative to the MS transduction τ. Similarly C_2MS transductions will induce a C_2MS logic formula $\beta^{\#}$ on $T \in \tau(S)$ for a C_2MS logic formula β on S.

We describe two terminologies. For an MS transduction $\tau : \{\text{all } R\text{-structures}\} \to 2^{\{\text{all } Q\text{-structures}\}}$ and a set $\mathcal{C}$ of R-structures, the set $\cup_{S\in\mathcal{C}}\tau(S)$ is called the *image* of $\mathcal{C}$ under τ. For two MS (or, C_2MS) transductions $\tau_1 : \{\text{all } R\text{-structures}\} \to 2^{\{\text{all } Q\text{-structures}\}}$ and $\tau_2 : \{\text{all } Q\text{-structures}\} \to 2^{\{\text{all } P\text{-structures}\}}$, we define the *composition* of τ_1 and τ_2 as a function $\tau_2 \circ \tau_1 : \{\text{all } R\text{-structures}\} \to 2^{\{\text{all } P\text{-structures}\}}$ such that $(\tau_2 \circ \tau_1)(S) = \cup_{T\in\tau_1(S)}\tau_2(T)$.

Proposition 4.18 (Courcelle [13, 15]).

(1) *If a set of relational structures has a decidable MS satisfiability problem (respectively,* C_2*MS satisfiability problem), then so does its image under an MS transduction (respectively, under a* C_2*MS transduction).*

(2) *The composition of two MS transductions (respectively, of two* C_2*MS transductions) is an MS transduction (respectively, a* C_2*MS transduction).*

Proof. We only prove (1). Let $\mathcal{C}$ be a set of relational structures having a decidable MS satisfiability problem, and τ be an MS transduction with parameters $Y_1, \ldots, Y_p$. For a given closed MS formula β, we want to know whether there exist $S \in \mathcal{C}$ and $T \in \tau(S)$ such that β is true on T. Since β has no free variables, it is equivalent to ask whether there exists $S \in \mathcal{C}$ such that $\exists Y_1 \exists Y_2 \cdots \exists Y_p\, \beta^{\#}(Y_1, Y_2, \ldots, Y_p)$ is true on S. Since $\mathcal{C}$ has a decidable MS satisfiability problem, there is an algorithm that answers this problem. □

4.3 Evaluation of CMS formulas

In this section, we review why and how CMS formulas can be evaluated in linear time on a set of graphs of bounded clique-width if graphs are given by their k-expressions.

The *quantifier height* $\mathrm{qh}(\varphi)$ of a CMS formula φ is defined recursively as follows.

(i) $\mathrm{qh}(\varphi) = 0$ if φ is atomic, which means that φ is of the form $x = y$ or $x \in X$ or $\mathsf{Card}_p(X)$ or $A(u_1, \cdots, u_n)$ or $A(U_1, \cdots, U_n)$.

(ii) $\mathrm{qh}(\neg\varphi) = \mathrm{qh}(\varphi)$.

(iii) $\mathrm{qh}(\varphi_1 \wedge \varphi_2) = \mathrm{qh}(\varphi_1 \vee \varphi_2) = \max\{\mathrm{qh}(\varphi_1), \mathrm{qh}(\varphi_2)\}$.

(iv) $\mathrm{qh}(\exists u\,\varphi) = \mathrm{qh}(\forall u\,\varphi) = \mathrm{qh}(\exists U\,\varphi) = \mathrm{qh}(\forall U\,\varphi) = 1 + \mathrm{qh}(\varphi)$.

Let $\mathrm{C}_p\mathrm{MS}^h(R,\emptyset)$ be the set of all closed CMS formulas on R-structures having quantifier height at most h with no Card_q for all q larger than p. Clearly this set is infinite because if it contains a formula φ, then it contains also all formulas of the form $\varphi \vee \varphi \vee \cdots \vee \varphi$. However all these formulas are equivalent. In [21, Proposition A.8], it is explained that there is an algorithm to transform every formula φ in $\mathrm{C}_p\mathrm{MS}^h(R,\emptyset)$ to its *canonical formula* $\mathrm{Can}(\varphi)$ in $\mathrm{C}_p\mathrm{MS}^h(R,\emptyset)$ such that φ and $\mathrm{Can}(\varphi)$ have the same truth value for every R-structure and moreover the set of canonical formulas, $\mathrm{Can}(\mathrm{C}_p\mathrm{MS}^h(R,\emptyset))$, is finite. However, the cardinality of $\mathrm{Can}(\mathrm{C}_p\mathrm{MS}^h(R,\emptyset))$ is a tower of exponentials of height proportional to h.

For every p, R, h as above and every R-structure S, we let

$$\mathrm{Th}_{p,R,h}(S) = \{\varphi \in \mathrm{Can}(\mathrm{C}_p\mathrm{MS}^h(R,\emptyset)) : S \text{ satisfies } \varphi\}.$$

We call it the (p,R,h)-*theory* of S. There are thus finitely many (p,R,h)-theories because it is a subset of a finite set, and each of them is a finite set of formulas.

A k-graph $G = (V_G, E_G, \mathrm{lab}_G)$ may be represented by the relational structure

$$\langle V_G, \mathsf{edg}_G, \mathsf{p}_{1G}, ..., \mathsf{p}_{kG}\rangle,$$

(also denoted by G) such that edg_G is the edge relation and $\mathsf{p}_{iG}(x)$ holds if and only if $\mathrm{lab}(x) = i$. The following proposition summarizes well-known results.

Proposition 4.19 (Courcelle [15, Theorem 5.7.5]). *Let k be a fixed positive integer.*

(1) *Let $R = \{\mathsf{edg}, \mathsf{p}_1, ..., \mathsf{p}_k\}$ with edg of arity two and p_i of arity one. For all positive integers p, h, i, j (where $i, j \in \{1, 2, \ldots, k\}$ and $i \neq j$), there exist mappings $f_{k,\oplus}$, $f_{k,\eta_{i,j}}$, $f_{k,\rho_{i\to j}}$ on subsets of $\mathrm{Can}(\mathrm{C}_p\mathrm{MS}^h(R,\emptyset))$ such that for all k-graphs G and H,*

$$\begin{aligned}
\mathrm{Th}_{p,R,h}(\eta_{i,j}(G)) &= f_{k,\eta_{i,j}}(\mathrm{Th}_{p,R,h}(G)),\\
\mathrm{Th}_{p,R,h}(\rho_{i\to j}(G)) &= f_{k,\rho_{i\to j}}(\mathrm{Th}_{p,R,h}(G)),\\
\mathrm{Th}_{p,R,h}(G \oplus H) &= f_{k,\oplus}(\mathrm{Th}_{p,R,h}(G), \mathrm{Th}_{p,R,h}(H)).
\end{aligned}$$

(2) *If a graph G is given as $\mathrm{val}(t)$ for a k-expression t, then $\mathrm{Th}_{p,R,h}(G)$ can be computed in time proportional to the size of t.*

(3) *For every closed CMS logic formula on $\{\mathsf{edg}\}$-structures, there is a $O(n)$-time algorithm that evaluates this formula on the n-vertex input graph of clique-width at most k, if the input graph is given by its k-expression.*

Proof. (1) Let us observe that the mapping $\eta_{i,j}$ is a *quantifier-free transduction*, which means that its definition scheme consists of MS logic formulas without quantifiers and without parameters. From the proof of Proposition 4.17, it follows that the backwards translation (denoted by $^{\#}$) associated with $\eta_{i,j}$ does not increase quantifier height and does not introduce new Card_p set predicates. Hence for every formula φ in $\mathrm{C}_p\mathrm{MS}^h(R,\emptyset)$, we have $\eta_{i,j}(G)$ satisfies φ if and only if G satisfies $\varphi^{\#}$. It is also equivalent to a statement that G satisfies $\mathrm{Can}(\varphi^{\#})$. Note that $\varphi^{\#}$ belongs to $\mathrm{C}_p\mathrm{MS}^h(R,\emptyset)$.

Hence, we can take, for every subset Φ of $\mathrm{Can}(\mathrm{C}_p\mathrm{MS}^h(R,\emptyset))$,

$$f_{k,\eta_{i,j}}(\Phi) = \{\varphi \in \mathrm{Can}(\mathrm{C}_p\mathrm{MS}^h(R,\emptyset)) : \mathrm{Can}(\varphi^{\#}) \in \Phi\}.$$

The proof is similar for $\rho_{i\to j}$.

The case of $\oplus$ is a particular case of a result by Feferman, Vaught and Shelah. The proof is in [12, Lemma (4.5)]. There is a nice survey by Makowsky [40] dealing with the history and the numerous consequences of this result.

(2) Consider a graph $G = \mathrm{val}(t)$ where t is a k-expression.

Each set $\mathrm{Th}_{p,R,h}(\mathrm{val}(\cdot_i))$ can be computed from the definitions. Then, using (1) one can compute $\mathrm{Th}_{p,R,h}(\mathrm{val}(t))$ by induction on the structure of t.

(3) To know whether G satisfies φ, we compute the set $\mathrm{Th}_{p,R,h}(\mathrm{val}(t))$ by (2) where p and h are the smallest integers such that $\varphi \in \mathrm{C}_p\mathrm{MS}^h(R,\emptyset)$. Then one determines whether $\mathrm{Can}(\varphi)$ belongs to $\mathrm{Th}_{p,R,h}(\mathrm{val}(t))$, which gives the answer. □

This method applies to optimization and enumeration (counting) problems formalized in monadic second-order logic. We refer the reader to [40].

4.4 Vertex-minors through isotropic systems

We describe relational structures for expressing isotropic systems. Let $S = (V, L)$ be an isotropic system. Let $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ be vectors in K^V such that $\bar{\alpha}(v) = \alpha$, $\bar{\beta}(v) = \beta$, and $\bar{\gamma}(v) = \gamma$ for all $v \in V$. A triple (X, Y, Z) of pairwise disjoint subsets of V is called a *set representation* of $a \in K^V$ if $a = \bar{\alpha}[X] + \bar{\beta}[Y] + \bar{\gamma}[Z]$.

Let Member be a set predicate on V with arity three such that $\mathsf{Member}(X, Y, Z)$ is true if and only if (X, Y, Z) is a set representation of a vector in L. Then, the isotropic system S is represented by a relational structure $\langle V, \mathsf{Member}\rangle$.

We will show that there is a $\mathrm{C}_2\mathrm{MS}$ transduction that maps a graph to the set of its all vertex-minors by using isotropic systems. This will imply that for a fixed graph H, there is a $\mathrm{C}_2\mathrm{MS}$ logic formula that describes whether H is isomorphic to a vertex-minor of G.

4.4.1 Fundamental graphs by $\mathrm{C}_2\mathrm{MS}$ logic formulas

We briefly recall Subsection 4.1.2. We know that a graph $G = (V, E)$ with two supplementary vectors $a, b \in K^V$ determines the isotropic system $S = (V, L)$ such that L is a subspace of K^V spanned by $\{a[n_G(v)] + b[\{v\}] : v \in V\}$. We call (G, a, b) a graphic presentation of the isotropic system S and at the same time G is called a fundamental graph of S. Conversely, an isotropic system $S = (V, L)$ with its Eulerian vector $a \in K^V$ determines the fundamental graph G of S.

In this section, we have two main objectives. First, we show that there is a C_2MS transduction that maps a graph G to the set of all isotropic systems having G as a fundamental graph. Second, we show that there is an MS transduction that maps an isotropic system S to the set of all fundamental graphs of S.

Proposition 4.20. *There is a C_2MS transduction*

$$\tau_g : \{\textit{all } \{\mathsf{edg}\}\textit{-structures}\} \to 2^{\{\textit{all } \{\mathsf{Member}\}\textit{-structures}\}}$$

with six parameters $X_a, Y_a, Z_a, X_b, Y_b, Z_b$ such that for a graph G, $\tau_g(G)$ is the set of all isotropic systems having G as a fundamental graph.

Proof. It is enough to show that given a $\{\mathsf{edg}\}$-structure of a graph $G = (V, E)$ with arbitrary two supplementary vectors a and b in K^V, we can describe the $\{\mathsf{Member}\}$-structure of the isotropic system having (G, a, b) as a graphic presentation. In other words, we need to show a C_2MS definition scheme for this C_2MS transduction.

Let $X_a, Y_a, Z_a, X_b, Y_b, Z_b$ be six parameters of the C_2MS transduction. We have a C_2MS logic formula answering whether (X_a, Y_a, Z_a), (X_b, Y_b, Z_b) are set representations of supplementary vectors a and b respectively as follows:

$$\begin{aligned}\varphi = (X_a \cap Y_a = \emptyset) \wedge (Y_a \cap Z_a = \emptyset) \wedge (Z_a \cap X_a = \emptyset) \\ \wedge (X_b \cap Y_b = \emptyset) \wedge (Y_b \cap Z_b = \emptyset) \wedge (Z_b \cap X_b = \emptyset) \\ \wedge (\forall x, x \in X_a \vee x \in Y_a \vee x \in Z_a) \wedge (\forall x, x \in X_b \vee x \in Y_b \vee x \in Z_b) \\ \wedge (X_a \cap X_b = \emptyset) \wedge (Y_a \cap Y_b = \emptyset) \wedge (Z_a \cap Z_b = \emptyset).\end{aligned}$$

Note that we write $X \cap Y = \emptyset$ instead of $\forall x, \neg(x \in X \wedge x \in Y)$ to simplify the formula.

Now we want to express $\mathsf{Member}(X, Y, Z)$ in terms of edg of G by using (X_a, Y_a, Z_a) and (X_b, Y_b, Z_b). By definition, $\mathsf{Member}(X, Y, Z)$ is true if and only if X, Y, Z are pairwise disjoint subsets and $w = \bar{\alpha}[X] + \bar{\beta}[Y] + \bar{\gamma}[Z] \in L$. To have $w \in L$, there should exist a linear combination of vectors in the basis $\{a[n_G(v)] + b[\{v\}] : v \in V\}$, and so there should exist $U \subseteq V$ such that $\sum_{v \in U} a[n_G(v)] + \sum_{v \in U} b[\{v\}] = w$. Since K^V is a vector space over GF(2), we do not need a scalar product.

Suppose we have a C_2MS logic formula $\mu_1(U, X_a, Y_a, Z_a, X_c, Y_c, Z_c)$ on $\{edg\}$-structures expressing that (X_c, Y_c, Z_c) is a set representation of $\sum_{v \in U} a[n_G(v)]$ and we also have a C_2MS logic formula $\mu_2(U, X_b, Y_b, Z_b, X_d, Y_d, Z_d)$ on $\{edg\}$-structures expressing that (X_d, Y_d, Z_d) is a set representation of $\sum_{v \in U} b[\{v\}]$. We claim that we have a C_2MS logic formula $\mu(U, X_a, Y_a, Z_a, X_b, Y_b, Z_b, X, Y, Z)$ expressing that (X, Y, Z) is a set representation of $\sum_{v \in U} a[n_G(v)] + \sum_{v \in U} b[\{v\}]$. Simply we can encode addition of elements in K into C_2MS logic formulas. Let

$$\begin{aligned}&\sigma(X, X_c, Y_c, Z_c, X_d, Y_d, Z_d) \\ &= \forall x,\, x \in X \Leftrightarrow \big((x \in Z_c \wedge x \in Y_d) \vee (x \in Y_c \wedge x \in Z_d) \\ &\qquad \vee (\neg(x \in X_c \cup Y_c \cup Z_c) \wedge x \in X_d) \vee (\neg(x \in X_d \cup Y_d \cup Z_d) \wedge x \in X_c)\big)\end{aligned}$$

be a C_2MS logic formula expressing that if w is a sum of two vectors of set representations (X_c, Y_c, Z_c) and (X_d, Y_d, Z_d), then $X = \{v : w(v) = \alpha\}$. By symmetry of the addition table of K, we can write $\mu(U, X_a, Y_a, Z_a, X_b, Y_b, Z_b, X, Y, Z)$ as follows:

$$\exists X_c \exists Y_c \exists Z_c \exists X_d \exists Y_d \exists Z_d\, \mu_1(U, X_a, Y_a, Z_a, X_c, Y_c, Z_c) \wedge \mu_2(U, X_b, Y_b, Z_b, X_d, Y_d, Z_d) \wedge$$
$$\sigma(X, X_c, Y_c, Z_c, X_d, Y_d, Z_d) \wedge \sigma(Y, Y_c, Z_c, X_c, Y_d, Z_d, X_d) \wedge \sigma(Z, Z_c, X_c, Y_c, Z_d, X_d, Y_d).$$

So we can express $\mathsf{Member}(X, Y, Z)$ as $\theta = \exists U\, \mu(U, X_a, Y_a, Z_a, X_b, Y_b, Z_b, X, Y, Z)$.

number of α's	number of β's	number of γ's	sum in K
even	even	even	0
odd	odd	odd	0
odd	even	even	α
even	odd	odd	α
even	odd	even	β
odd	even	odd	β
even	even	odd	γ
odd	odd	even	γ

Table 4.1: Addition table of K

Now it is enough to show μ_1 and μ_2. Let $\nu(U_\alpha, x, X_a, U) = \forall v(v \in U_\alpha \Leftrightarrow x \in X_a \wedge \mathsf{edg}(x, v) \wedge v \in U)$ expressing that for fixed x, $U_\alpha = \{v \in U : (a[n(\{v\})])(x) = \alpha\}$. Let $\Sigma(A, B, C) = \neg\,\mathsf{Even}(A) \wedge \mathsf{Even}(B) \wedge \mathsf{Even}(C)) \vee (\mathsf{Even}(A) \wedge \neg\,\mathsf{Even}(B) \wedge \neg\,\mathsf{Even}(C))$ expressing that $|A|\alpha + |B|\beta + |C|\gamma = \alpha$ by using Table 4.1. Now we express $\mu_1(U, X_a, Y_a, Z_a, X_c, Y_c, Z_c)$ as follows:

$$\exists U_\alpha \exists U_\beta \exists U_\gamma \forall x\, \nu(U_\alpha, x, X_a, U) \wedge \nu(U_\beta, x, Y_a, U) \wedge \nu(U_\gamma, x, Z_a, U)$$
$$\wedge (x \in X_c \Leftrightarrow \Sigma(U_\alpha, U_\beta, U_\gamma)) \wedge (x \in Y_c \Leftrightarrow \Sigma(U_\beta, U_\gamma, U_\alpha)) \wedge (x \in Z_c \Leftrightarrow \Sigma(U_\gamma, U_\alpha, U_\beta)).$$

Similarly we can express $\mu_2(U, X_b, Y_b, Z_b, X_d, Y_d, Z_d)$ as follows:

$$\exists V_\alpha \exists V_\beta \exists V_\gamma \forall x\, (V_\alpha = U \cap X_b) \wedge (V_\beta = U \cap Y_b) \wedge (V_\gamma = U \cap Y_d) \wedge$$
$$(x \in X_d \Leftrightarrow \Sigma(V_\alpha, V_\beta, V_\gamma)) \wedge (x \in Y_d \Leftrightarrow \Sigma(V_\beta, V_\gamma, V_\alpha)) \wedge (x \in Z_d \Leftrightarrow \Sigma(V_\gamma, V_\alpha, V_\beta)).$$

Thus we obtain a C_2MS definition scheme $(\varphi, \mathsf{true}, \theta)$ that defines a C_2MS transduction τ_g mapping a graph G into all isotropic systems having G as a fundamental graph. □

We now consider the reverse direction.

Proposition 4.21. *There is an MS transduction*

$$\tau_s : \{\textit{all } \{\mathsf{Member}\}\textit{-structures}\} \to 2^{\{\textit{all } \{\mathsf{edg}\}\textit{-structures}\}}$$

with three parameters (X_e, Y_e, Z_e) such that for an isotropic system S, $\tau_s(S)$ is the set of all fundamental graphs of S.

Proof. We would like to show that given a $\{\mathsf{Member}\}$-structure of an isotropic system $S = (V, L)$ with a set representation (X_e, Y_e, Z_e) of an Eulerian vector a of S, we can describe the $\{edg\}$-structure of the fundamental graph of S with respect to a.

We have an MS logic formula expressing that (X_e, Y_e, Z_e) is a set representation of an Eulerian vector of S (Definition 4.4) as follows:

$$\varphi = (X_e \cap Y_e = Y_e \cap Z_e = Z_e \cap X_e = \emptyset) \wedge (\forall x,\ x \in X_e \vee x \in Y_e \vee x \in Z_e) \wedge$$
$$\forall X \forall Y \forall Z \big((X \subseteq X_e \wedge Y \subseteq Y_e \wedge Z \subseteq Z_e \wedge \mathsf{Member}_S(X, Y, Z)) \Rightarrow X = Y = Z = \emptyset\big).$$

By Proposition 4.7, for every v in V, there exists a unique vector b_v in L such that

$$b_v(v) \neq 0 \text{ for all } v \quad \text{and} \quad b_v(w) \in \{0, a(w)\} \text{ for } v \neq w.$$

These vectors satisfy the following properties: $a(v) \neq b_v(v) \neq 0$ for all v, and $b_v(w) \neq 0$ if and only if $b_w(v) \neq 0$ for $v \neq w$. The graph $G = (V, E)$ is called a fundamental graph with respect to a if $E = \{vw : b_v(w) \neq 0\}$. We may obtain different graphs using other Eulerian vectors, but they are locally equivalent.

We can easily translate this into MS logic formulas. We let $\nu_1(X, Y, Z, X_e, Y_e, Z_e, v)$ be the formula:

$$\mathsf{Member}(X, Y, Z) \wedge v \in X \cup Y \cup Z$$
$$\wedge \forall w[w \neq v \Rightarrow \{(w \in X \Rightarrow w \in X_e) \wedge (w \in Y \Rightarrow w \in Y_e) \wedge (w \in Z \Rightarrow w \in Z_e)\}],$$

expressing that (X, Y, Z) is a set representation of b_v. Now we can write an MS logic formula describing edg of the fundamental graph with respect to a in terms of Member as $\theta(v, w) = (v \neq w) \wedge \exists X \exists Y \exists Z[\nu_1(X, Y, Z, X_e, Y_e, Z_e, v) \wedge w \in X \cup Y \cup Z]$.

Hence we have constructed a definition scheme $(\varphi, \mathsf{true}, \theta)$ for the MS transduction τ_s with three parameters X_e, Y_e, Z_e such that τ_s transforms an isotropic system into the set of its fundamental graphs. □

4.4.2 Minors and vertex-minors by C_2MS logic formulas

Proposition 4.22. *There exists an MS transduction*

$$\tau_m : \{\textit{all } \{\mathsf{Member}\}\textit{-structures}\} \to 2^{\{\textit{all } \{\mathsf{Member}\}\textit{-structures}\}}$$

with three parameters $V_\alpha, V_\beta, V_\gamma$ that maps an isotropic system to the set of its minors.

Proof. From Definition 4.2, an isotropic system $S' = (V', L')$ is a minor of $S = (V, L)$ if there are three pairwise disjoint subsets $V_\alpha = \{x_1, x_2, \ldots, x_a\}$, $V_\beta = \{y_1, y_2, \ldots, y_b\}$, $V_\gamma = \{z_1, z_2, \ldots, z_c\}$ of V such that $S' = S|^{x_1}_\alpha|^{x_2}_\alpha \cdots |^{x_a}_\alpha|^{y_1}_\beta|^{y_2}_\beta \cdots |^{y_b}_\beta|^{z_1}_\gamma|^{z_2}_\gamma \cdots |^{z_c}_\gamma$. Then, $V' = V \setminus (V_\alpha \cup V_\beta \cup V_\gamma)$ and

$$L' = \{p_{V'}(a) : a \in L \text{ and for all } v \in V,\ \text{if } a(v) \neq 0, \text{then } v \in V_{a(v)}\}. \tag{4.1}$$

Note that the canonical projection function $p_{V'}(a)$ is defined in page 37.

We describe $\mathsf{Member}_{S'}(X, Y, Z)$ by an MS formula $\mu_1(V_\alpha, V_\beta, V_\gamma, X, Y, Z)$ on S.

A triple (X, Y, Z) is a set representation of a vector in L' if and only if there exists a set representation (X_a, Y_a, Z_a) of a vector a in L such that the following four conditions hold.

(i) X, Y, Z are pairwise disjoint,

(ii) $(X \cup Y \cup Z) \cap (V_\alpha \cup V_\beta \cup V_\gamma) = \emptyset$,

(iii) $X = X_a \setminus V_\alpha, Y = Y_a \setminus V_\beta, Z = Z_a \setminus V_\gamma$,

(iv) $V_\alpha \subseteq X_a \cup (V \setminus (Y_a \cup Z_a))$, $V_\beta \subseteq Y_a \cup (V \setminus (X_a \cup Z_a))$, and $V_\gamma \subseteq Z_a \cup (V \setminus (X_a \cup Y_a))$.

Conditions (i)–(iii) express that (X, Y, Z) is a set representation of $p_{V'}(a)$; condition (iv) translates condition (4.1) expressing that $p_{V'}(a) \in L'$. Hence, the desired formula $\mu_1(V_\alpha, V_\beta, V_\gamma, X, Y, Z)$ can be written as $\mu_2 \wedge \exists X_a \exists Y_a \exists Z_a(\mathsf{Member}(X_a, Y_a, Z_a) \wedge \mu_3)$ where μ_2 with free variables $V_\alpha, V_\beta, V_\gamma, X, Y, Z$ expresses conditions (i) and (ii) and μ_3 with free variables $V_\alpha, V_\beta, V_\gamma, X, Y, Z, X_a, Y_a, Z_a$ expresses conditions (iii) and (iv). □

Theorem 4.23.

(1) *There exists a C_2MS transduction with six parameters $V_\alpha, V_\beta, V_\gamma, X_e, Y_e, Z_e$ that maps a graph into the set of its vertex-minors.*

(2) *There exists a C_2MS transduction with three parameters X_e, Y_e, Z_e that maps a graph into the set of its locally equivalent graphs.*

Proof. (1) We have C_2MS transductions τ_g, τ_s, and τ_m from Proposition 4.20, 4.21, and 4.22. Then, the composition $\tau_s \circ \tau_m \circ \tau_g$ is a C_2MS transduction by Proposition 4.18 and it maps a graph to the set of its vertex-minors by Corollary 4.9. But this will give a C_2MS transduction with twelve parameters. However we can eliminate parameters of τ_g by choosing one particular pair of supplementary vectors, in other words, setting $X_a = Y_b = V$, $Y_a = Z_a = X_b = Z_b = \emptyset$. This is possible because we can choose one particular isotropic system in Corollary 4.9 to find all vertex-minors. Eliminating those parameters actually means that we obtain another C_2MS transduction τ'_g by replacing $x \in X_a$, $x \in Y_a$ by true and $x \in Y_a$, $x \in Z_a$, $x \in X_b$, and $x \in Z_b$ by false in the C_2MS definition scheme for τ_g.

(2) Since local complementations do not change the associated isotropic system, if two graphs are locally equivalent graphs then there is an isotropic system having both as fundamental graphs. So it is clear that $\tau_s \circ \tau_g$ is a C_2MS transduction that maps a graph to the set of its locally equivalent graphs. As we discussed in the proof of (1), we can also eliminate parameters of τ_g. □

Corollary 4.24. *For every graph H, there is a closed C_2MS logic formula φ_H expressing that a given graph contains a vertex-minor isomorphic to H.*

Proof. For every graph H with vertices $v_1, \ldots, v_n$, we can write a closed MS logic formula $\varkappa_H$ that is true on a graph G if and only if G is isomorphic to H as follows:

$$\begin{aligned}\exists x_1, \ldots, \exists x_n (&\text{“}x_1, \ldots, x_n \text{ are pairwise distinct”}\\ &\wedge \text{“every vertex is equal to } x_i \text{ for some } i\text{”}\\ &\wedge \text{“for all } i, j,\ \mathsf{edg}(x_i, x_j) \text{ holds if and only if } v_i v_j \in E(H)\text{”})\end{aligned}$$

Let τ_v be a C_2MS transduction that maps a graph into the set of its vertex-minors (Theorem 4.23). Its backwards translation (Proposition 4.17) relative to τ_v is a C_2MS formula $\varkappa_H^{\#}$ with free variables $V_\alpha, V_\beta, V_\gamma, X_e, Y_e, Z_e$. It is valid in a graph G if and only if its vertex-minor defined by the sets $V_\alpha, V_\beta, V_\gamma, X_e, Y_e, Z_e$ is isomorphic to H. Hence G has a vertex-minor isomorphic to H if and only if it satisfies $\exists V_\alpha \exists V_\beta \exists V_\gamma \exists X_e \exists Y_e \exists Z_e, \varkappa_H^{\#}$. □

Theorem 4.25. *For fixed k and fixed graph H, there exists a $O(|V(G)|)$-time algorithm that answers whether an input graph G of clique-width at most k has a vertex-minor isomorphic to H, if G is given by its k-expression.*

Proof. We combine the previous corollary with Proposition 4.19. □

In Chapter 7, we will discuss how to eliminate the requirement of k-expressions as an input by constructing it from the adjacency list of the input graph G in $O(|V(G)|^3)$ time.

Chapter 5

Seese's Conjecture

In this chapter, we prove a weakened statement of Seese's conjecture [52]. We express Seese's conjecture in terms of rank-width as following.

Conjecture 5.1 (Seese [52]). *If a set of graphs has a decidable monadic second-order (MS) theory, then it has bounded rank-width.*

The conjecture has been proved for various graph classes: planar graphs [52], graphs of bounded degree, graphs without a fixed graph as a minor, graphs of which every subgraph has the bounded average degree [16], interval graphs, line graphs [17]. We did not solve this conjecture, but we show a weaker statement: if a set of graphs has a decidable C_2MS theory, then it has bounded rank-width.

We briefly summarize the proof. Courcelle [17] showed that Seese's conjecture is true if and only if it is true for bipartite graphs. In Section 3.5, we have various connection relating branch-width of binary matroids to rank-width of bipartite graphs. Moreover, the grid theorem of binary matroids by Geelen, Gerards, and Whittle [28] implies the analogous one, Corollary 3.25, stating that bipartite graphs of sufficiently large rank-width contain a vertex-minor isomorphic to S_k (defined in page 32).

Theorem 6.28 shows that there is a C_2MS transduction that maps a graph into the set of all its vertex-minors. Combining with Proposition 4.18, we conclude that if a set $\mathcal{C}$ of bipartite graphs of unbounded rank-width has a decidable C_2MS theory, then its image under the above C_2MS transduction contains graphs isomorphic to S_k for all k.

We explicitly construct a C_2MS transduction τ_2 that maps a graph isomorphic to S_k into the $k \times k$ grid. Then, the image of $\mathcal{C}$ under $\tau_2 \circ \tau_1$ contains a graph isomorphic to the $k \times k$ grid for all k. We use the following theorem of Seese.

Theorem 5.2 (Seese [52, Theorem 5]). *Let K be a set of graphs such that for every planar graph H there is a planar graph $G \in K$ such that H is isomorphic to a minor of G. Then, K does not have a decidable monadic second-order theory.*

Therefore, we conclude that, by Proposition 4.18, a set of graphs of unbounded rank-width does not have a decidable monadic second-order theory.

5.1 Enough to consider bipartite graphs

Courcelle showed that Seese's conjecture is true if and only if it is true for bipartite graphs in [17] by using a certain graph transformation from graphs to bipartite graphs. We will see that his argument also works for our weakened problem obtained by relaxing "decidable MS theory" to "decidable C_2MS theory", but will use graph theoretic arguments to show that this transduction preserves boundedness of rank-width without using a deep theorem on MS transductions.

The following lemma describes a graph transformation from graphs G to bipartite graphs $B(G)$ found by Courcelle [17]. He proved that there exist two functions f_1 and f_2 such that $f_1(\operatorname{rwd}(G)) \le \operatorname{rwd}(B(G)) \le f_2(\operatorname{rwd}(G))$. We show that $\operatorname{rwd}(B(G)) = \max(2\operatorname{rwd}(G), 1)$ if $V(G) \neq \emptyset$.

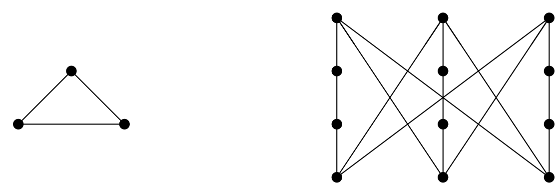

Figure 5.1: K_3 and $B(K_3)$

Lemma 5.3. *Let $G = (V, E)$ be a graph such that $V \neq \emptyset$. Let $B(G) = (V \times \{1, 2, 3, 4\}, E')$ be a bipartite graph obtained from G as follows:*

(i) *if $v \in V$ and $i \in \{1, 2, 3\}$, then (v, i) is adjacent to $(v, i+1)$ in $B(G)$,*

(ii) *if $vw \in E$, then $(v, 1)$ is adjacent to $(w, 4)$ in $B(G)$.*

Then we have $\operatorname{rwd}(B(G)) = \max(2\operatorname{rwd}(G), 1)$.

To show Lemma 5.3, we will use the following lemma, that appears in [26, Lemma 2.1] in terms of matroids. This lemma will be also used in Section 6.8.

Lemma 5.4. *Let G be a graph having at least three vertices. Let $(T, \mathcal{L})$ be a rank-decomposition of G of width k such that $k > 0$. If v is a vertex of T and e is an edge of T, we let $X_{ev} = \mathcal{L}^{-1}(\mathcal{X}_{ev})$ where $\mathcal{X}_{ev}$ is the set of leaves of T in the component of $T \setminus e$ not containing v. Let A be a subset of $V(G)$ such that $A \neq X_{ev}$ for every $v \in V(T)$ and each edge e incident to v.*

Suppose that for every partition (A_1, A_2, A_3) of A, there exists $i \in \{1, 2, 3\}$ such that $\rho_G(A_i) \ge \rho_G(X)$. Then, there exists a degree-3 vertex s of T such that

(i) *for each edge e of T, we have $\rho_G(X_{es} \setminus A) \le k$,*

(ii) *there is no edge f incident to s such that $A \subseteq X_{fs}$.*

Proof. We first claim that if (X_1, X_2) is a partition of $V(G)$ with $\rho_G(X_1) \leq k$, then either $\rho_G(X_1 \setminus A) \leq k$ or $\rho_G(X_2 \setminus A) \leq k$. From the partition $(A \cap X_1, A \cap X_2, \emptyset)$ of A, either $\rho_G(A \cap X_1) \geq \rho_G(A)$ or $\rho_G(A \cap X_2) \geq \rho_G(A)$. We may assume that $\rho_G(A \cap X_1) \geq \rho_G(A)$. By submodularity, $\rho_G(A \cup X_1) \leq \rho_G(A) + \rho_G(X_1) - \rho_G(A \cap X_1) \leq k$. So, $\rho_G(X_2 \setminus A) = \rho_G(A \cup X_1) \leq k$. Thus we showed the claim.

Now, we construct an orientation of T. Let e be an edge of T, and let u and v be the ends of e. If $\rho_G(X_{ev} \setminus A) \leq k$, then we orient e from u to v. By the previous claim, each edge receives at least one orientation.

First, assume that there exists a node v of T such that every other node can be connected to v by a directed path on T. Since $k \geq 1$, each edge incident with a leaf has been oriented away from that leaf. Hence we may assume that v has degree 3. If there is an edge $f = vw$ incident to v such that $A \subseteq X_{fv}$, then $X_{fw} = V(G) \setminus X_{fv}$, $\rho_G(X_{fw} \setminus A) = \rho_G(X_{fw}) \leq k$, and therefore f has been oriented for both directions. So we may replace v by w. Since $A \neq X_{ev}$ for every vertex $v \in V(T)$ and each edge e incident to v, this process will terminate and we may assume that there is no edge f incident to v such that $A \subseteq X_{fv}$. Then the lemma follows with $s = v$.

Next, we assume that there is no vertex reachable from every other vertex. Then there exists a pair of edges e and f and a vertex w on the path connecting e and f such that neither e nor f is oriented toward w. Let $Y_1 = X_{ew}$, $Y_3 = X_{fw}$, and $Y_2 = V(G) \setminus (Y_1 \cup Y_2)$. Since e and f are oriented away from w, $\rho_G((Y_2 \cup Y_3) \setminus A) \leq k$ and $\rho_G((Y_1 \cup Y_2) \setminus A) \leq k$. By submodularity,

$$\rho_G(Y_1 \setminus A) + \rho_G(Y_3 \setminus A) \leq \rho_G((Y_2 \cup Y_3) \setminus A) + \rho_G((Y_1 \cup Y_2) \setminus A) \leq 2k.$$

This contradicts the fact that neither e nor f is oriented toward w. □

Proof of Lemma 5.3. (1) Let us show that $\operatorname{rwd}(B(G)) \leq \max(2\operatorname{rwd}(G), 1)$.

If $\operatorname{rwd}(G) = 0$, then G has no edges, and therefore $B(G)$ is a disjoint union of paths of three edges. Since a path of three edges has rank-width 1, we deduce that $\operatorname{rwd}(B(G)) = 1$ if $\operatorname{rwd}(G) = 0$.

We now assume that $\operatorname{rwd}(G) > 0$. Let $(T, \mathcal{L})$ be a rank-decomposition of G of width k. Let N be the set of leaves of T. Let T' be a tree having $(V(T) \times \{0\}) \cup (N \times \{1, 2, 3, 4, 12, 34\})$ as the set of vertices such that

(i) if $vw \in E(T)$, then $(v, 0)$ is adjacent to $(w, 0)$ in T',

(ii) for all $v \in N$, $(v, 12)$ is adjacent to both $(v, 1)$ and $(v, 2)$,

(iii) for all $v \in N$, $(v, 34)$ is adjacent to both $(v, 3)$ and $(v, 4)$,

(iv) for all $v \in N$, $(v, 0)$ is adjacent to both $(v, 12)$ and $(v, 34)$.

Informally speaking, we obtain T' from T by replacing each leaf with a rooted binary tree having four leaves. For each leaf (v, i) of T', we define $\mathcal{L}'(v, i) = (\mathcal{L}(v), i) \in V(B(G))$. Then $(T', \mathcal{L}')$ is a rank-decomposition of $B(G)$.

We claim that the width of $(T', \mathcal{L}')$ is at most $2k$.

For each edge $e = vw \in E(T)$, let (X, Y) be a partition of N induced by the connected components of $T \setminus e$. Then, the edge $(v, 0)(w, 0)$ of $E(T')$ induces a partition $(X \times \{1, 2, 3, 4\}, Y \times \{1, 2, 3, 4\})$ of $N \times \{1, 2, 3, 4\}$. We observe that $\mathcal{L}'^{-1}(X \times \{1, 2, 3, 4\}) = \mathcal{L}^{-1}(X) \times \{1, 2, 3, 4\}$. It is easy to see that

$$\rho_{B(G)}(\mathcal{L}'^{-1}(X \times \{1, 2, 3, 4\})) = 2\rho_G(\mathcal{L}^{-1}(X)) \leq 2k.$$

We now consider the remaining edges of T'. Each of them induces a partition (X, Y) of the leaves of T' such that $|X| \leq 2$ or $|Y| \leq 2$. So, $\rho_{B(G)}(\mathcal{L}'^{-1}(X)) \leq 2$. Therefore we deduce that the width of $(T', \mathcal{L}')$ is at most $2k$. Thus, rank-width of $B(G)$ is at most $2k$.

(2) We show that $\mathrm{rwd}(B(G)) \geq \max(2\,\mathrm{rwd}(G), 1)$. We may assume that $\mathrm{rwd}(G) > 0$, otherwise it is trivial. For each $v \in V(G)$, let $P_v = \{(v, 1), (v, 2), (v, 3), (v, 4)\} \subseteq V(B(G))$.

(a) We claim that if $X \subseteq P_v$ and $|X| \geq 2$, then $\rho_{B(G)}(X) \geq \rho_{B(G)}(P_v)$. We may assume that $X \neq P_v$. If $X = \{(v, 2), (v, 3)\}$ or $X = \{(v, 1), (v, 4)\}$, then $\rho_{B(G)}(X) = 2$. By our construction, we have $\rho_{B(G)}(P_v) = 0$ or 2. We may assume that $\rho_{B(G)}(P_v) = 2$, otherwise it is trivial. Therefore we deduce that there is a vertex not in P_v, that is adjacent to $(v, 1)$. So, if $X = \{(v, 1), (v, 2)\}$, $X = \{(v, 1), (v, 3)\}$, or $X = \{(v, 1), (v, 2), (v, 3)\}$, then $\rho_{B(G)} = 2$. By symmetry, we deduce our claim. In particular, this claim implies that for every partition (X_1, X_2, X_3) of P_v, there exists $i \in \{1, 2, 3\}$ such that $\rho_{B(G)}(X_i) \geq \rho_{B(G)}(P_v)$.

(b) We say that an edge e of T *crosses* P_v if for a partition (X, Y) of the set of leaves of T induced by $T \setminus e$, the following four sets are nonempty: $\mathcal{L}^{-1}(X) \cap P_v$, $\mathcal{L}^{-1}(X) \setminus P_v$, $\mathcal{L}^{-1}(Y) \cap P_v$, and $\mathcal{L}^{-1}(Y) \setminus P_v$.

(c) Let $k = \mathrm{rwd}(B(G))$. Let $(T, \mathcal{L})$ be a rank-decomposition of $B(G)$ of width at most k with the minimum number of vertices v of $V(G)$ having an edge of T crossing P_v.

We claim that no edge of T crosses P_v for all $v \in V(G)$. Suppose there is an edge of T that crosses P_v for some $v \in V(G)$. Let s be a vertex satisfying Lemma 5.4, let e_1, e_2, and e_3 be the edges of T incident with s, and let X_i denote $X_{e_i s}$ for each $i \in \{1, 2, 3\}$. We may assume that $\rho_{B(G)}(X_1 \cap P_v) \geq \rho_{B(G)}(P_v)$. Then by submodularity,

$$\begin{aligned} &\rho_{B(G)}((X_2 \cup X_3) \setminus P_v) \\ &\quad = \rho_{B(G)}(X_1 \cup P_v) \leq \rho_{B(G)}(X_1) + \rho_{B(G)}(P_v) - \rho_{B(G)}(X_1 \cap P_v) \leq \rho_{B(G)}(X_1) \leq k. \end{aligned}$$

Now we construct a rank-decomposition $(T', \mathcal{L}')$ of $B(G)$; let T' be a tree obtained from the minimum subtree of T containing both e_1 and leaves in $\mathcal{L}(V(B(G)) \setminus P_v)$ by

(i) subdividing e_1 with a new vertex b,

(ii) adding new vertices r_1, r_2, r_3, r_4, r_{12}, r_{34},

(iii) adding new edges br_{12}, br_{34}, $r_{12}r_1$, $r_{12}r_2$, $r_{34}r_3$, $r_{34}r_4$, and

(iv) contracting one of incident edges of each degree-2 vertex until no degree-2 vertices are left.

For each $x \in V(B(G)) \setminus P_v$, we define $\mathcal{L}'(x)$ to be a leaf of T' induced by $\mathcal{L}(x)$. For $i \in \{1, 2, 3, 4\}$, we define $\mathcal{L}'((v, i)) = r_i$.

Then $(T', \mathcal{L}')$ is a rank-decomposition of $B(G)$. It is easy to see that the width of $(T', \mathcal{L}')$ is at most k by Lemma 5.4. Moreover, the number of vertices w of $V(G)$ having an edge of T' crossing P_w is exactly one less than that for T. This is a contradiction, because we choose $(T, \mathcal{L})$ to have the minimum number of those vertices.

(d) Therefore, for every vertex $v \in V(G)$, there exists an edge e_v of T such that $\mathcal{L}(P_v)$ is exactly the set of leaves in one component X_v of $T \setminus e_v$. Let b_v be one end of e_v in X.

Let T_G be the minimal subtree of T containing b_v for all $v \in V(G)$. Let $\mathcal{L}_G$ be a function from $V(G)$ to the set of leaves of T_G such that $\mathcal{L}_G(v) = b_v$. It is easy to see that $(T_G, \mathcal{L}_G)$ is a rank-decomposition of G.

(e) We claim that the width of $(T_G, \mathcal{L}_G)$ is at most $k/2$. Let e be an edge of T_G and (X, Y) be a partition of leaves of T_G induced by $T_G \setminus e$. We note that $T \setminus e$ induces a partition (X', Y') of leaves of T such that $\mathcal{L}^{-1}(X') = \mathcal{L}_G^{-1}(X) \times \{1, 2, 3, 4\}$. We deduce that $2\rho_G(\mathcal{L}_G^{-1}(X)) = \rho_{B(G)}(\mathcal{L}_G^{-1}(X) * \{1, 2, 3, 4\}) \leq k$.

(f) Therefore, $k \geq 2\,\mathrm{rwd}(G)$. □

Lemma 5.5 (Courcelle [17, Proposition 3.2, 3.3]). *Let $B(G)$ be the function defined in Lemma 5.3. Let $\tau(G) = \{B(G)\}$. Then τ is an MS transduction.*

Sketch of proof. In order to simplify the paper, we skipped the general definition of MS transductions in this paper. In general, the definition of MS transductions allows duplicating a fixed number of times (here four times) a given structure before defining the new structure inside it by a definition scheme. For detailed definition, see [17]. From that definition, it is clear. □

5.2 Proof using vertex-minors

In this section, we prove the following theorem.

Theorem 5.6. *If a set of graphs has a decidable C_2MS theory, then it has bounded rank-width.*

The proof will use a family of bipartite graphs S_k and we will build the $k \times (2k-2)$ grid by a fixed MS transduction from S_k. The graph S_k was used in Corollary 3.25.

Lemma 5.7. *Let $\mathcal{C}$ be a set of bipartite graphs of unbounded rank-width. Then there are infinitely many values of k such that S_k is isomorphic to a vertex-minor of a graph in $\mathcal{C}$.*

Proof. Suppose not. There exists an integer k such that no graph in $\mathcal{C}$ has a vertex-minor isomorphic to S_k. This implies, by Corollary 3.25, that $\mathcal{C}$ has bounded rank-width. A contradiction. □

Proposition 5.8. *There exists an MS transduction τ on graphs such that the $k \times (2k-2)$ grid belongs to $\tau(S_k)$ for all $k > 1$.*

Proof. Our objective is to find an MS transduction on graphs such that its image of S_k contains the $k \times (2k-2)$ grid for all k. Suppose we are given a graph G isomorphic to S_k for some k as a relational structure $\langle V, \mathsf{edg} \rangle$.

Let (A, B) be a bipartition of G such that A has a vertex of degree one. Let s be a neighbor of a vertex of degree one.

Two vertices v and w of B are called *consecutive* if $|n_G(v) \setminus n_G(w)| = 1$ and $|n_G(w) \setminus n_G(v)| = 1$. A subset X of B is called the *tail* of v if it is a maximal subset of B satisfying the following two conditions:

(i) $v, s \notin X$,

(ii) for all $x \in X$ and $y \in B$, if x, y are consecutive and $y \neq v$, then $y \in X$.

We call that v is a *successor* of w if v and w are consecutive and the tail of w is a subset of the tail of v. Two vertices $v \in A$ and $w \in B$ are called *matched* if (informally) they have the same number in Figure 5.2. We may define it as follows:

(i) they are adjacent,

(ii) for all y, if y is a successor of w, then y is not adjacent to v,

(iii) if $v' \in A$ satisfies the above two conditions and $n_G(v) \subseteq n_G(v')$, then $v = v'$.

A vertex $w \in B$ is called a *far successor* of $v \in B$ if (informally) the number given to w is the number given to v added by k. Even though we do not know k by an MS logic formula, we can define this as follows: there exist $x \in A$, $y \in B$, and $z \in A$ such that

(i) v is not adjacent to z but adjacent to x,

(ii) x and y are matched,

(iii) w and z are matched,

(iv) w is a successor of y.

Let T be the minimal subset of B containing s such that if $x \in T$ then the far successor of x is in T.

We are now ready to describe edges of the $k \times (2k-2)$ grid by an MS logic formula in terms of edg of G. We define the set of vertices of the grid as the set of vertices of G having a matched vertex. In fact, each vertex of S_k has either one matched vertex or none. Two vertices v, w of the grid are adjacent if and only if one of the following four conditions is true:

(i) $v, w \in B$, and v is a successor of w, and $v \notin T$,

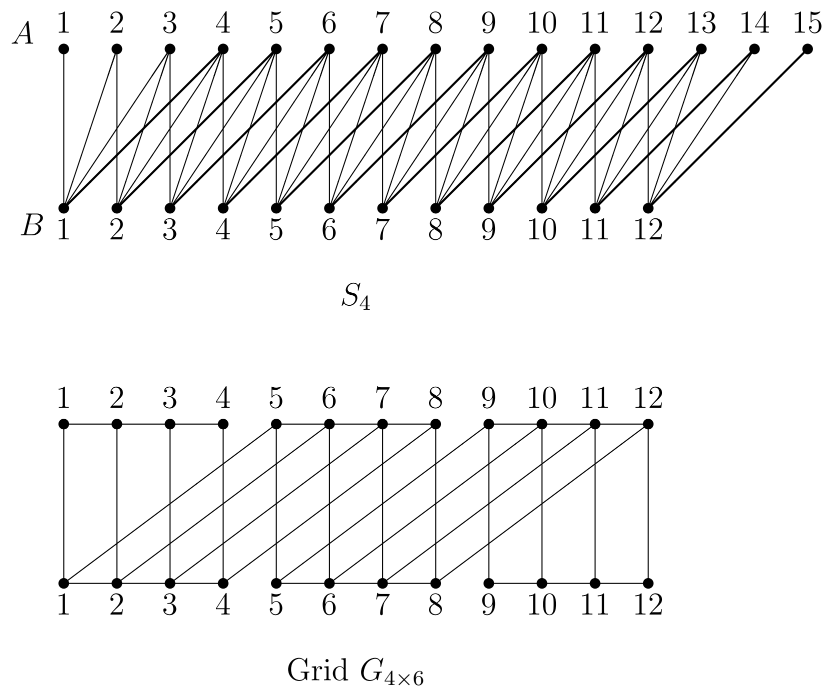

Figure 5.2: Getting the grid from S_k

(ii) $v, w \in A$, and their matched vertices are adjacent in the grid by the previous condition,

(iii) $v \in A$, $w \in B$, and they are either matched or the matched vertex of w is a far successor of v,

(iv) after swapping v and w, one of above conditions is true.

We skip the detailed MS logic formula, but since each step of this proof can be written as an MS logic formula, we show that there exists an MS transduction τ on graphs such that $\tau(S_k)$ contains the $k \times (2k-2)$ grid. □

We are now ready to prove our main theorem of this chapter.

Proof of Theorem 5.6. Suppose that $\mathcal{C}$ is a set of graphs having unbounded rank-width. Let τ_1 be an MS transduction given by Lemma 5.5, that maps a graph G to $\{B(G)\}$. Let τ_2 be a C_2MS transduction given by Theorem 4.23 that maps a graph to the set of its vertex-minors. Let τ_3 be an MS transduction on graphs given by Proposition 5.8, such that $\tau_3(S_k)$ contains the $k \times (2k-2)$ grid. Let $\tau = \tau_3 \circ \tau_2 \circ \tau_1$. By (2) of Proposition 4.18, τ is a C_2MS transduction. Let $\mathcal{I}$ be the image of $\mathcal{C}$ under the C_2MS transduction τ.

Let $\mathcal{B} = \{B(G) : G \in \mathcal{C}\}$. By Lemma 5.3, we know that $\mathcal{B}$ has unbounded rank-width. Since $\mathcal{B}$ is a set of bipartite graphs and has unbounded rank-width, there are infinitely many values of k such that S_k is isomorphic to a vertex-minor of a graph

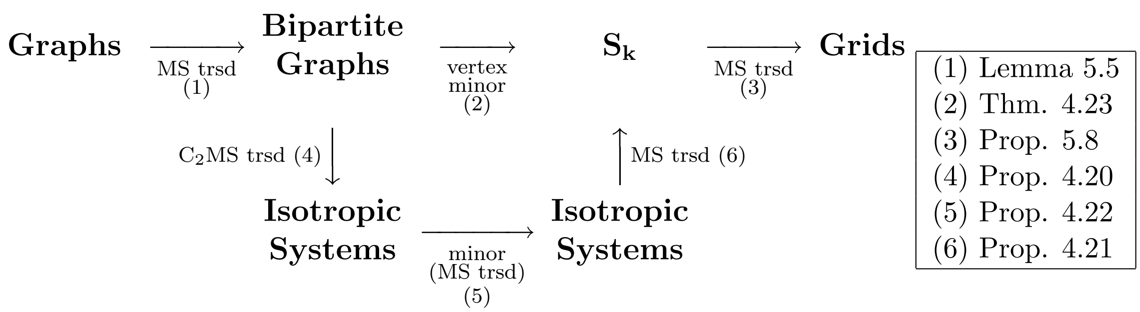

Figure 5.3: Sketch of the proof via vertex-minors

in $\mathcal{B}$ by Corollary 3.25, and therefore there are infinitely many values of k such that the $k \times (2k-2)$ grid is contained in $\mathcal{I}$. Furthermore, every planar graph is a minor of the $k \times (2k-2)$ grid in $\mathcal{I}$ for sufficiently large k (Lemma 3.24).

By Theorem 5.2 of Seese, $\mathcal{I}$ does not have a decidable MS theory and therefore $\mathcal{I}$ does not have a decidable C_2MS theory, because every MS logic formula is a C_2MS logic formula.

By (1) of Proposition 4.18, $\mathcal{C}$ does not have a decidable C_2MS theory. □

5.3 Proof using matroid minors

We give another proof of Theorem 5.6 based on binary matroids instead of isotropic systems and using results by Hliněný and Seese [35]. They showed that if a set of matroids representable over a fixed finite field has a decidable monadic second-order theory, then it has bounded branch-width. We assume that matroids are given by their $\{\mathsf{Indep}\}$-structures, described in Example 4.13.

Since binary matroids are closely related to bipartite graphs, it is natural to show the following proposition.

Proposition 5.9. *There is a C_2MS transduction with two parameters A and B that maps a bipartite graph G to the set of all binary matroids having G as a fundamental graph.*

Proof. Let N be the adjacency matrix of G. Suppose that (A, B) is a bipartition of G and $\mathcal{M} = Bin(G, A, B)$. ($Bin$ is defined in Section 3.5.) The binary matroid $\mathcal{M}$ has a standard representation $P = \begin{pmatrix} I_A & N[A,B] \end{pmatrix}$. It is enough to show that we can express $\mathsf{Indep}(U)$ of $\mathcal{M}$ by a C_2MS logic formula in terms of the edg relation of G.

A subset U of $V(G)$ is independent in $\mathcal{M}$ if and only if columns of P are linearly independent. Thus, it is equivalent to say that there is no subset W of U such that the sum of column vectors of P indexed by elements of W is zero. We claim that we can write a C_2MS logic formula $\mathsf{Zero}(W)$ expressing that the sum of column vectors of P indexed by elements of W is zero. Since each row of P corresponds to an element of A, $\mathsf{Zero}(W)$ is true if and only if for each $x \in A$, the number of neighbors of x in

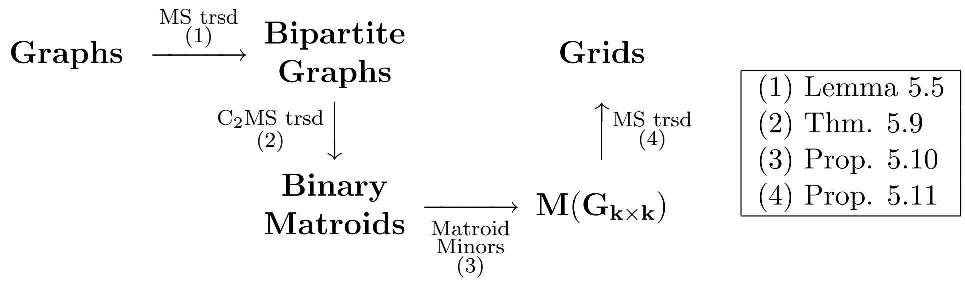

Figure 5.4: Sketch of the proof via matroid minors

W is odd if $x \in W$, and even otherwise. We may easily write this in a C$_2$MS logic formula. □

The following two proposition is proved in [35] but stated in different terminologies. We recall the notation $G_{k\times k}$ for the $k \times k$ grid.

Proposition 5.10 (Hliněný and Seese [35, Lemma 6.4, 6.5]). *There is an MS transduction that maps a matroid to the set of its minors.*

Proposition 5.11 (Hliněný and Seese [35, Lemma 6.6, 6.7]). *Let $M(G_{k\times k})$ be the cycle matroid of the $k \times k$ grid. There is an MS transduction $\tau_p : \{matroids\} \to 2^{\{graphs\}}$ such that $\tau_p(M(G_{k\times k}))$ contains the $(k-2)\times(k-2)$ grid when $k > 6$ and k is even.*

Second proof of Theorem 5.6. Suppose that $\mathcal{C}$ is a set of graphs having unbounded rank-width. Let τ_1 be an MS transduction given by Lemma 5.5, that maps a graph G to $\{B(G)\}$. Let τ_2 be a C$_2$MS transduction given by Proposition 5.9 that maps a graph to the set of binary matroids having it as a fundamental graph. Let τ_3 be an MS transduction that maps a matroid to the set of its minors. Let τ_4 be an MS transduction from matroids to graphs such that $\tau_3(M(G_{k\times k}))$ contains the $(k-2)\times(k-2)$ grid when k is even and $k > 6$.

By Corollary 3.18 and Lemma 5.3, $\tau_2 \circ \tau_1(\mathcal{C})$ has unbounded branch-width because $\mathcal{C}$ has unbounded rank-width.

By Theorem 3.23, $\tau_3 \circ \tau_2 \circ \tau_1(\mathcal{C})$ contains cycle matroids $M(G_{k\times k})$ for infinitely many values of k. Since $\tau_3 \circ \tau_2 \circ \tau_1(\mathcal{C})$ is minor-closed, we know that $\tau_3 \circ \tau_2 \circ \tau_1(\mathcal{C})$ contains $M(G_{k\times k})$ for all k.

Therefore $\mathcal{I} = \tau_4 \circ \tau_3 \circ \tau_2 \circ \tau_1(\mathcal{C})$ contains the $(k-2)\times(k-2)$ grid for infinitely many values of k.

By Theorem 5.2 of Seese, $\mathcal{I}$ does not have a decidable MS theory and therefore $\mathcal{I}$ does not have a decidable C$_2$MS theory, because every MS logic formula is a C$_2$MS logic formula.

By (2) of Proposition 4.18, $\tau_4 \circ \tau_3 \circ \tau_2 \circ \tau_1$ is a C$_2$MS transduction. By (1) of Proposition 4.18, we conclude that $\mathcal{C}$ does not have a decidable C$_2$MS theory. □

Chapter 6

Well-quasi-ordering with Vertex-minors

In this chapter, our main objective is to prove the following.

Theorem 6.1. *Let k be a constant. If $\{G_1, G_2, G_3, \cdots\}$ is an infinite sequence of graphs of rank-width at most k, then there exist $i < j$ such that G_i is isomorphic to a pivot-minor of G_j, and therefore isomorphic to a vertex-minor of G_j.*

In general, we call a binary relation $\leq$ on X a *quasi-order* if it is reflexive and transitive. For a quasi-order $\leq$, we say "$\leq$ is a *well-quasi-ordering*" or "X is *well-quasi-ordered* by $\leq$" if for every infinite sequence $a_1, a_2, \ldots$ of elements of X, there exist $i < j$ such that $a_i \leq a_j$. We may reiterate Theorem 6.1 as follows: a set of graphs of rank-width at most k is well-quasi-ordered by a vertex-minor relation (or pivot-minor relation) up to isomorphisms.

Here is a corollary of Theorem 6.1. Note that this corollary has an elementary proof in Section 6.8, and will be used to construct a polynomial-time algorithm to recognize graphs of rank-width at most k for a fixed k in Chapter 7.

Corollary 6.2. *For a fixed k, there is a finite list of graphs $G_1, G_2, \ldots, G_m$ such that for every graph H, rank-width of H is at most k if and only if G_i is not isomorphic to a vertex-minor of H for all i.*

Proof. Let $X = \{G_1, G_2, \ldots\}$ be a set of graphs satisfying that for every graph H, rank-width of a graph H is at most k if and only if G_i is not isomorphic to a vertex-minor of H for all i. We choose X minimal by set inclusion. There are no $G_i, G_j \in S$ such that G_i is isomorphic to a vertex-minor of G_j, because if so, then we may remove G_j from X. By assumption, the rank-width of $G \setminus v$ for $v \in V(G)$ is at most k, and therefore the rank-width of G_i is at most $k+1$. By Theorem 6.1, X is finite. □

We say that an isotropic system $S_1 = (V_1, L_1)$ is *simply isomorphic* to another isotropic system $S_2 = (V_2, L_2)$ if there exists a bijection $\mu : V_1 \to V_2$ such that $L_1 = \{a \circ \mu : a \in L_2\}$. A bijection μ is called a *simple isomorphism*. It is clear that if S_1 is simply isomorphic to S_2, then every fundamental graph of S_1 is isomorphic to a graph locally equivalent to a fundamental graph of S_2.

We say that an isotropic system S_1 is an *$\alpha\beta$-minor* of an isotropic system $S = (V, L)$ if there are a subset $X \subseteq V$ and a vector $a \in K^X$ such that

$$a(v) \in \{\alpha, \beta\} \text{ for all } v \in X \quad \text{and} \quad S_1 = S|_a^X.$$

Every $\alpha\beta$-minor of an isotropic system S is a minor of S, but not vice versa. Similarly we may define a *$\beta\gamma$-minor* and an *$\alpha\gamma$-minor*, but by symmetry among nonzero elements of K, it is enough to consider an $\alpha\beta$-minor in this paper. By restricting an elementary minor operation, we will prove the following lemma in Section 6.6, which links pivot-minors of graphs and $\alpha\beta$-minors of isotropic systems.

Lemma 6.22. *For $i \in \{1, 2\}$, let S_i be the isotropic system whose graphic presentation is (G_i, a_i, b_i) such that*

$$a_i(v), b_i(v) \in \{\alpha, \beta\}$$

for all $v \in V(G_i)$. If S_1 is an $\alpha\beta$-minor of S_2, then G_1 is a pivot-minor of G_2.

Instead of dealing with graphs, we will prove the following stronger proposition on isotropic systems.

Proposition 6.20. *Let k be a constant. If $\{S_1, S_2, S_3, \cdots\}$ is an infinite sequence of isotropic systems of branch-width at most k, then there exist $i < j$ such that S_i is simply isomorphic to an $\alpha\beta$-minor of S_j.*

By using Proposition 6.20, Theorem 6.1 is deduced.

Proof of Theorem 6.1. Let S_i be an isotropic system whose graphic presentation is (G_i, a_i, b_i) where $a_i(v) = \alpha$, $b_i(v) = \beta$ for all $v \in V(G_i)$. Each S_i has branch-width at most k, since its branch-width is equal to rank-width of G_i. By Proposition 6.20, there exist $i < j$ such that S_i is simply isomorphic to a $\alpha\beta$-minor of S_j, and therefore by Lemma 6.22, G_i is isomorphic to a pivot-minor of G_j. □

We recall a *linked* branch-decomposition from Section 2.1. Let $f : V \to \mathbb{Z}$ be a symmetric submodular function. For a branch-decomposition $(T, \mathcal{L})$ of f, let e_1 and e_2 be two edges of T. Let E be the set of leaves of T in the component of $T \setminus e_1$ not containing e_2, and let F be the set of leaves of T in the component of $T \setminus e_2$ not containing e_1. Let P be the shortest path in T containing e_1 and e_2. We call e_1 and e_2 *linked* if

$$\min_{h \in E(P)} (\text{width of } h \text{ of } (T, \mathcal{L})) = \min_{\mathcal{L}^{-1}(E) \subseteq Z \subseteq V \setminus \mathcal{L}^{-1}(F)} f(Z).$$

We call a branch-decomposition $(T, \mathcal{L})$ is *linked* if each pair of edges of T is linked. Since we define the branch-decomposition of isotropic systems and the rank-width of graphs as branch-decompositions of the connectivity functions and the cut-rank functions respectively, we may define linkedness for branch-decompositions of isotropic systems as well as rank-decompositions of graphs. The following lemma was shown by Geelen, Gerards, and Whittle [27]. It was the first step to prove well-quasi-ordering of matroids representable over a fixed finite field having bounded branch-width. Its analogous result by Thomas [54] was used to prove well-quasi-ordering of graphs of bounded tree-width in Robertson and Seymour [47].

Lemma 6.3 (Geelen et al. [27, Theorem (2.1)]). *An isotropic system* (V, L) *of branch-width* n *has a linked branch-decomposition of width* n *if* $|V| > 1$. *Equivalently, a graph* (V, E) *of rank-width* n *has a linked rank-decomposition of width* n *if* $|V| > 1$.

We also use Robertson and Seymour's "lemma on trees," proved in [47]. It enabled them to prove that a set of graphs of bounded tree-width are well-quasi-ordered by the graph minor relation. It was also used by Geelen et al. [27] to prove that a set of matroids representable over a fixed finite field and having bounded branch-width is well-quasi-ordered by the matroid minor relation. We need a special case of "lemma on trees," in which a given forest is subcubic, that was also useful for branch-decompositions of matroids in Geelen et al. [27].

The following definitions are in Geelen et al. [27]. A *rooted tree* is a finite directed tree where all but one of the vertices have indegree 1. A *rooted forest* is a collection of countably many vertex disjoint rooted trees. Its vertices with indegree 0 are called *roots* and those with outdegree 0 are called *leaves*. Edges leaving a root are *root edges* and those entering a leaf are *leaf edges*.

An n*-edge labeling* of a graph F is a map from the set of edges of F to the set $\{0, 1, \ldots, n\}$. Let λ be an n-edge labeling of a rooted forest F and let e and f be edges in F. We say that e is λ*-linked* to f if F contains a directed path P starting with e and ending with f such that $\lambda(g) \geq \lambda(e) = \lambda(f)$ for edge g on P.

A *binary forest* is a rooted orientation of a subcubic forest with a distinction between left and right outgoing edges. More precisely, we call a triple (F, l, r) a *binary forest* if F is a rooted forest where roots have outdegree 1 and l and r are functions defined on non-leaf edges of F, such that the head of each non-leaf edge e of F has exactly two outgoing edges, namely $l(e)$ and $r(e)$.

Lemma 6.4 (Lemma on subcubic trees; Robertson and Seymour [47]). *Let* (F, l, r) *be an infinite binary forest with an* n*-edge labeling* λ. *Moreover, let* $\leq$ *be a quasi-order on the set of edges of* F *with no infinite strictly descending sequences, such that* $e \leq f$ *whenever* f *is* λ*-linked to* e. *If the set of leaf edges of* F *is well-quasi-ordered by* $\leq$ *but the set of root edges of* F *is not, then* F *contains an infinite sequence* $(e_0, e_1, \ldots)$ *of non-leaf edges such that*

(i) $\{e_0, e_1, \ldots\}$ *is an antichain with respect to* $\leq$,

(ii) $l(e_0) \leq l(e_1) \leq l(e_2) \leq \cdots$,

(iii) $r(e_0) \leq r(e_1) \leq r(e_2) \leq \cdots$.

Proof. See Geelen et al. [27, (3.2)]. □

Informally speaking, at the last stage of proving Proposition 6.20, we need an object describing a piece of isotropic systems such that the number of ways to merge those objects into one isotropic system is finite up to simple isomorphisms. More precisely, we call a triple $P = (V, L, B)$ a *scrap* if V is a finite set, L is a totally isotropic subspace of K^V, and B is an ordered basis of $L^\perp/L$. An *ordered basis* is a basis with a linear ordering, and therefore B is of the form $\{b_1+L, b_2+L, \ldots, b_k+L\}$ with $b_i \in L^\perp$. We denote $V(P) = V$. Note that $L^\perp/L$ is a vector space containing vectors of the form $a+L$ with $a \in L^\perp$ and $a+L = b+L$ if and only if $a-b \in L$. Also note that $|B| = \dim(L^\perp/L) = \dim(L^\perp) - \dim(L) = 2(|V| - \dim(L)) = 2\lambda(L)$.

Two scraps $P_1 = (V, L, B)$ and $P_2 = (V', L', B')$ are called *isomorphic* if there exists a bijection $\mu : V \to V'$ such that $L = \{a \circ \mu : a \in L'\}$ and $b_i + L = (b'_i \circ \mu) + L$ where $B = \{b_1+L, b_2+L, \ldots, b_k+L\}$ and $B' = \{b'_1+L', b'_2+L', \ldots, b'_k+L'\}$.

For $x \in K \setminus \{0\}$ and $v \in V$, let $\delta^v_x \in K^V$ such that $\delta^v_x(v) = x$ and $\delta^v_x(w) = 0$ for all $w \neq v$. We will slightly abuse δ^v_x without referring V if it is not ambiguous. If $P = (V, L, B)$ is a scrap with $\delta^v_x \notin L^\perp \setminus L$, then we denote

$$P|^v_x = (V \setminus \{x\}, L|^v_x, \{p_{V\setminus\{v\}}(b_i) + L|^v_x\}_i)$$

where each $b_i \in L^\perp$ is chosen to satisfy that $B = \{b_i + L\}_i$ and $b_i(v) \in \{0, x\}$. We will prove that $P|^v_x$ is a well-defined scrap in Proposition 6.10. Note that $\delta^v_x \notin L^\perp \setminus L$ is required to write $P|^v_x$.

A scrap P' is called a *minor* of a scrap P if $P' = P|^{v_1}_{x_1}|^{v_2}_{x_2}\cdots|^{v_l}_{x_l}$ for some v_i and x_i. Similarly a scrap P' is called an *$\alpha\beta$-minor* of a scrap P if $P' = P|^{v_1}_{x_1}|^{v_2}_{x_2}\cdots|^{v_l}_{x_l}$ for some v_i and $x_i \in \{\alpha, \beta\}$.

Two scraps $P_1 = (V, L, B)$ and $P_2 = (V', L', B')$ are called *disjoint* if $V \cap V' = \emptyset$. A scrap $P = (V, L, B)$ is called a *sum* of two disjoint scraps $P_1 = (V_1, L_1, B_1)$ and $P_2 = (V_2, L_2, B_2)$ if

$$V = V_1 \cup V_2,\ L_1 = L|_{\subseteq V_1},\ \text{and } L_2 = L|_{\subseteq V_2}.$$

A sum of two disjoint scraps is not uniquely determined; we, however, will define the *connection types* that will determine a sum of two disjoint scraps such that there are only finitely many connection types. Moreover, we will prove the following.

Lemma 6.19. *Let P_1, P_2, Q_1, Q_2 be scraps. Let P be the sum of P_1 and P_2 and Q be the sum of Q_1 and Q_2. If P_i is a minor of Q_i for $i = 1, 2$ and the connection type of P_1 and P_2 is equal to the connection type of Q_1 and Q_2, then P is a minor of Q.*

Moreover, if P_i is an $\alpha\beta$-minor of Q_i for $i \in \{1, 2\}$ and the connection type of P_1 and P_2 is equal to the connection type of Q_1 and Q_2, then P is an $\alpha\beta$-minor of Q.

Another requirement to apply Lemma 6.4 is that $e \leq f$ whenever f is λ-linked to e. This condition will be satisfied by the following lemma, which is a generalization of Tutte's linking theorem. Tutte's linking theorem for matroids was used by Geelen et al. [27] and is a generalization of Menger's theorem. Robertson and Seymour also used Menger's theorem in [47].

Theorem 6.12. *Let V be a finite set and X be a subset of V. Let L be a totally isotropic subspace of K^V. Let k be a constant. Let b be a complete vector of $K^{V\setminus X}$.*

For all $Z \supseteq X$, $\lambda(L|_{\subseteq Z}) \geq k$ if and only if there is a complete vector $a \in K^{V\setminus X}$ such that $\lambda(L|^{V\setminus X}_a) \geq k$ and $a(v) \neq b(v)$ for all $v \in V \setminus X$.

The actual proof of Proposition 6.20 is based on a construction of a forest with a certain k-labeling from branch-decompositions of isotropic systems, and applying lemmas described above.

In subsequent sections, we will prove those lemmas and we will prove Proposition 6.20 in Section 6.5.

6.1 Lemmas on totally isotropic subspaces

In this section, L is a totally isotropic subspace of K^V, not necessarily $\dim(L) = |V|$. We prove some general results on totally isotropic subspaces.

Lemma 6.5. *Let L be a totally isotropic subspace of K^V and $v \in V$, $x \in K \setminus \{0\}$. Then, $(L|_x^v)^\perp = L^\perp|_x^v$.*

Proof. Suppose that $y \in L^\perp|_x^v$. There exists $\bar{y} \in L^\perp$ such that $\bar{y}(v) \in \{0, x\}$ and $y = p_{V\setminus\{v\}}(\bar{y})$. For every $z \in L|_x^v$, there exists $\bar{z} \in L$ such that $\bar{z}(v) \in \{0, x\}$ and $p_{V\setminus\{v\}}(\bar{z}) = z$. Since $\langle y, z\rangle = \langle \bar{y}, \bar{z}\rangle - \langle \bar{y}(v), \bar{z}(v)\rangle = 0$, $y \in (L|_x^v)^\perp$.

Conversely, suppose that $y \notin L^\perp|_x^v$. Let $y \oplus x \in K^V$ be such that $p_{V\setminus\{v\}}(y\oplus x) = y$ and $(y \oplus x)(v) = x$. By assumption, $y \oplus x \notin L^\perp$. Therefore, there exists $z \in L$ such that

$$\langle y \oplus x, z\rangle = 1 = \langle y, p_{V\setminus\{v\}}(z)\rangle + \langle x, z(v)\rangle.$$

If $\langle x, z(v)\rangle = 0$, then $p_{V\setminus\{v\}}(z) \in L|_x^v$ and $\langle y, p_{V\setminus\{v\}}(z)\rangle = 1$, and therefore $y \notin (L|_x^v)^\perp$. So, we may assume that $\langle x, z(v)\rangle = 1$.

Let $y \oplus 0 \in K^V$ such that $p_{V\setminus\{v\}}(y \oplus 0) = y$ and $(y \oplus 0)(v) = 0$. By assumption, $y \oplus 0 \notin L^\perp$. Therefore, there exists $w \in L$ such that $\langle y \oplus 0, w\rangle = 1 = \langle y, p_{V\setminus\{v\}}(w)\rangle$. If $w(v) \in \{0, x\}$, then $p_{V\setminus\{v\}}(w) \in L|_x^v$ and $y \notin (L|_x^v)^\perp$. Hence we may assume that $\langle x, w(v)\rangle = 1$.

Now, we obtain that $\langle x, w(v) + z(v)\rangle = 0$, and so $w(v) + z(v) \in \{0, x\}$. Therefore $p_{V\setminus\{v\}}(w + z) \in L|_x^v$. Furthermore $\langle p_{V\setminus\{v\}}(w + z), y\rangle = 1$. So, $y \notin (L|_x^v)^\perp$. □

Lemma 6.6. *If L is a totally isotropic subspace of K^V and $X \subseteq V$, then*

$$(L|_{\subseteq X})^\perp = L^\perp|_X.$$

Proof. We use an induction on $|V \setminus X|$. If $|X| < |V| - 1$, then we pick $v \notin X$, and deduce that $(L|_{\subseteq V\setminus\{v\}}|_{\subseteq X})^\perp = (L|_{\subseteq V\setminus\{v\}})^\perp|_X = L^\perp|_{V\setminus\{v\}}|_X = L^\perp|_X$. Therefore we may assume that $V \setminus X = \{v\}$.

For $x \in K^X$ and $y \in K$, we let $x \oplus y$ denote a vector in K^V such that $p_X(x \oplus y) = x$ and $(x \oplus y)(v) = y$.

(1) We claim that $L^\perp|_X \subseteq (L|_{\subseteq X})^\perp$.

Suppose that there exists $a \in L^\perp|_X$. There is $b \in K$ such that $a \oplus b \in L^\perp$. For any $c \in L|_{\subseteq X}$, $\langle a \oplus b, c \oplus 0\rangle = 0$, and therefore $\langle a, c\rangle = 0$. Thus, $a \in (L|_{\subseteq X})^\perp$.

(2) We claim that $(L|_{\subseteq X})^\perp \subseteq L^\perp|_X$.

Suppose that there exists $a \in (L|_{\subseteq X})^\perp$ such that $a \notin L^\perp|_X$.

For every $x \in K$, $a \oplus x \notin L^\perp$, and therefore there exists $a_x \oplus c_x \in L$ such that $\langle a_x \oplus c_x, a \oplus x\rangle = \langle a_x, a\rangle + \langle c_x, x\rangle = 1$. Thus, $\langle a_0, a\rangle = 1$.

If $c_x = 0$, then $a_x \in L|_{\subseteq X}$ and so $\langle a_x, a\rangle = 0$ and $\langle c_x, x\rangle = 0$, contrary to the assumption that $\langle a_x, a\rangle + \langle c_x, x\rangle = 1$. Therefore $c_x \neq 0$ for all $x \in K$.

If $c_x = c_y$ for $x \neq y$, then $a_x + a_y \in L|_{\subseteq X}$. Thus, $0 = \langle a_x + a_y, a\rangle = 1 + \langle c_x, x\rangle + 1 + \langle c_y, y\rangle = \langle c_x, x + y\rangle$. Since $c_x \neq 0$ and $x + y \neq 0$, $c_x = c_y = x + y$ and $\langle a_x, a\rangle = 1 + \langle x + y, x\rangle = 1 + \langle x, y\rangle$.

If $c_x = c_y = c_z$ for distinct x, y, z, then $x + y = y + z = z + x$. So, $x = y = z$, which is a contradiction.

If $c_x = c_y$, $c_z = c_w$ for distinct x, y, z, w, then $a_x = x + y = z + w = a_z$. So, $x = y = z = w$. This is a contradiction.

Therefore, there are exactly one pair $x, y \in K$ such that $c_x = c_y$. Let $\{z, w\} = K \setminus \{x, y\}$.

Since $c_z \neq c_w$ and $c_z, c_w \in K \setminus \{0, x + y\}$, $c_z + c_w = x + y = c_x = c_y$. Therefore, $a_z + a_w + a_x \in L|_{\subseteq X}$ and $\langle a_z + a_w + a_x, a\rangle = 0$. Since $\langle a_z, a\rangle + \langle a_w, a\rangle = \langle c_z, z\rangle + \langle c_w, w\rangle$,

$$0 = \langle a_z + a_w + a_x, a\rangle = 1 + \langle x, y\rangle + \langle c_z, z\rangle + \langle c_w, w\rangle.$$

If $x = 0$, then $c_z, c_w \in \{z, w\}$. So, $\langle c_z, z\rangle + \langle c_w, w\rangle = 0$. Thus, $\langle a_z + a_w + a_x, a\rangle = 1$. A contradiction.

So we may assume that $x \neq 0$, $y \neq 0$, $z = 0$, and then $x + y = w$ and $\langle c_w, w\rangle = 0$. But, this implies that $c_w = w = x + y = c_x$. A contradiction. □

Proposition 6.7. *Let V be a finite set and L be a totally isotropic subspace of K^V and $v \in V$.*

$$\dim(L|_x^v) = \begin{cases} \dim(L) & \text{if } \delta_x^v \in L^\perp \setminus L \\ \dim(L) - 1 & \text{otherwise.} \end{cases}$$

In other words, $\lambda(L|_x^v) = \begin{cases} \lambda(L) & \text{if } \delta_x^v \notin L^\perp \setminus L \\ \lambda(L) - 1 & \text{otherwise.} \end{cases}$

Proof. For $w \in K^{V\setminus\{v\}}$ and $u \in K$, let $w \oplus u$ denote a vector in K^V such that $p_{V\setminus\{v\}}(w \oplus u) = w$ and $(w \oplus u)(v) = u$.

A basis of $L|_x^v$ extends to a set of independent vectors in L. Thus, $\dim(L|_x^v) \leq \dim(L)$.

Suppose C is a basis of L. We may assume that at most one vector of C has x on v. Let us choose $y \in K \setminus \{0, x\}$ such that at most one, possibly none, of C has y on v and all other vectors in C have either 0 or x on v.

(1) If $\delta_x^v \in L^\perp \setminus L$, then, no vector in L has y on v. Thus, for every $z \in C$, $z(v) \in \{0, x\}$. Since $\delta_x^v \notin L$, $p_{V\setminus\{v\}}(C)$ is linearly independent and $p_{V\setminus\{v\}}(C) \subseteq L|_x^v$. So, $\dim(L) \leq \dim(L|_x^v)$.

(2) If $\delta_x^v \notin L^\perp$, then there exists $z \in C$ with $z(v) \notin \{0, x\}$. Since $\delta_x^v \notin L$, $p_{V\setminus\{v\}}(C \setminus \{z\})$ is linearly independent and $p_{V\setminus\{v\}}(C \setminus \{z\}) \subseteq L|_x^v$. So, $\dim(L|_x^v) \geq \dim(L) - 1$. Conversely, let D be a basis of $L|_x^v$. Let $z \in L$ be such that $z(v) \notin \{0, x\}$. For each $w \in L|_x^v$, there exists a unique $\bar{w} \in L$ such that $\bar{w} = w \oplus 0$ or $w \oplus x$, because $\delta_x^v \notin L$. Let $D' = \{\bar{w} : w \in D\} \cup \{z\}$. Then, D' is linearly independent. So, $\dim(L) \geq \dim(L|_x^v) + 1$.

(3) If $\delta_x^v \in L$, then we may assume $\delta_x^v \in C$. For all $z \in C$, if $z \neq \delta_x^v$, then $z(v) = 0$. Thus, $p_{V\setminus\{v\}}(C \setminus \{z\})$ is linearly independent and $p_{V\setminus\{v\}}(C \setminus \{z\}) \subseteq L|_x^v$. So, $\dim(L|_x^v) \geq \dim(L) - 1$. Conversely, let D be a basis of $L|_x^v$. For any vector $w \in L|_x^v$, $w \oplus 0, w \oplus x \in L$ because $\delta_x^v \in L$. Since every vector of L has either 0 or x on v, $\{w \oplus 0 : w \in D\} \cup \{\delta_x^v\}$ is linearly independent in L. So, $\dim(L) \geq \dim(L|_x^v)+1$. □

Corollary 6.8. *Let V be a finite set and L be a totally isotropic subspace of K^V and $v \in V$. Let $C \subseteq K \setminus \{0\}$, $|C| = 2$. Then, either there is $x \in C$ such that $\lambda(L|_x^v) = \lambda(L)$ or for all $y \in K \setminus \{0\}$,*

$$L|_y^v = L|_{\subseteq V\setminus\{v\}} \quad \textit{and} \quad \lambda(L|_y^v) = \lambda(L) - 1.$$

Proof. Let $C = \{a, b\}$. Suppose there is no such $x \in C$. $\delta_a^v, \delta_b^v \in L^\perp \setminus L$. Therefore, for all $z \in L$, $z(v) = 0$. Thus, $L|_y^v = L|_{\subseteq V\setminus\{v\}}$ and $\lambda(L|_y^v) = \lambda(L) - 1$ for all $y \in K \setminus \{0\}$. □

6.2 Scraps

In this section, we prove that a minor of a scrap is well-defined. Definitions related to scraps were described in the beginning of this chapter.

Lemma 6.9. *Let $P = (V, L, B)$ be a scrap and $v \in V$. If $\delta_x^v \notin L^\perp \setminus L$, then there is a sequence $b_1, b_2, \ldots, b_m \in L^\perp$ such that $b_i(v) \in \{0, x\}$ and $B = \{b_1 + L, b_2 + L, \ldots, b_m + L\}$.*

Proof. Let $B = \{a_1+L, a_2+L, \ldots, a_m+L\}$ with $a_i \in L^\perp$. If $\delta_x^v \in L$, then $a_i(v) \in \{0, x\}$ for all i. Hence we may assume that $\delta_x^v \notin L$ and so $\delta_x^v \notin L^\perp$.

There is $y \in L$ such that $\langle y, \delta_x^v \rangle = 1$. Thus, $y(v) \notin \{0, x\}$. Let

$$b_i = \begin{cases} a_i & \text{if } a_i(v) \in \{0, x\}, \\ a_i + y & \text{otherwise.} \end{cases}$$

Then, $b_i + L = a_i + L$ and $b_i(v) \in \{0, x\}$. □

Proposition 6.10. *Let $P = (V, L, B)$ be a scrap. If $\delta_x^v \notin L^\perp \setminus L$, then $P|_x^v$ is well-defined and is a scrap.*

Proof. Let us first show that it is well-defined. Let $b_1, b_2, \ldots, b_k \in L^\perp$ be such that $b_i(v) \in \{0, x\}$ and $B = \{b_i + L : i = 1, 2, \ldots, k\}$. We claim that the choice of b_i does not change $P|_x^v$. Suppose $b_i - b_i' \in L$ and $b_i(v), b_i'(v) \in \{0, x\}$. Since $b_i - b_i' \in L$ and $(b_i - b_i')(v) \in \{0, x\}$, $p_{V\setminus\{v\}}(b_i - b_i') \in L|_x^v$. Therefore, $p_{V\setminus\{v\}}(b_i) + L|_x^v = p_{V\setminus\{v\}}(b_i') + L|_x^v$.

Now, we claim that $P|_x^v$ is a scrap.

First, we show that $L|_x^v$ is a totally isotropic subspace of $K^{V\setminus\{v\}}$. For all $a, b \in L|_x^v$, there are $\bar{a}, \bar{b} \in L$ such that $\bar{a}(v), \bar{b}(v) \in \{0, x\}$, $p_{V\setminus\{v\}}(\bar{a}) = a$, $p_{V\setminus\{v\}}(\bar{b}) = b$, and $\bar{a}, \bar{b} \in L$. Hence $\langle a, b \rangle = \langle \bar{a}, \bar{b} \rangle = 0$.

Next, we show that $\{p_{V\setminus\{v\}}(b_i) + L|^v_x : i = 1, 2, \ldots, k\}$ is a basis of $(L|^v_x)^\perp/(L|^v_x)$. Since $b_i(v) \in \{0, x\}$, we have $p_{V\setminus\{v\}}(b_i) \in (L|^v_x)^\perp = (L^\perp)|^v_x$. Suppose that there exists $C \neq \emptyset$ such that

$$\sum_{i\in C}(p_{V\setminus\{v\}}(b_i) + L|^v_x) = 0 + L|^v_x.$$

Since $\sum_{i\in C} p_{V\setminus\{v\}}(b_i) \in L|^v_x$, there exists $z \in L \subseteq L^\perp$ such that $z(v) \in \{0, x\}$ and $p_{V\setminus\{v\}}(z) = \sum_{i\in C} p_{V\setminus\{v\}}(b_i)$. By assumption, $\sum_{i\in C} b_i \notin L$. Since $p_{V\setminus\{v\}}(\sum_{i\in C} b_i - z) = 0$, $\sum_{i\in C} b_i - z = \delta^v_x \in L^\perp \setminus L$. A contradiction. Therefore, $\{p_{V\setminus\{v\}}(b_i) + L|^v_x : i = 1, 2, \ldots, k\}$ is linearly independent. Moreover, $\dim((L|^v_x)^\perp/(L|^v_x)) = 2(|V| - 1 - \dim(L|^v_x)) = 2(|V| - \dim(L)) = \dim(L^\perp/L)$ because $\delta^v_x \notin L^\perp \setminus L$. □

6.3 Generalization of Tutte's linking theorem

In this section, we show an extension of Tutte's linking theorem [57]. We note that we already have one generalization of Tutte's linking theorem into graphs in Section 3.7.

The following inequality is analogous to Lemma 3.27.

Lemma 6.11. *Let V be a finite set and $v \in V$. Let L be a totally isotropic subspace of K^V. Let X_1, $Y_1 \subseteq V \setminus \{v\}$. Let $x, y \in K \setminus \{0\}$, $x \neq y$.*

$$\dim(L|_{\subseteq X_1\cap Y_1}) + \dim(L|_{\subseteq X_1\cup Y_1\cup\{v\}}) \geq \dim(L|^v_x|_{\subseteq X_1}) + \dim(L|^v_y|_{\subseteq Y_1}).$$

In other words,

$$\lambda(L|^v_x|_{\subseteq X_1}) + \lambda(L|^v_y|_{\subseteq Y_1}) \geq \lambda(L|_{\subseteq X_1\cap Y_1}) + \lambda(L|_{\subseteq X_1\cup Y_1\cup\{v\}}) - 1.$$

Proof. We may assume that $V = X_1 \cup Y_1 \cup \{v\}$ by taking $L' = L|_{\subseteq X\cup Y\cup\{v\}}$.

Let B be a minimum set of vectors in L such that $p_{X_1\cap Y_1}(B)$ is a basis of $L|_{\subseteq X_1\cap Y_1}$ and for every $z \in B$, $z(w) = 0$ for all $w \notin X_1 \cap Y_1$.

Let C be a minimum set of vectors in L such that $p_{X_1}(B\cup C)$ is a basis of $L|^v_x|_{\subseteq X_1}$ and for every $z \in C$, $z(w) = 0$ for all $w \notin X_1 \cup \{v\}$ and $z(v) \in \{0, x\}$. We may assume at most one vector in C has x on v.

Let D be a minimum set of vectors in L such that $p_{Y_1}(B\cup D)$ is a basis of $L|^v_y|_{\subseteq Y_1}$ and for every $z \in D$, $z(w) = 0$ for all $w \notin Y_1 \cup \{v\}$ and $z(v) \in \{0, y\}$. We may assume at most one vector in D has y on v.

We claim that $B \cup C \cup D$ is linearly independent. Suppose there is $B' \subseteq B$, $C' \subseteq C$, $D' \subseteq D$ such that

$$\sum_{b\in B'} b + \sum_{c\in C'} c + \sum_{d\in D'} d = 0.$$

No element of C' has x on v, because the LHS has 0 on v. Since $\sum_{c\in C'} c(w) = 0$ for all $w \in V \setminus (X_1 \cap Y_1)$, $p_{X_1\cap Y_1}(\sum_{c\in C'} c) \in L|_{\subseteq X_1\cap Y_1}$. Since $p_{X_1\cap Y_1}(B)$ is a basis, there is $B'' \subseteq B$ such that $p_{X_1\cap Y_1}(\sum_{c\in C'} c) = p_{X_1\cap Y_1}(\sum_{b\in B''} b)$. So,

$$\sum_{c\in C'} c + \sum_{b\in B''} b = 0.$$

This means that $C' = \emptyset$ because $C \cup B$ is a basis.

Similarly $D' = \emptyset$ and so $B' = \emptyset$.

$\dim(L) \geq |B| + |C| + |D| = \dim(L|_x^v|_{\subseteq X_1}) + \dim(L|_y^v|_{\subseteq Y_1}) - \dim(L|_{\subseteq X_1\cap Y_1})$. □

Now, we translate Tutte's linking theorem into isotropic subspaces. As a matter of fact, we are proving Theorem 3.28 in terms of isotropic systems.

Theorem 6.12. *Let V be a finite set and X be a subset of V. Let L be a totally isotropic subspace of K^V. Let k be a constant. Let b be a complete vector of $K^{V\setminus X}$.*

For all $Z \supseteq X$, $\lambda(L|_{\subseteq Z}) \geq k$ if and only if there is a complete vector $a \in K^{V\setminus X}$ such that $\lambda(L|_a^{V\setminus X}) \geq k$ and $a(v) \neq b(v)$ for all $v \in V \setminus X$.

Proof. ($\Leftarrow$) Let Z be a subset of V such that $X \subseteq Z$. Let $a_1 = p_{V\setminus Z}(a)$, $a_2 = p_{Z\setminus X}(a)$. Since $L|_{\subseteq Z} \subseteq L|_{a_1}^{V\setminus Z}$, $\lambda(L|_{\subseteq Z}) \geq \lambda(L|_{a_1}^{V\setminus Z})$.

$$k \leq \lambda(L|_a^{V\setminus X}) = \lambda(L|_{a_1}^{V\setminus Z}|_{a_2}^{Z\setminus X}) \leq \lambda(L|_{a_1}^{V\setminus Z}) \leq \lambda(L|_{\subseteq Z}).$$

($\Rightarrow$) Induction on $|V \setminus X|$. Suppose that there is no such complete vector $a \in K^{V\setminus X}$. We may assume that $|V \setminus X| \geq 1$.

Pick $v \in V \setminus X$. Let $K \setminus \{0, b(v)\} = \{x, y\}$. Since there is no complete vector $a' \in K^{V\setminus\{v\}\setminus X}$ such that $\lambda(L|_x^v|_{a'}^{V\setminus\{v\}\setminus X}) \geq k$, there exists X_1 such that $X \subseteq X_1 \subseteq V \setminus \{v\}$ and $\lambda(L|_x^v|_{\subseteq X_1}) < k$.

Similarly, there exists Y_1 such that $X \subseteq Y_1 \subseteq V \setminus \{v\}$ and $\lambda(L|_y^v|_{\subseteq Y_1}) < k$. By Lemma 6.11, either $\lambda(L|_{\subseteq X_1\cap Y_1}) < k$ or $\lambda(L|_{\subseteq X_1\cup Y_1\cup\{v\}}) < k$. A contradiction. □

Corollary 6.13. *Let V be a finite set and X be a subset of V. Let L be a totally isotropic subspace of K^V. Let b be a complete vector of $K^{V\setminus X}$.*

If $\lambda(L|_{\subseteq Z}) \geq \lambda(L|_{\subseteq X})$ for all $Z \supseteq X$, then there is a complete vector $a \in K^{V\setminus X}$ such that $L|_a^{V\setminus X} = L|_{\subseteq X}$ and $a(v) \neq b(v)$ for all $v \in V \setminus X$.

Proof. By Theorem 6.12, there exists a complete vector $a \in K^{V\setminus X}$ such that

$$\lambda(L|_a^{V\setminus X}) = \lambda(L|_{\subseteq X}) \text{ and } a(v) \neq b(v) \text{ for all } v \in V \setminus X.$$

Since $L|_{\subseteq X} \subseteq L|_a^{V\setminus X}$ and $\dim(L|_{\subseteq X}) = \dim(L|_a^{V\setminus X})$, $L|_{\subseteq X} = L|_a^{V\setminus X}$. □

Corollary 6.14. *Let $P = (V, L, B)$ be a scrap and $X \subseteq V$. If*

$$\lambda(P) = \lambda(L|_{\subseteq X}) = \min_{X\subseteq Z\subseteq V} \lambda(L|_{\subseteq Z}),$$

then there is an ordered set B' such that $Q = (X, L|_{\subseteq X}, B')$ is a scrap and an $\alpha\beta$-minor of P.

Proof. By applying Corollary 6.13 with $b(v) = \gamma$ for all $v \in V \setminus X$, there is a complete vector $a \in K^{V\setminus X}$ such that $L|_a^{V\setminus X} = L|_{\subseteq X}$ and $a(v) \in \{\alpha, \beta\}$ for all $v \in V \setminus X$. Let $V\setminus X = \{y_1, y_2, \ldots, y_m\}$ and $a_i = a(y_i)$. Then, $L|_{\subseteq X} = L|_{a_1}^{y_1}|_{a_2}^{y_2}\cdots|_{a_m}^{y_m}$. Let $L_0 = L$ and $L_i = L_{i-1}|_{a_i}^{y_i}$. By Proposition 6.7, $\lambda(L|_{\subseteq X}) = \lambda(L) = \lambda(L_i)$ implies $\delta_{a_{i+1}}^{y_{i+1}} \notin L_i^\perp \setminus L_i$. So, $P|_{a_1}^{y_1}|_{a_2}^{y_2}\cdots|_{a_m}^{y_m} = (X, L|_{\subseteq X}, B')$ is well-defined and is an $\alpha\beta$-minor of P. □

6.4 Sum

A scrap $P = (V, L, B)$ is called a *sum* of two disjoint scraps $P_1 = (V_1, L_1, B_1)$ and $P_2 = (V_2, L_2, B_2)$ if $V = V_1 \cup V_2$, $L_1 = L|_{\subseteq V_1}$, and $L_2 = L|_{\subseteq V_2}$. For given two disjoint scraps, there could be many scraps that are sums of those. In this section, we define the *connection type*, which determines a sum uniquely. Let $[n]$ denote the set $\{1, 2, 3, \ldots, n\}$.

Definition 6.15. *Let $P = (V, L, B)$ be a sum of two disjoint scraps $P_1 = (V_1, L_1, B_1)$ and $P_2 = (V_2, L_2, B_2)$ where $B = \{b_1 + L, b_2 + L, \ldots, b_n + L\}$, $B_1 = \{b_1^1 + L_1, b_2^1 + L_1, \ldots, b_m^1 + L_1\}$, and $B_2 = \{b_1^2 + L_2, b_2^2 + L_2, \ldots, b_l^2 + L_2\}$. For $x_1 \in K^{V_1}$ and $x_2 \in K^{V_2}$, let $x_1 \oplus x_2$ denote a vector in K^V such that $p_{V_i}(x_1 \oplus x_2) = x_i$ for $i = 1, 2$. Let*

$$C_0 = \left\{ (X, Y) : X \subseteq [m], Y \subseteq [l], \left(\sum_{i\in X} b_i^1\right) \oplus \left(\sum_{j\in Y} b_j^2\right) \in L \right\}$$

$$C_s = \left\{ (X, Y) : X \subseteq [m], Y \subseteq [l], \left(\sum_{i\in X} b_i^1\right) \oplus \left(\sum_{j\in Y} b_j^2\right) - b_s \in L \right\} \quad s = 1, \ldots, n$$

A sequence $C(P, P_1, P_2) = (C_0, C_1, C_2, \ldots, C_n)$ is called the connection type *of this sum.*

It is easy to see that if $\lambda(P), \lambda(P_1), \lambda(P_2) \le k$, then the number of distinct connection types is bounded by a function of k, because $|B| = 2\lambda(P) \le 2k$ and $|B_i| = 2\lambda(P_i) \le 2k$ for $i = 1$ and 2.

Proposition 6.16. *The connection type is well-defined.*

Proof. It is enough to show that the choice of b_i, b_i^1, and b_i^2 does not affect C_i. Suppose $b_i + L = d_i + L$, $b_i^1 + L_1 = d_i^1 + L_1$, and $b_i^2 + L_2 = d_i^2 + L_2$. For any (X, Y) such that $X \subseteq [m]$ and $Y \subseteq [l]$, we have $\sum_{i\in X}(b_i^1 - d_i^1) \oplus \sum_{j\in Y}(b_j^2 - d_j^2) \in L$ and $b_s - d_s \in L$, and therefore C_0 and C_s are well-defined. □

Proposition 6.17. *The connection type uniquely determines the sum of two disjoint scraps P_1 and P_2.*

Proof. Suppose not. Let $P = (V, L, B)$, $Q = (V, L', B')$ be two distinct sums of $P_1 = (V_1, L_1, B_1)$ and $P_2 = (V_2, L_2, B_2)$ by the same connection type. Let $B_1 = \{b_1^1 + L_1, b_2^1 + L_1, \ldots, b_m^1 + L_1\}$, and $B_2 = \{b_1^2 + L_2, b_2^2 + L_2, \ldots, b_k^k + L_2\}$.

We claim that $L = L'$. To show this, it is enough to show that $L \subseteq L'$. For any $a \in L$, $p_{V_1}(a) \in (L|_{\subseteq V_1})^{\perp}$ and $p_{V_2}(a) \in (L|_{\subseteq V_2})^{\perp}$. Therefore there is (X, Y) such that

$$x_1 = \sum_{i \in X} b_i^1 - p_{V_1}(a) \in L_1 \quad \text{and} \quad x_2 = \sum_{i \in Y} b_i^2 - p_{V_2}(a) \in L_2.$$

Since $x_1 \oplus 0, 0 \oplus x_2 \in L$, $x_1 \oplus x_2 \in L$. We deduce that $\sum_{i \in X} b_i^1 \oplus \sum_{i \in Y} b_i^2 = a + x_1 \oplus x_2 \in L$. Therefore, $(X, Y) \in C_0$ and $a + x_1 \oplus x_2 \in L'$. Since $x_1 \oplus 0, 0 \oplus x_2 \in L'$, $x_1 \oplus x_2 \in L'$, and so $a \in L'$.

Now, we show that $B = B'$. Let $b_s + L$ be the s-th element of B with $b_s \in L^{\perp}$. Let $b'_s + L$ be the s-th element of B' with $b'_s \in L^{\perp}$. Since $p_{V_i}(b_s) \in (L|_{\subseteq V_i})^{\perp} = L^{\perp}|_{V_i}$, there is (X, Y) such that

$$x_1 = \sum_{i \in X} b_i^1 - p_{V_1}(b_s) \in L_1 \quad \text{and} \quad x_2 = \sum_{i \in Y} b_i^2 - p_{V_2}(b_s) \in L_2.$$

Since $x_1 \oplus 0, 0 \oplus x_2 \in L$, $x_1 \oplus x_2 \in L$, and therefore $\sum_{i \in X} b_i^1 \oplus \sum_{i \in Y} b_i^2 - b_s \in L$. Thus, $(X, Y) \in C_s$, and

$$\sum_{i \in X} b_i^1 \oplus \sum_{i \in Y} b_i^2 - b'_s \in L' = L.$$

Thus, $b_s + L = b'_s + L = b_s + L'$. □

Proposition 6.18. *Let $P_1 = (V_1, L_1, B_1)$, $P_2 = (V_2, L_2, B_2)$ be two disjoint scraps. Let P be the sum of P_1 and P_2 by connection type $C(P, P_1, P_2)$. If $v \in V_1$ and $\delta_x^v \notin L_1^{\perp} \setminus L_1$. then, $\delta_x^v \notin L^{\perp} \setminus L$ and $P|_x^v$ is the sum of $P_1|_x^v$ and P_2 by connection type $C(P, P_1, P_2)$.*

Proof. If $\delta_x^v \in L^{\perp} \setminus L$, then $\delta_x^v \in (L^{\perp})|_{V_1} = (L|_{\subseteq V_1})^{\perp} = L_1^{\perp}$ and $\delta_x^v \notin L|_{\subseteq V_1}$. This contradicts to $\delta_x^v \notin L_1^{\perp} \setminus L_1$. So, $\delta_x^v \notin L^{\perp} \setminus L$.

First, we claim that $P|_x^v$ is a sum of $P_1|_x^v$ and P_2. It is equivalent to show that

$$L|_x^v|_{\subseteq V_1 \setminus \{v\}} = L|_{\subseteq V_1}|_x^v \quad \text{and} \quad L|_x^v|_{\subseteq V_2} = L|_{\subseteq V_2}.$$

It is easy to see that $L|_x^v|_{\subseteq V_1 \setminus \{v\}} = L|_{\subseteq V_1}|_x^v$ and $L|_{\subseteq V_2} \subseteq L|_x^v|_{\subseteq V_2}$. Therefore, it is enough to show that

$$L|_x^v|_{\subseteq V_2} \subseteq L|_{\subseteq V_2}.$$

Suppose $z \in L|_x^v|_{\subseteq V_2}$. Let $\bar{z} \in K^V$ such that $p_{V_2}(\bar{z}) = z$, $\bar{z}(v) \in \{0, x\}$, and $p_{V_1 \setminus \{v\}}(\bar{z}) = 0$. If $\bar{z}(v) = 0$, then $z \in L|_{\subseteq V_2}$. If $\bar{z}(v) = x$, then $p_{V_1}(z) = \delta_x^v \in L^{\perp}|_{V_1} = L_1^{\perp}$, and therefore $\delta_x^v \in L_1$. So, $\delta_x^v \in L$ and $z + \delta_x^v \in L$. Since $(z + \delta_x^v)(v) = 0$, $z \in L|_{\subseteq V_2}$.

Now, let us show that $C(P, P_1, P_2) = C(P|_x^v, P_1|_x^v, P_2)$. Let $B_1 = \{b_1^1 + L_1, b_2^1 + L_1, \ldots, b_m^1 + L_1\}$, and $B_2 = \{b_1^2 + L_2, b_2^2 + L_2, \ldots, b_k^2 + L_2\}$. For $x \in K^{V_1}$ and $y \in K^{V_2}$, let $x \oplus y$ denote a vector in K^V such that $p_{V_1}(x \oplus y) = x$ and $p_{V_2}(x \oplus y) = y$. We may assume that $b_i^1(v) \in \{0, x\}$ for all i by Lemma 6.9. Let $b \in L^{\perp}$ be such that $b(v) \in \{0, x\}$. Let $a(X, Y) = \left(\sum_{i \in X} b_i^1\right) \oplus \left(\sum_{j \in Y} b_j^2\right) - b$. Suppose we have (X, Y)

such that $X \subseteq [m]$, $Y \subseteq [k]$, and $a(X,Y) \in L$. Since $\left(\sum_{i\in X} b_i^1(v)\right) - b(v) \in \{0, x\}$,

$$p_{V_1\setminus\{v\}}(a(X,Y)) = \left(\sum_{i\in X} p_{V\setminus\{v\}}(b_i^1)\right) \oplus \left(\sum_{j\in Y} b_j^2\right) - p_{V\setminus\{v\}}(b) \in L|_x^v.$$

Conversely, let us suppose that there is (X,Y) such that $X \subseteq [m]$, $Y \subseteq [k]$, and

$$\left(\sum_{i\in X} p_{V_1\setminus\{v\}}(b_i^1)\right) \oplus \left(\sum_{j\in Y} b_j^2\right) - p_{V\setminus\{v\}}(b) \in L|_x^v.$$

Then, either $a(X,Y) \in L$ or $a(X,Y) + \delta_x^v \in L$. If $\delta_x^v \in L$, then $a(X,Y) \in L$. If $\delta_x^v \notin L^\perp$, then $a(X,Y) + \delta_x^v \notin L^\perp$ by $a(X,Y) \in L^\perp$, and therefore $a(X,Y) \in L$. □

Lemma 6.19. *Let P_1, P_2, Q_1, Q_2 be scraps. Let P be the sum of P_1 and P_2 and Q be the sum of Q_1 and Q_2. If P_i is a minor of Q_i for $i = 1, 2$ and the connection type of P_1 and P_2 is equal to the connection type of Q_1 and Q_2, then P is a minor of Q.*

Moreover, if P_i is an $\alpha\beta$-minor of Q_i for $i \in \{1, 2\}$ and the connection type of P_1 and P_2 is equal to the connection type of Q_1 and Q_2, then P is an $\alpha\beta$-minor of Q.

Proof. Induction on $|V(Q_1) \setminus V(P_1)| + |V(Q_2) \setminus V(P_2)|$. We may assume $|V(Q_1) \setminus V(P_1)| + |V(Q_2) \setminus V(P_2)| > 0$ and $V(Q_1) \neq V(P_1)$ by symmetry. There are $v \in V(Q_1) \setminus V(P_1)$, $x \in K \setminus \{0\}$, $X = V(Q_1) \setminus V(P_1) \setminus \{v\}$, and a complete vector $a \in K^X$ such that $P_1 = Q_1|_x^v|_a^X$. If P_1 is an $\alpha\beta$-minor of Q_1, then we may assume $x \in \{\alpha, \beta\}$ and $a(w) \in \{\alpha, \beta\}$ for all $w \in X$.

$Q|_x^v$ is the sum of $Q_1|_x^v$ and Q_2. P_1 is a minor of $Q_1|_x^v$. $C(Q|_x^v, Q_1|_x^v, Q_2) = C(Q, Q_1, Q_2) = C(P, P_1, P_2)$. So, P is a minor of $Q|_x^v$ by induction. Thus, P is a minor of Q.

Similarly if P_1 is an $\alpha\beta$-minor of Q_1 and P_2 is an $\alpha\beta$-minor of Q_2, then by induction P is an $\alpha\beta$-minor of Q. □

6.5 Well-quasi-ordering

Proposition 6.20. *Let k be a constant. If $\{S_1, S_2, S_3, \ldots\}$ is an infinite sequence of isotropic systems of branch-width at most k, then there exist $i < j$ such that S_i is simply isomorphic to an $\alpha\beta$-minor of S_j.*

Proof. We may assume that each $S_i = (V_i, L_i)$ satisfies that $|V_i| > 1$. By Lemma 6.3, there is a linked branch-decomposition $(T_i, \mathcal{L}_i)$ of S_i of width at most k for each i. Let F be a forest such that the i-th component is T_i. In T_i, we pick an edge and attach a root and direct every edge so that each leaf has a directed path from the root.

For each edge e of T_i, let X_e be the set of leaves of T_i having a directed path from e. Let $A_e = \mathcal{L}_i^{-1}(X_e)$. We associate e with a scrap $P_e = (A_e, L_i|_{\subseteq A_e}, B_e)$ and $\lambda(e) = \lambda(L_i|_{\subseteq A_e}) \leq k$ where B_e is chosen to satisfy the following:

If f is λ-linked to e, then P_e is an $\alpha\beta$-minor of P_f.

We claim that we can choose B_e satisfying the above property. We prove it by induction on the length of directed path from the root edge to e. If no other edge is λ-linked to e, let B_e be a basis of $(L_i|_{\subseteq A_e})^\perp/(L_i|_{\subseteq A_e})$ in an arbitrary order. If f, other than e, is λ-linked to e, choose f such that the distance between e and f is minimal. We assign B_e given by B_f by Corollary 6.14.

For $e, f \in E(F)$, let $e \leq f$ denote that a scrap P_e is isomorphic to an $\alpha\beta$-minor of a scrap P_f. Clearly, $\leq$ has no infinite strictly descending sequences, since there are finitely many scraps of bounded number of elements up to isomorphism. By construction if f is λ-linked to e, then $e \leq f$.

The leaf edges of F are well-quasi-ordered, because there are only finitely many distinct scraps of one element up to isomorphisms.

Suppose the root edges are not well-quasi-ordered. By Lemma 6.4, F contains an infinite sequence $(e_0, e_1, \ldots)$ of non-leaf edges such that

(i) $\{e_0, e_1, \ldots\}$ is an antichain with respect to $\leq$,

(ii) $l(e_0) \leq l(e_1) \leq \cdots$,

(iii) $r(e_0) \leq r(e_1) \leq \cdots$.

Since $\lambda(e_i) \leq k$ for all i, we may assume that $\lambda(e_i)$ is a constant for all i, by taking a subsequence.

Since the number of distinct connection types $C(P_{e_i}, P_{l(e_i)}, P_{r(e_i)})$ is finite, we may assume that the connection types are same for all i also by taking a subsequence.

Then, by Lemma 6.19, P_{e_0} is isomorphic to an $\alpha\beta$-minor of P_{e_1}, which means $e_0 \leq e_1$. This contradicts that $\{e_0, e_1, \ldots,\}$ is an antichain with respect to $\leq$.

Therefore, root edges are well-quasi-ordered, and there exist $i < j$ such that a scrap $(V_i, L_i, \emptyset)$ is isomorphic to a $\alpha\beta$-minor of a scrap $(V_j, L_j, \emptyset)$. Thus, S_i is simply isomorphic to an $\alpha\beta$-minor of S_j. □

6.6 Pivot-minors and $\alpha\beta$-minors

In this section, we shall show a relation between a pivot-minor of graphs and an $\alpha\beta$-minor of isotropic systems.

Proposition 6.21. *For $i \in \{1, 2\}$, let S_i be an isotropic system whose graphic presentation is (G_i, a_i, b_i) such that*

$$a_i(v), b_i(v) \in \{\alpha, \beta\} \text{ for all } v \in V(G_i).$$

If $S_1 = S_2$, then G_1 can be obtained from G_2 by applying a sequence of pivoting.

Proof. Let $V = V(G_1) = V(G_2)$ and let $S = S_1 = S_2 = (V, L)$ be an isotropic system. We show this by induction on $N(a_1, a_2) = |\{v \in V : a_1(v) \neq a_2(v)\}|$.

Suppose that $N(a_1, a_2) > 1$. Let $u \in V$ with $a_1(u) \neq a_2(u)$.

We first claim that there exists $v \in V$ such that $uv \in E(G_2)$ and $a_1(v) \neq a_2(v)$. Suppose not. By Proposition 4.6, there is a vector c in L such that

(i) $c(u) = b_2(u) = a_1(u)$,

(ii) $c(w) \in \{0, a_2(u)\}$ for all $w \neq u$.

In G_2, u and w are adjacent if $c(w) \neq 0$. Therefore if $c(w) = a_2(u)$, then $c(w) = a_1(u)$ by our assumption. Thus, for all $x \in V$, $c(x) \in \{0, a_1(x)\}$ and $c \neq 0$. A contradiction, because a_1 is an Eulerian vector.

Now, we apply pivoting uv to G_2, and we obtain another graphic presentation of S, that is,

$$(G_2 \wedge uv, a_2', b_2')$$

where $a_2' = a_2[V \setminus \{u, v\}] + b_2[\{u, v\}]$ and $b_2' = b_2[V \setminus \{u, v\}] + a_2[\{u, v\}]$. Since $N(a_1, a_2') = N(a_1, a_2) - 2$, by induction G_1 can be obtained from $G_2 \wedge uv$ by applying a sequence of pivoting, and so it can be obtained from G_2 as well.

If $N(a_1, a_2) = 0$, then $b_1 = b_2$ and $G_1 = G_2$.

Hence we may assume that $N(a_1, a_2) = 1$. We claim that this is impossible. Let $v \in V$ be such that $a_1(v) \neq a_2(v)$. By Proposition 4.6, we choose a unique vector $c \in L$ such that $c(v) = b_1(v) = a_2(v)$ and $c(w) \in \{0, a_2(w)\}$ for all $w \neq v$. Then $c = a_2[\{w \in V : c(w) \neq 0\}]$ and we obtain a contradiction, because a_2 is an Eulerian vector of S. □

Lemma 6.22. *For $i \in \{1, 2\}$, let S_i be the isotropic system whose graphic presentation is (G_i, a_i, b_i) such that $a_i(v), b_i(v) \in \{\alpha, \beta\}$ for all $v \in V(G_i)$. If S_1 is an $\alpha\beta$-minor of S_2, then G_1 is a pivot-minor of G_2.*

Proof. We use induction on $|V(G_2)| - |V(G_1)|$.

If $V(G_2) = V(G_1)$, then G_1 is a pivot-minor of G_2 by Proposition 6.21. Therefore we may assume that $|V(G_2)| > |V(G_1)|$.

Let $v \in V(G_2) \setminus V(G_1)$, $x \in \{\alpha, \beta\}$ and $y \in K^{V(G_2) \setminus V(G_1) \setminus \{v\}}$ be such that $y(w) \in \{\alpha, \beta\}$ for all $w \in V(G_2) \setminus V(G_1) \setminus \{v\}$ and $S_1 = S_2|_x^v|_y^{V(G_2) \setminus V(G_1) \setminus \{v\}}$. Note that S_1 is an $\alpha\beta$-minor of $S_2|_x^v$.

If $a_2(v) = x$, then

$$(G_2 \setminus v, p_{V \setminus \{v\}}(a_i), p_{V \setminus \{v\}}(b_i))$$

is a graphic presentation of $S_2|_x^v$. Thus by induction, G_1 is a pivot-minor of $G_2 \setminus v$, and so is a pivot-minor of G_2.

Now let us assume that $a_2(v) \neq x$, ans so $a_2(v) = b_2(v)$ since $b_2(v), a_2(v) \in \{\alpha, \beta\}$ and $a_2(v) \neq b_2(v)$. Suppose there is $u \in V(G_2)$ adjacent to v. Then

$$(G_2 \wedge uv \setminus v, p_{V \setminus \{v\}}(a_i[V(G_2) \setminus \{u, v\}] + b_i[\{u, v\}]), p_{V \setminus \{v\}}(b_i[V(G_2) \setminus \{u, v\}] + a_i[\{u, v\}]))$$

is a graphic presentation of $S_2|_x^v$. Thus by induction, G_1 is a pivot-minor of $G_2 \wedge uv \setminus v$, and so is a pivot-minor of G_2.

Hence we may assume that v has no adjacent vertex in G_2. Then δ_x^v is a vector of v in the fundamental basis of S_2 with respect to a_2. Let L_2 be such that $S_2 = (V(G_2), L_2)$. It follows that $\delta_x^v \in L_2$ and so $S_2|_x^v = S_2|_{a_2(v)}^v$. Thus in this case, we may let x be $a_2(v)$. □

6.7 Application to binary matroids

We would like to show that Theorem 6.1 implies the well-quasi-ordering theorem of Geelen, Gerards, and Whittle [27] for binary matroids. The proof uses the following theorems.

(1) (Seymour [53]) If $\mathcal{M}_1$, $\mathcal{M}_2$ are connected binary matroids on E, with the same connecitivy function, then $\mathcal{M}_1 = \mathcal{M}_2$ or $\mathcal{M}_1 = \mathcal{M}_2^*$.

(2) (Higman's lemma) Let $\leq$ be a quasi-order on X. For finite subsets $A, B \subseteq X$, we write $A \leq B$ if there is an injective mapping $f : A \to B$ such that $a \leq f(a)$ for all $a \in A$. Then $\leq$ is a well-quasi-ordering on the set of all finite subsets of X. (For proof, see Diestel's book [22, Lemma 12.1.3].)

For a binary matroid $\mathcal{M}$ with a fixed base B, we define a bipartite graph $Bip(\mathcal{M}, B)$ such that $V(Bip(\mathcal{M}, B)) = E(M)$ and $v \in E(M) \setminus B$ is adjacent to $w \in B$ if and only if w is in the fundamental circuit of v with respect to B. For a bipartite graph $G = (V, E)$ with a bipartition $V = A \cup B$, $Bin(G, A, B)$ is a binary matroid on V, represented by a $A \times V$ matrix $\begin{pmatrix} I_A & M[A, B] \end{pmatrix}$, where I_A is a $A \times A$ identity matrix and M is the adjacency matrix of G.

Lemma 6.23. *Let $\mathcal{M}_1$, $\mathcal{M}_2$ be binary matroids and let B_i be a fixed base of $\mathcal{M}_i$. If $\mathcal{M}_1$ is connected and $Bip(\mathcal{M}_1, B_1)$ is a pivot-minor of $Bip(\mathcal{M}_2, B_2)$, then $\mathcal{M}_1$ is a minor of either $\mathcal{M}_2$ or $\mathcal{M}_2^*$.*

Proof. Let $H = Bip(\mathcal{M}_1, B_1)$ and $G = Bip(\mathcal{M}_2, B_2)$. In Corollary 3.22, it was shown that if H is a pivot-minor of a bipartite graph G, then there is a bipartition (A', B') of H such that a binary matroid $\mathcal{M}_3 = Bin(H, A', B')$ is a minor of $\mathcal{M}_2 = Bin(G, B_2, V(G) \setminus B_2)$.

Since $\mathcal{M}_1$ and $\mathcal{M}_3$ have the same connecitivity function and $\mathcal{M}_1$ is connected, $\mathcal{M}_3$ is connected. By Seymour's theorem [53], $\mathcal{M}_1 = \mathcal{M}_3$ or $\mathcal{M}_1 = \mathcal{M}_3^*$. □

Corollary 6.24. *Let k be a constant. If $\{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \cdots\}$ is an infinite sequence of binary matroids of branch-width at most k, then there exist $i < j$ such that $\mathcal{M}_i$ is isomorphic to a minor of $\mathcal{M}_j$.*

Proof. First, we claim that if $\mathcal{M}_i$ is connected for all i, then the statement is true. Let B_i be a fixed base of $\mathcal{M}_i$ and $G_i = Bip(\mathcal{M}_i, B_i)$ for all i. The rank-width of G_i is at most $k - 1$, since rank-width of G_i is equal to (branch-width of $\mathcal{M}_i$)-1. By Theorem 6.1, there is an infinite subsequence $G_{a_1}, G_{a_2}, G_{a_3}, \ldots$ such that G_{a_i} is isomorphic to a pivot-minor of $G_{a_{i+1}}$ for all i. By Lemma 6.23, $\mathcal{M}_{a_1}$ is isomorphic to a minor of either $\mathcal{M}_{a_2}$ or $\mathcal{M}_{a_2}^*$ and $\mathcal{M}_{a_2}$ is isomorphic to a minor of either $\mathcal{M}_{a_3}$ or $\mathcal{M}_{a_3}^*$. It follows that $\mathcal{M}_{a_1}$ is isomorphic to a minor of $\mathcal{M}_{a_2}$ or $\mathcal{M}_{a_2}$ is isomorphic to a minor of $\mathcal{M}_{a_3}$ or $\mathcal{M}_{a_1}$ is isomorphic to a minor of $\mathcal{M}_{a_3}$. This proves the above claim.

Now, we prove the main statement. We may consider each $\mathcal{M}_i$ as a set of disjoint connected matroids and then $\mathcal{M}_i$ is isomorphic to a minor of $\mathcal{M}_j$ if and only if there is an injective function f from components of $\mathcal{M}_i$ to components of $\mathcal{M}_j$ such that a is isomorphic to a minor of $f(a)$ for every component a of $\mathcal{M}_i$. By Higman's lemma, there exist $i < j$ such that $\mathcal{M}_i$ is isomorphic to a minor of $\mathcal{M}_j$. □

6.8 Excluded vertex-minors

In this section, we show that Corollary 6.2 has an elementary proof not using isotropic systems. In other words, we show that for any fixed k, there is a finite set $\mathcal{C}_k$ of graphs such that for every graph G, $\text{rwd}(G) \le k$ if and only if no graph in $\mathcal{C}_k$ is isomorphic to a vertex-minor of G. Since the number of graphs with bounded number of vertices is finite up to isomorphism, it is enough to show that if a graph G has rank-width larger than k but every proper vertex-minor of G has rank-width at most k, then $|V(G)|$ is bounded by a function of k. We prove a stronger statement that if $\text{rwd}(G) > k$ and every proper pivot-minor has rank-width at most k, then $|V(G)|$ is bounded by a function of k. The analogous result for matroids was proved by Geelen, Gerards, Robertson, and Whittle [26] and we extend their method to graphs.

Let us begin with some additional definitions from [26]. Let G be a graph and (A, B) a partition of $V(G)$. A *branching* of B is a triple $(T, r, \mathcal{L})$ where T is a ternary tree with a fixed leaf node r and $\mathcal{L}$ is a bijection from B to the set of leaf nodes of T different from r. For an edge e of T of the branching $(T, r, \mathcal{L})$, let T_e be the set of vertices in B mapped by $\mathcal{L}$ to nodes in the component of $T \setminus e$ not containing r. We say B is *k-branched* if there is a branching $(T, r, \mathcal{L})$ of B such that for each edge e of T, $\rho_G(T_e) \le k$. Note that if both A and B are k-branched, then the rank-width of G is at most k.

The following lemma is proved in [26, Lemma 2.1] in terms of matroids. But their proof relies on the fact that $\lambda_{\mathcal{M}}$ is integer-valued submodular, and since cut-rank also has these properties, we can use basically the same argument.

Lemma 6.25. *Let G be a graph of rank-width k. Let (A, B) be a partition of $V(G)$ such that $\rho_G(A) \le k$. If there is no partition (A_1, A_2, A_3) of A such that $\rho(A_i) < \rho(A)$ for all $i \in \{1, 2, 3\}$, then B is k-branched.*

Proof. (Obvious modification of the proof of Geelen et al. [26, Lemma 2.1]) Let $(T, \mathcal{L})$ be a rank-decomposition of G of width k. We may assume that T has degree-3 nodes, as otherwise it is trivial. We may also assume that $k > 0$. If v is a vertex of T and e is an edge of T, we let $X_{ev} = \mathcal{L}^{-1}(\mathcal{X}_{ev})$ where $\mathcal{X}_{ev}$ is the set of leaves of T in the component of $T \setminus e$ not containing v (as defined in Lemma 5.4). We may assume that $X_{ev} \ne A$ for every $v \in V(T)$ and every edge e incident to v, otherwise B is k-branched.

Let s be a vertex satisfying Lemma 5.4, let e_1, e_2, and e_3 be the edges of T incident with s, and let X_i denote $X_{e_i s}$ for each $i \in \{1, 2, 3\}$. Note that $\rho_G(X_i \cap A) \ge \rho_G(A)$ for some $i \in \{1, 2, 3\}$; suppose that $\rho_G(X_1 \cap A) \ge \rho_G(A)$. Then by submodularity,

$$\begin{aligned}\rho_G((X_2 \cup X_3) \cap B) &= \rho_G(X_1 \cup A) \\ &\le \rho_G(X_1) + \rho_G(A) - \rho_G(X_1 \cap A) \\ &\le \rho_G(X_1) \le k.\end{aligned}$$

Now we construct a branching $(T', r, \mathcal{L}')$ of B; let T' be a tree obtained from the minimum subtree of T containing both e_1 and nodes in $\mathcal{L}(B)$ by subdividing e_1 with a vertex b, adding a new leaf r adjacent to b, and contracting one of incident edges of each degree-2 vertex until no degree-2 vertices are left. For each $x \in B$, we define $\mathcal{L}'(x)$ to be a leaf of T' induced by $\mathcal{L}(x)$. Then $(T', r, \mathcal{L}')$ is a branching.

It is easy to see that $\rho_G(T'_e) \leq k$ for all e in T' by Lemma 5.4. So, B is k-branched. □

We continue to follow [26]. Let $\mathbb{Z}^+$ be the set of nonnegative integers. Let $g : \mathbb{Z}^+ \to \mathbb{Z}^+$ be a function. A graph G is called *(m, g)-connected* if for every partition (A, B) of $V(G)$, $\rho_G(A) = l < m$ implies either $|A| \leq g(l)$ or $|B| \leq g(l)$.

Lemma 6.26. *Let $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ be a nondecreasing function. Let G be a (m, f)-connected graph and let $v \in V(G)$ and $vw \in E(G)$. Then either $G \setminus v$ or $G \wedge vw \setminus v$ is $(m, 2f)$-connected.*

Proof. The proof for matroids in Geelen et al. [26, Lemma 3.1] works for general graphs. For the completeness of this paper, the proof is included here.

Suppose not. There are partitions (X_1, X_2), (Y_1, Y_2) of $V(G) \setminus \{v\}$ such that

$$\begin{aligned} a &= \rho_{G\setminus v}(X_1) < m, & |X_1| &> 2f(a), & |X_2| &> 2f(a), \\ b &= \rho_{G\wedge vw\setminus v}(Y_1) < m, & |Y_1| &> 2f(b), & |Y_2| &> 2f(b). \end{aligned}$$

We may assume that $a \geq b$ by replacing G by $G \wedge vw$. We may assume that $|X_1 \cap Y_1| > f(a)$ by swapping Y_1 and Y_2.

By Lemma 3.27, we obtain

$$\rho_G(X_1 \cap Y_1) + \rho_G(X_2 \cap Y_2) \leq a + b + 1.$$

Thus, either $\rho_G(X_1 \cap Y_1) \leq a$ or $\rho_G(X_2 \cap Y_2) \leq b$. So, either $|X_1 \cap Y_1| \leq f(a)$ or $|X_2 \cap Y_2| \leq f(b)$. By assumption, $|X_2 \cap Y_2| \leq f(b)$.

Similarly we apply the same inequality after swapping X_1 and X_2. Either $|X_2 \cap Y_1| \leq f(a)$ or $|X_1 \cap Y_2| \leq f(b)$. Since $|X_1 \cap Y_2| = |Y_2| - |Y_2 \cap X_2| > f(b)$, $|X_2 \cap Y_1| \leq f(a)$.

Then $|X_2| = |X_2 \cap Y_1| + |X_2 \cap Y_2| \leq f(a) + f(b) \leq 2f(a)$. This is a contradiction. □

Let $g(n) = (6^n - 1)/5$. Note that $g(0) = 0$, $g(1) = 1$, and $g(n) = 6g(n-1) + 1$ for all $n \geq 1$.

Lemma 6.27. *Let $k \geq 1$. If G has rank-width larger than k but every proper pivot-minor of G has rank-width at most k, then G is $(k + 1, g)$-connected.*

Proof. We continue to follow the proof of Geelen et al. [26, Lemma 4.1] with a slight modification.

It is easy to see that G is $(1, g)$-connected, because if G is disconnected, then the rank-width of G is the maximum of the rank-width of each component.

Suppose that $m \leq k$ and G is (m, g)-connected and G is not $(m+1, g)$-connected. Then there exists a partition (A, B) with $\rho_G(A) = m$ such that $|A|, |B| > g(m) = 6g(m-1)+1$. Since G has rank-width greater than k, either A or B is not k-branched. We may assume that B is not k-branched. Let $v \in A$. Since G is connected, there is a neighbor w of v in G.

By Lemma 6.26, either $G \setminus v$ or $G \wedge vw \setminus v$ is $(m, 2g)$-connected. Since both $G \setminus v$ and $G \wedge vw \setminus v$ are proper pivot-minors of G, they have rank-width at most k.

We may assume that $G \setminus v$ is $(m, 2g)$-connected by swapping G and $G \wedge vw$. Let (A_1, A_2, A_3) be a partition of $A \setminus \{v\}$. Since $|A| > 6g(m-1)+1$, $|A_i| > 2g(m-1)$ for some $i \in \{1, 2, 3\}$. Since $G \setminus v$ is $(m, 2g)$-connected and $|B| > 2g(m-1)$,

$$\rho_{G\setminus v}(A_i) \geq m \geq \rho_{G\setminus v}(A \setminus \{v\}).$$

Therefore by Lemma 6.25, B is k-branched in $G \setminus v$. Since B is not k-branched in G, there exists $X \subseteq B$ such that $\rho_G(X) = \rho_{G\setminus v}(X)+1$. Let $M = A(G)$ be the adjacency matrix of G over GF(2). By submodular inequality (Proposition 3.2), we obtain

$$\begin{aligned}\rho_{G\setminus v}(B) + \rho_G(X) &= \mathrm{rk}(M[B, V(G) \setminus B \setminus \{v\}]) + \mathrm{rk}(M[X, V(G) \setminus X]) \\ &\geq \mathrm{rk}(M[B, V(G) \setminus B]) + \mathrm{rk}(M[X, V(G) \setminus X \setminus \{v\}]) \\ &= \rho_G(B) + \rho_{G\setminus v}(X) \\ &= \rho_G(B) + \rho_G(X) - 1,\end{aligned}$$

and therefore $\rho_{G\setminus v}(B) = \rho_G(B) - 1 = m - 1$. But this is a contradiction because $G \setminus v$ is $(m, 2g)$-connected. □

Theorem 6.28. *Let $k \geq 1$. If G has rank-width larger than k but every proper pivot-minor of G has rank-width at most k, then $|V(G)| \leq (6^{k+1} - 1)/5$.*

Proof. Let $v \in V(G)$. Since G is connected, pick w such that $vw \in E(G)$. We may replace G by $G \wedge vw$, and hence we may assume that $G \setminus v$ is $(k+1, 2g)$-connected. Since $G \setminus v$ has rank-width k, there exists a partition (X_1, X_2) of $V(G) \setminus \{v\}$ such that $|X_1|, |X_2| \geq \frac{1}{3}(|V(G)| - 1)$ and $\rho_{G\setminus v}(X_1) \leq k$. By $(k+1, 2g)$-connectivity, either $|X_1| \leq 2g(k)$ or $|X_2| \leq 2g(k)$. Therefore, $|V(G)| - 1 \leq 6g(k)$ and consequently $|V(G)| \leq 6g(k) + 1 = g(k+1)$. □

One of the main corollary of the above theorem is the following corollary. This corollary will be used in Chapter 7 to construct a polynomial-time algorithm to recognize graphs of rank-width at most k.

Corollary 6.29. *For each $k \geq 0$, there is a finite list $\mathcal{C}_k$ of graphs having at most $\max((6^{k+1} - 1)/5, 2)$ vertices such that a graph has rank-width at most k if and only if no graph in $\mathcal{C}_k$ is isomorphic to a vertex-minor of G.*

Proof. If $k = 0$, then we let K_2 be a graph with two vertices and one edge joining them and let $\mathcal{C}_0 = \{K_2\}$. Since a graph G has rank-width 0 if and only if G has no edge, the rank-width of G is 0 if and only if K_2 is not isomorphic to a vertex-minor of G. Now we may assume that $k \geq 1$.

Let $\mathcal{C}_k$ be the set of graphs H with $V(H) = \{1, 2, \ldots, n\}$ for some integer n such that $\mathrm{rwd}(H) > k$ and every proper vertex-minor has rank-width at most k. By Theorem 6.28, $\mathcal{C}_k$ is finite and each graph in $\mathcal{C}_k$ has at most $(6^{k+1} - 1)/5$ vertices.

Suppose the rank-width of a graph G is at most k. Since every graph in $\mathcal{C}_k$ has rank-width larger than k, no graph in $\mathcal{C}_k$ is isomorphic to a vertex-minor of G.

Conversely, suppose that the rank-width of a graph G is larger than k. Let H be a proper vertex-minor of G with the minimum number of vertices such that $\mathrm{rwd}(H) > k$. Then there exists a graph $H' \in \mathcal{C}_k$ isomorphic to H. □

Let us discuss this corollary when $k = 1$. We obtain $\mathcal{C}_1$ such that every graph in $\mathcal{C}_1$ has at most 7 vertices. Then what is $\mathcal{C}_1$? In Section 3.3, we proved that a graph has rank-width at most 1 if and only if it is distance-hereditary. Bouchet [4, 6] proved that a graph is distance-hereditary if and only if it has no vertex-minor isomorphic to the 5-cycle. So, $C_1 = \{\text{5-cycle}\}$.

By Corollary 3.22, Theorem 6.28 implies the following corollary, which is a special case of Geelen et al. [26, Theorem 1.1].

Corollary 6.30. *Let $k \geq 2$. If a binary matroid $\mathcal{M}$ has branch-width larger than k but every proper minor of $\mathcal{M}$ has branch-width at most k, then $|E(\mathcal{M})| \leq (6^k - 1)/5$.*

Chapter 7

Recognizing Rank-width

7.1 Approximating rank-width quickly

In this section, we show that, for fixed k, there is a $O(n^4)$-time algorithm that, with a n-vertex graph, outputs a rank-decomposition of width at most $3k+1$ or confirms that the input graph has rank-width larger than k. Since rank-width is defined as branch-width of the cut-rank function, it is easy to see from Corollary 2.13 that we have a $O(n^9 \log n)$-time algorithm using algorithms that can minimize any submodular functions. To obtain a $O(n^4)$-time algorithm, we construct a direct combinatorial algorithm that minimizes the cut-rank function so that we can obtain it faster. The main idea of this section was due to Jim Geelen (private communication).

We first define a *blocking sequence*, defined by J. Geelen [25]. Let G be a graph and A, B be two disjoint subsets of $V(G)$. A sequence $v_1, v_2, \ldots, v_m$ of vertices in $V(G) \setminus (A \cup B)$ is called a *blocking sequence* for (A, B) in G if it satisfies the following:

(i) $\rho^*_G(A, B \cup \{v_1\}) > \rho^*_G(A, B)$.

(ii) $\rho^*_G(A \cup \{v_i\}, B \cup \{v_{i+1}\}) > \rho^*_G(A, B)$ for all $i \in \{1, 2, \ldots, m-1\}$.

(iii) $\rho^*_G(A \cup \{v_m\}, B) > \rho^*_G(A, B)$.

(iv) No proper subsequence satisfies (i)—(iii).

The following proposition is used in most applications of blocking sequences.

Proposition 7.1. *Let G be a graph and A, B be two disjoint subsets of $V(G)$. The following are equivalent:*

(i) *There is no blocking sequence for (A, B) in G.*

(ii) *There exists Z such that $A \subseteq Z \subseteq V(G) \setminus B$ and $\rho_G(Z) = \rho^*_G(A, B)$.*

Proof. (i)→(ii): We assume that $a, b \notin V(G) \setminus (A \cup B)$ by relabeling. Let $k = \rho^*_G(A, B)$. We construct the *auxiliary digraph* $D = (\{a, b\} \cup (V(G) \setminus (A \cup B)), E)$ from G such that for $x, y \in V(G) \setminus (A \cup B)$,

i) $(a, x) \in E$ if $\rho^*_G(A, B \cup \{x\}) > k$,

ii) $(x, b) \in E$ if $\rho^*_G(A \cup \{x\}, B) > k$,

iii) $(x, y) \in E$ if $\rho^*_G(A \cup \{x\}, B \cup \{y\}) > k$.

Since there is no blocking sequence for (A, B) in G, there is no directed path from a to b in D. Let J be a set of vertices in $V(G) \setminus (A \cup B)$ having a directed path from a in D. We show that $Z = J \cup A$ satisfies $\rho_G(Z) = k$.

To prove this, we claim that $\rho^*_G(A \cup X, B \cup Y) = k$ for all $X \subseteq J$, $Y \subseteq V(G) \setminus (Z \cup B)$. We proceed by induction on $|X| + |Y|$.

If $|X| \leq 1$ and $|Y| \leq 1$, then we have $\rho^*_G(A \cup X, B \cup Y) = k$ by the construction of J.

If $|X| > 1$, then for all $x \in X$ we have

$$\rho^*_G(A \cup X, B \cup Y) + \rho^*_G(A, B \cup Y) \leq \rho^*_G(A \cup (X \setminus \{x\}), B \cup Y) + \rho_G(A \cup \{x\}, B \cup Y) = 2k,$$

because $\rho^*_G(A \cup \{x\}, B \cup Y) = k$ by induction. So, $\rho^*_G(A \cup X, B \cup Y) = k$.

Similarly if $|Y| > 1$, then for all $y \in Y$ we have $\rho^*_G(A \cup X, B \cup Y) + \rho^*_G(A \cup X, B) \leq \rho^*_G(A \cup X, B \cup (Y \setminus \{y\})) + \rho_G(A \cup X, B \cup \{y\}) = 2k$, and therefore $\rho^*_G(A \cup X, B \cup Y) = k$.

(ii)→(i): Suppose that there is a blocking sequence $v_1, v_2, \ldots, v_m$. Then, $v_m \notin Z$ because $\rho^*_G(A \cup \{v_m\}, B) > \rho_G(Z)$. Similarly $v_1 \in Z$ because $\rho^*_G(A, B \cup \{v_1\}) > \rho_G(Z)$. Therefore there exists $i \in \{1, 2, \ldots, m-1\}$ such that $v_i \in Z$ but $v_{i+1} \notin Z$. But this is a contradiction, because $\rho_G(Z) < \rho^*_G(A \cup \{v_i\}, B \cup \{v_{i+1}\}) \leq \rho^*_G(Z, V(G) \setminus Z) = \rho_G(Z)$. □

Lemma 7.2. *Let G be a graph (V, E) and A, B be two disjoint subsets of V such that $\rho^*_G(A, B) = k$ and $|A|, |B| \leq l$. Let $n = |V|$. There is a polynomial-time algorithm to either*

- *obtain a graph G' locally equivalent to G with $\rho^*_{G'}(A, B) > k$, or*
- *obtain a set Z such that $A \subseteq Z \subseteq V \setminus B$ and $\rho_G(Z) = k$.*

The running time of this algorithm is $O(n^3)$ if l is fixed or $O(n^4)$ if l is not fixed.

Proof. If there is no blocking sequence for (A, B) in G, then $\min_{A \subseteq Z \subseteq V \setminus B} \rho(Z) = k$ by Proposition 7.1. In this case, we obtain Z by finding a set of vertices reachable from A in the auxiliary graph.

Therefore, we may assume that there is a blocking sequence $v_1, v_2, \ldots, v_m$. We will find another graph G' locally equivalent to G such that $\mathrm{rk}_{G'}(A, B) > k$. Since $\mathrm{rk}_G(A \cup \{v_m\}, B) = k + 1$, there is a vertex $w \in B$ adjacent to v_m.

(1) We claim that $v_1, v_2, \ldots, v_{m-1}$ is a blocking sequence of (A, B) in $G \wedge v_m w$ if $m > 1$.

By applying Lemma 3.27 for $G[A \cup B \cup \{v_1, v_m\}]$, a subgraph of G induced by $A \cup B \cup \{v_1, v_m\}$, we have

$$\rho^*_{G \wedge v_m w}(A, B \cup \{v_1\}) + \rho^*_G(A \cup \{v_1\}, B) \geq \rho^*_G(A, B \cup \{v_1, v_m\}) + \rho^*_G(A \cup \{v_1, v_m\}, B) - 1.$$

Since $\rho^*_G(A, B \cup \{v_1, v_m\} \geq \rho^*_G(A, B \cup \{v_1\}) \geq k+1$, $\rho^*_G(A \cup \{v_1, v_m\}, B) \geq \rho^*_G(A \cup \{v_m\}, B) \geq k+1$, and $\rho^*_G(A \cup \{v_1\}, B) = k$, we obtain that $\rho^*_{G\wedge v_m w}(A, B \cup \{v_1\}) \geq k+1$.

By applying the same inequality we obtain that $\rho^*_{G\wedge v_m w}(A \cup \{v_i\}, B \cup \{v_{i+1}\}) + \rho^*_G(A \cup \{v_i, v_{i+1}\}, B) \geq \rho^*_G(A \cup \{v_i\}, B \cup \{v_{i+1}, v_m\}) + \rho^*_G(A \cup \{v_i, v_{i+1}, v_m\}, B) - 1 \geq 2k+1$ for each $i \in \{1, 2, 3, \ldots, m-2\}$ and therefore $\rho^*_{G\wedge v_m w}(A \cup \{v_i\}, B \cup \{v_{i+1}\}) \geq k+1$.

Moreover, $\rho^*_{G\wedge v_m w}(A \cup \{v_{m-1}\}, B) + \rho^*_G(A \cup \{v_{m-1}\}, B) \geq \rho^*_G(A \cup \{v_{m-1}\}, B \cup \{v_m\}) + \rho^*_G(A \cup \{v_{m-1}, v_m\}, B) - 1 \geq 2k+1$ and therefore $\rho^*_{G\wedge v_m w}(A \cup \{v_{m-1}\}, B) \geq k+1$.

We prove one lemma to be used later. If X and Y are disjoint subsets of V such that $A \subseteq X$, $B \subseteq Y$, $v_m \notin X \cup Y$ and $\rho^*_G(X, Y) = k$, then $\rho^*_{G\wedge v_m w}(X, Y) = \rho^*_G(X, Y \cup \{v_m\})$ because

$$\begin{aligned}
\rho^*_{G\wedge v_m w}(X, Y) + \rho^*_G(X, Y) &\geq \rho^*_G(X, Y \cup \{v_m\}) + \rho^*_G(X \cup \{v_m\}, Y) - 1 \\
&\geq \rho^*_G(X, Y \cup \{v_m\}) + k \\
&= \rho^*_{G\wedge v_m w}(X, Y \cup \{v_m\}) + \rho^*_G(X, Y).
\end{aligned}$$

By letting $X = A \cup \{v_{m-1}\}$ and $Y = B$, we obtain that $\rho^*_{G\wedge v_m w}(A \cup \{v_{m-1}\}, B) = \rho^*_G(A \cup \{v_{m-1}\}, B \cup \{v_m\}) \geq k+1$. We also obtain $\rho^*_{G\wedge v_m w}(A, B \cup \{v_i\}) = k$ for each $i > 1$ by letting $X = A$, $Y = B \cup \{v_i\}$. Similarly we obtain $\rho^*_{G\wedge v_m w}(A \cup \{v_i\}, B \cup \{v_j\}) = k$ for i, j such that $1 \leq i < i+1 < j \leq m-1$.

Therefore, $v_1, v_2, \ldots, v_{m-1}$ is a blocking sequence for (A, B) in $G \wedge v_m w$.

(2) If $m = 1$ then we obtain $\rho^*_{G\wedge v_1 w}(A, B) \geq k+1$, by applying the previous lemma with letting $X = A$ and $Y = B$.

(3) For each k, we claim that we can obtain another graph G' locally equivalent to G with $\rho^*_{G'}(A, B) > k$ or find Z satisfying $A \subset Z \subseteq V \setminus B$ and $\rho_G(Z) = k$.

If l is fixed, then we can test an adjacency in the auxiliary graph (defined in the proof of Proposition 7.1) in constant time by calculating rank of matrices of size no bigger than $(l+1) \times (l+1)$, and therefore it takes $O(n^2)$ time to construct the auxiliary digraph. If l is not fixed, then it takes $O(n^4)$ time to construct the auxiliary digraph for finding a blocking sequence. We first obtain the diagonalized matrix R obtained by applying elementary row operations to the matrix $M[A, B]$ in $O(n^3)$ time. For each vertex v not in $A \cup B$, we calculate the rank of $M[A \cup \{v\}, B]$ by using the stored matrix in $O(n^2)$ time. Similarly we calculate the rank of $M[A, B \cup \{v\}]$ by storing the matrix obtained by applying elementary column operations to $M[A, B]$. To check whether $\rho^*_G(A \cup \{x\}, B \cup \{y\}) > k$, it is enough to see when $\rho^*_G(A \cup \{x\}, B) = \rho^*_G(A, B \cup \{y\}) = k$. We first store the rows of the original matrices to each column of R and then we obtain the linear combination of rows of $M[A, B]$ giving $M[\{x\}, B]$. By the same linear combination, we check whether rows of $M[A, \{y\}]$ gives $M[\{x\}, \{y\}]$. It takes $O(n^2)$ time for each $x, y \in V \setminus (A \cup B)$ and therefore we construct the auxiliary digraph in $O(n^4)$ time (if l is not fixed).

To find a blocking sequence, it is enough to find a shortest path in this digraph and it takes $O(n^2)$ time. If there is no blocking sequence, then we find Z in $O(n^2)$ time by choosing all vertices reachable from A by a directed path.

We pick a neighbor of v_m in B and obtain $G \wedge v_m w$ in $O(n^2)$ time. By (1), $G \wedge v_m w$

has a blocking sequence $v_1, v_2, \ldots, v_{m-1}$ for (A, B). We apply this kind of pivoting m times so that in the new graph G' we have $\rho^*_{G'}(A, B) > k$. Since $m \leq n$, we obtain G' in $O(n^3)$ time. □

Theorem 7.3. *Let G be a graph (V, E) and A, B be two disjoint subsets of V. Then, there is a $O(n^5)$-time algorithm to find Z with $A \subseteq Z \subseteq V \setminus B$ having the minimum cut-rank.*

Proof. We apply the algorithm given by Lemma 7.2 until it finds a cut. We will call the algorithm at most n times, and therefore the running time is at most $O(n^5)$. □

Theorem 7.4. *Let l be a fixed constant. Let G be a graph (V, E) and A, B be two disjoint subsets of V such that $|A|, |B| \leq l$. Then, there is a $O(n^3)$-time algorithm to find Z with $A \subseteq Z \subseteq V \setminus B$ having the minimum cut-rank.*

Proof. We apply the algorithm given by Lemma 7.2 until it finds a cut. We will call the algorithm at most l times, and therefore the running time is at most $O(n^3)$. □

Now we pay attention to our rank-width approximation algorithm, described in Corollary 2.13. We continue running time analysis of Theorem 2.10 done in Section 2.4. For rank-width, we are given the natural interpolation ρ^*_G of the cut-rank function ρ_G. It takes $O(n^2)$ time to find a set $X \subseteq V(G) \setminus B$ such that $\rho^*_G(X, B) = \rho_G(B)$, because we know that $\rho_G(B) \leq k$. To show that X is not well-linked, we use Theorem 7.4 and this can be done in $O(n^3)$ time. Since the process is cycled through at most $O(n)$ times, it follows that the time complexity of obtaining a rank-decomposition or a well-linked set is $O(n^4)$.

Theorem 7.5. *For given k, there is an algorithm, for the input graph $G = (V, E)$, that either concludes that $\mathrm{rwd}(G) > k$ or outputs a rank-decomposition of G of width at most $3k + 1$; and its running time is $O(|V|^4)$.*

7.2 Approximating rank-width more quickly

In this section, we show another algorithm that approximate rank-width as in the previous section, but in $O(n^3)$ time with a worse approximation ratio. The algorithm in Section 7.1 was based on the idea of Theorem 2.10 with a quick method to find a minimum of cut-rank functions. However, in this section we take a different approach based on simple observation in Section 5.1. We use the following algorithm developed by Hliněný [32].

Theorem 7.6 (Hliněný [32, Theorem 4.12]). *For fixed k, there is a $O(n^3)$-time algorithm that, for a given binary matroid with n elements, obtains a branch-decomposition of width at most $3k+1$ or confirms that the given matroid has branch-width larger than $k+1$. We assume that binary matroids are given by their matrix representations.*

This algorithm can be used to approximate rank-width of a bipartite graph G because we can run this algorithm for binary matroids having G as a fundamental graph. By Lemma 5.3, we obtain a bipartite graph $B(G)$ for each graph G such that $\operatorname{rwd}(B(G)) = \max(2\operatorname{rwd}(G), 1)$. Moreover we can construct $B(G)$ in $O(n^2)$ time when $n = |V(G)|$. It is unclear whether we can transform the rank-decomposition of $B(G)$ of width k into a rank-decomposition of G of width at most $k/2$ in $O(n^3)$ time. Instead we show that it is easy to transform the rank-decomposition of $B(G)$ of width k into a rank-decomposition of G of width at most $4k$.

Lemma 7.7. *Let G be a graph (V, E). Let $(T, \mathcal{L})$ be a rank-decomposition of $B(G)$ of width k and T' be the minimum subtree of T containing every leaf in $\mathcal{L}^{-1}(V(G)\times\{1\})$. Let $\mathcal{L}'(v) = \mathcal{L}((v, 1))$. Then, $(T', \mathcal{L}')$ is a rank-decomposition of G of width at most $4k$.*

Proof. Let e be an edge of T, and (X, Y) be a partition of leaves of T induced by connected components of $T \setminus e$.

For four subsets A_1, A_2, A_3, A_4 of V, we denote $A_1|A_2|A_3|A_4 = (A_1 \times \{1\}) \cup (A_2 \times \{2\}) \cup (A_3 \times \{3\}) \cup (A_4 \times \{4\})$ to simplify our notation. Let $\mathcal{L}^{-1}(X) = A_1|A_2|A_3|A_4$. Let $B_i = V \setminus A_i$ for $i \in \{1, 2, 3, 4\}$.

It is easy to observe, for each $i \in \{1, 2, 3\}$, that $\rho^*_{B(G)}((A_i \times \{i\}) \cup (A_{i+1} \times \{i + 1\}), (B_i \times \{i\}) \cup (B_{i+1} \times \{i + 1\}) = |A_i \cap B_{i+1}| + |B_i \cap A_{i+1}| = |A_i \Delta A_{i+1}|$. Since $\rho_{B(G)}(A_1|A_2|A_3|A_4) = \rho^*_{B(G)}(A_1|A_2|A_3|A_4, B_1|B_2|B_3|B_4) \le k$, we have, for each $i \in \{1, 2, 3\}$,

$$|A_i \Delta A_{i+1}| \le \rho_{B(G)}(A_1|A_2|A_3|A_4) \le k.$$

By adding these inequalities for all i, we obtain that $|A_1 \Delta A_4| \le 3k$.

Let M be an adjacency matrix of G. We observe that $\operatorname{rk}(M[A_4, B_1]) = \rho_{B(G)}(A_4 \times \{4\}, B_1 \times \{1\}) \le k$. Then we have the following upper bound of $\rho_G(A_1)$:

$$\begin{aligned}\rho_G(A_1) &= \operatorname{rk}(M[A_1, B_1])\\ &\le \operatorname{rk}(M[A_4 \cup (A_4 \Delta A_1), B_1])\\ &\le \operatorname{rk}(M[A_4, B_1]) + \operatorname{rk}(M[A_4 \Delta A_1, B_1])\\ &\le 4k.\end{aligned}$$

So $(T', \mathcal{L}')$ is a rank-decomposition of G and its width is at most $4k$. □

Therefore, we obtain the following algorithm.

Corollary 7.8. *For fixed k, there is a $O(n^3)$-time algorithm that, for a given graph with n vertices, obtains a rank-decomposition of width at most $24k$ (while confirming that the rank-width of the input graph is at most $3k$) or confirms that the rank-width of the input graph is larger than k.*

Proof. Let $G = (V, E)$ be the input graph. We may assume that $E(G) \ne \emptyset$. First we construct $B(G)$ in $O(n^2)$ time. We run the algorithm of Theorem 7.6 with an input $\mathcal{M} = Bin(B(G), V \times \{1, 3\}, V \times \{2, 4\})$ and a constant $2k$.

If it confirms that branch-width of $\mathcal{M}$ is larger than $2k+1$, then rank-width of $B(G)$ is larger than $2k$, and therefore the rank-width of G is larger than k.

If it outputs the branch-decomposition of $\mathcal{M}$ of width at most $6k+1$, then the output is a rank-decomposition of $B(G)$ of width at most $6k$. This confirms that the rank-width of G is at most $3k$. This can be transformed into a rank-decomposition of G of width at most $24k$ in linear time by Lemma 7.7. (We use the depth-first-search algorithm from one leaf of T corresponding to a vertex in $V(G) \times \{1\}$.) □

7.3 Recognizing rank-width

By Corollary 6.2, for a fixed k, there are only finitely many graphs, such that a graph does not contain any of them as a vertex-minor if and only if it has rank-width at most k. By Theorem 4.23.2, for any fixed graph H, there is a C_2MS formula expressing that H is isomorphic to a vertex-minor of an input graph. Let n be the number of vertices in the input graph. By Corollary 7.8, we have a $O(n^3)$-time algorithm that either confirms the input graph has rank-width at least $k+1$ or outputs a rank-decomposition of width at most $24k$. In Proposition 3.4, we develop a $O(n^2)$-time algorithm that converts the rank-decomposition into a k-expression. In Section 4.3, we recall that any property specified by a CMS formula can be checked in linear time on graphs given by k-expressions.

By combining all of these, we obtain the following theorem.

Theorem 7.9. *For fixed k, there is a $O(n^3)$-time algorithm to check that the input graph with n vertices has rank-width at most k.*

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