VERTEX-MINORS, MONADIC SECOND-ORDER LOGIC, AND A CONJECTURE BY SEESE.

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Abstract. We prove that one can express the vertex-minor relation on finite undirected graphs by formulas of Monadic Second-order logic with a predicate expressing that a set has even cardinality. We obtain a slight weakening of a conjecture by D. Seese stating that sets of graphs having a decidable satisfiability problem for monadic second-order logic have bounded clique-width. We also obtain a polynomial-time algorithm to decide rank-width at most k for any fixed k. The proofs use the notion of isotropic system.

1. Introduction

The notion of *tree-width* of a graph, based on that of *tree-decomposition*, plays an essential role in the theory of graph minors because a set of graphs has bounded tree-width iff some planar graph is not a minor of any graph in this set, and also because the set of graphs of tree-width at most k, for any fixed k, is well-quasi-ordered by minor inclusion.

tree-width
tree-decomposition

Tree-width is also important for the construction of polynomial-time algorithms because many hard problems (in particular NP complete problems like 3-colorability) have polynomial-time algorithms on the set of graphs of tree-width at most k, for each k. Every graph problem specified by a formula of *Monadic Second-order logic* has such algorithms. (Monadic Second-order logic, MS logic in short, is the extension of First-Order logic with set variables. In this language, one can write properties of the form "there exist sets of vertices such that ...". This result actually holds for the strong version of MS logic, denoted by MS_2, called *Monadic Second-order logic with edge set quantifications* that uses also variables denoting sets of edges. (For the main definitions and results on MS logic and detailed examples of formulas, the reader is refereed to the book chapter [Cou97]. The preliminary sections of any of the articles [Cou94, Cou96, Cou03, Cou04b, CE95] also contain definitions and examples.)

Monadic Second-order logic
Monadic Second-order logic with edge set quantifications

Finally, MS_2 logic is decidable on the set of graphs of tree-width at most k. There is even a kind of converse, that we will call Seese's Theorem [See91], stating that if a set of graphs has a decidable satisfiability problem for MS_2 formulas, then it has bounded tree-width. The proof rests upon the result by Robertson and Seymour [RS86] that if a set of finite graphs has unbounded tree-width, every square grid is isomorphic to a minor of some of its graphs.

The *clique-width* of a graph is also an important notion for the construction of polynomial-time graph algorithms. It is based on certain hierarchical graph decompositions, and every graph problem specified by a formula of MS logic (without edge set quantifications) has a polynomial-time algorithm on the set of graphs of

clique-width

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Counting Monadic Second-order logic

clique-width at most k. MS logic is also decidable on the set of graphs of clique-width at most k. These results actually hold for the extension of MS logic, called *Counting Monadic Second-order logic* (CMS logic in short) using atomic formulas of the form $Card_p(X)$, expressing that the cardinality of a set X is a multiple of p, where p is an integer > 1.

The statement analogous to Seese's Theorem for MS formulas (without edge set quantifications) is a conjecture, also made by D. Seese in [See91], for which we prove here a weaker statement. This conjecture says that if a set of graphs has a decidable satisfiability problem for MS formulas, then it has bounded clique-width. Its hypothesis concerns less formulas, hence is weaker than that of Seese's Theorem. Since a set of graphs has bounded clique-width if it has bounded tree-width, Seese's Theorem is actually a weakening of the Conjecture.

C_2MS

We will actually prove a slight weakening of the Conjecture, by assuming that the considered sets of graphs has a decidable satisfiability problem for *C_2MS* formulas, i.e., for MS formulas that can be written with the set predicate $Card_2(X)$, that we will write $Even(X)$ for simplicity. Hence, $\mathrm{C_2MS}$ is a sublanguage of CMS, strictly more expressive than MS.

rank-width

vertex-minors

Our proof uses actually the notion of *rank-width*, introduced by S. Oum and P. Seymour [OS04]. It is equivalent to clique-width in the sense that a set of graphs has bounded rank-width iff it has bounded clique-width. Furthermore, the set of graphs of rank-width at most k is characterized by a finite set of excluded *vertex-minors*, a crucial notion that has for rank-width the good properties that minors have for tree-width.

local complementations

defined inside G by C_2MS formulas

isotropic system

A graph G contains H as a vertex-minor if H is an induced subgraph of some graph K, that can be obtained from G by a sequence of *local complementations*. A local complementation consists of choosing a vertex x and edge-complementing the subgraph induced by the neighbors of x. We prove that the vertex-minors H of G can be *defined inside G by C_2MS formulas*. This is not at all obvious because local complementations relative to neighbors can interact in quite complicated ways. However, we can do so by using the notion of *isotropic system*, introduced by A. Bouchet [Bou87, Bou88a] which represents graphs by certain vector spaces over GF(2) and makes it possible to handle algebraically local complementations. The corresponding computations can be formalized in $\mathrm{C_2MS}$ logic. The summations in GF(2) necessitate the use of the even cardinality set predicate.

Two main results follow from these constructions. First, the set of graphs of rank-width at most k, for any k, is characterized by a $\mathrm{C_2MS}$ formula. With results of [OS04], this gives a polynomial-time algorithm for deciding whether a graph has rank-width at most k. This can be contrasted with the case of clique-width: it can be decided in polynomial time if a graph has clique-width at most k for each $k \leq 3$. For $k > 3$, the complexity of the problem is still unknown.

The second result is the above discussed weakening of Seese's Conjecture. This latter result extends to countable graphs.

This article is organized as follows. Sections 2, 3, 4 review definitions, notation and results on graphs, matroids, isotropic systems, and the relationships between these different notions. Section 5 reviews Monadic Second-order logic and its use for expressing properties and transformations of graphs, matroids ans isotropic systems. The various forms of Seese's Conjecture are recalled in this section. In section 6, we show how the notion of a vertex-minor can be formalized in $\mathrm{C_2MS}$

logic. This formalization is done via a logical formalization of isotropic systems, of their minors and of their so-called *fundamental graphs*. The application to the recognition of graphs of given rank-width follows then. These constructions are applied in Section 7 to the proof of our weakening of Seese's Conjecture. In Section 8 we give an alternative proof of it based on binary matroids and using results by Geelen, Gerards, and Whittle [GGW03] and Hliněny and Seese [HS04]. Section 9 is a conclusion. *fundamental graphs*

2. Graphs, clique-width and rank-width

In this section, we review the notion of clique-width, and give a survey of results about rank-width, which will be necessary to understand this paper. We assume graphs are undirected, loop-free, without multiple edges and finite, except at the end of section 7 where we discuss countable graphs.

2.1. **Definitions of clique-width and rank-width.** A *graph* is defined as a pair (V, E) where V is the set of vertices and E is the set of edges. We write $V(G)$ and $E(G)$, or sometimes V_G and E_G to specify the graph under consideration. *graph*

Clique-width is, like tree-width and branch-width, a graph complexity measure. It is defined in terms of algebraic expressions denoting graphs up to isomorphism. The operations used in these expressions have been introduced in [CER93] for denoting hypergraphs. Their restriction to graphs yields the notion of clique-width which has been defined and investigated first in Courcelle and Olariu [CO00], and then in subsequent papers among which we quote Corneil et al. [CHL$^+$00]. *Clique-width*

Let k be a positive integer. A *k-graph* is a graph given with a total mapping from its vertices to $[k] = \{1, ..., k\}$, denoted by lab. Hence it is defined as a triple (V, E, lab). We call $lab(v)$ the *label* of a vertex v. The operations on k-graphs are the following ones: *k-graph* *label*

(1) For each $i = 1, ..., k$, we define a constant $\mathbf{i}$ for denoting an isolated vertex labeled by i.
(2) For $i, j \in [k]$ with $i \neq j$, we define a unary function $\eta_{i,j}$ such that
$$\eta_{i,j}(V, E, lab) = (V, E', lab)$$
where E' is E augmented with the set of all edges linking a vertex labeled i and one labeled j.
(3) We let also $\rho_{i \to j}$ be the unary function such that
$$\rho_{i \to j}(V, E, lab) = (V, E, lab')$$
where
$$lab'(v) = \begin{cases} j & \text{if } lab(v) = i, \\ lab(v) & \text{otherwise.} \end{cases}$$
This mapping relabels into j every vertex labeled by i.
(4) Finally, we use the binary operation $\oplus$ that makes the union of two disjoint copies of its arguments. Hence $G \oplus G \neq G$. Its size is twice that of G.

A well-formed expression t over these symbols will be called a *k-expression*. Its *value* is a k-graph $G = val(t)$. The set of vertices of $val(t)$ can be defined as the set of occurrences of the constant symbols in t. However, we will also consider that an expression t designates all graphs isomorphic to $val(t)$. A graph is considered as a 1-graph whose vertices are (necessarily) labeled by 1. The *clique-width* of a graph *k-expression* *value* *clique-width*

G, denoted by $\mathrm{cwd}(G)$ is the minimal k such that $G = val(t)$ for some k-expression t.

Example 2.1. The set of graphs of clique-width 1 is the set of graphs without edges. The set of graphs of clique-width at most 2 is the set of cographs, which are the graphs with no P_4 induced subgraph, see [CO00]. (P_4 is a path of 4 vertices.)

In this paper, the notion of rank-with, introduced in [OS04], is useful. Since it is rather new, we describe it in detail.

Definition 2.2. Let's define rank, cut-rank, rank-decomposition, and rank-width.

Let G be a graph and A, B be disjoint subsets of $V(G)$. Let $M_A^B(G)$ be an $A \times B$ matrix $(m_{ij})_{i\in A, j\in B}$ over GF(2) with

$$m_{ij} = \begin{cases} 1 & \text{if } i \text{ is adjacent to } j, \\ 0 & \text{otherwise.} \end{cases}$$

rank of (A, B)
cut-rank

We define the *rank of* (A, B), $\mathrm{rk}_G(A, B)$, as $\mathrm{rank}(M_A^B(G))$, where rank is the matrix rank, calculated over GF(2). The *cut-rank*, $\mathrm{cutrk}_G(A)$ of $A \subseteq V(G)$, is defined by $\mathrm{cutrk}_G(A) = \mathrm{rk}_G(A, V(G) - A)$.

rank-decomposition
width
rank-width

A cubic tree T with a bijection from $V(G)$ to a set of leaves of T is called a *rank-decomposition* of G. For a pair of adjacent nodes u, v in T, let $V_u^v(T)$ be the set of vertices, mapped into leaves linked to v by a path in T not using the edge uv. The *width* of a rank-decomposition T is $\max_{uv\in E(T)} \mathrm{cutrk}_G(V_u^v(T))$. The *rank-width* of a graph G, denoted by $\mathrm{rwd}(G)$, is a minimum k such that there is a rank-decomposition T of width k.

branch-width
connectivity function

Remark. Informally, its definition is a modification of that of *branch-width*, introduced in [RS91]. A. Bouchet defined the cut-rank function under the name of *connectivity function* in [Bou90].

The most important reason why the rank-width is useful to study the clique-width is the following.

Proposition 2.3 ([OS04])**.** *For a graph G, $\mathrm{rwd}(G) \le \mathrm{cwd}(G) \le 2^{\mathrm{rwd}(G)+1} - 1$.*

Furthermore, there is a polynomial-time algorithm to convert any rank-decomposition of width k of G into a $(2^{k+1} - 1)$-expression defining the same graph.

By this inequality, a class $\mathcal{C}$ of graphs is of bounded clique-width iff it is of bounded rank-width.

distance-hereditary graph

Example 2.4. A graph G is a *distance-hereditary graph* if the distance function is the same in any connected induced subgraph and in G. These graphs are those of rank-width at most 1, see [Oum04a]. This gives a new proof of the theorem by [GR99] stating that every distance-hereditary graph has clique-width at most 3 by Proposition 2.3.

2.2. **Algorithmic aspects.** The notion of clique-width has been studied, mainly motivated by the fact that for graphs of clique-width $\le k$, if an input graph is given by the k-expression, then many hard problems can be solved in polynomial time, assuming k is a constant, even if they are NP-complete. For instance, there are polynomial-time algorithms to decide whether a graph has a Hamiltonian path or circuit [EGW01, Wan94] , find the chromatic number [KR03], and more strikingly, for all problems specified in CMS logic, see 5.5.

This approach requires the k-expression be given as an input. This requirement is removed in [OS04].

Theorem 2.5 ([OS04]). *Let k be fixed. There is a $O(n^9 \log n)$-time algorithm that either confirms that the input graph has rank-width at least $k+1$ or outputs some rank-decomposition of width at most $3k+1$.*

By using also Proposition 2.3, the above algorithm can give a (8^k-1)-expression, which now can be used as an input to algorithms based on the k-expression to construct polynomial-time algorithms.

We now have an "approximation" algorithm saying that either the input graph has clique-width at most $f(k)$ or it has clique-width $> k$, where $f(k) = 8^k - 1$. How about recognizing graphs of clique-width at most k? It's easy for $k = 1$. Recognizing cographs [CPS85] was done before the birth of clique-width. For $k = 3$, there is a polynomial-time algorithm [CHL$^+$00]. The complexity of deciding $\mathrm{cwd}(G) \leq k$ is still open for $k > 3$. In this paper, we describe a polynomial-time algorithm to recognize graphs of rank-width at most k for a fixed k. It'll be discussed in Section 6.

2.3. **Vertex-minor and well-quasi-ordering.** It's now well-known that the *minor* relation is crucial to study the properties of graphs with respect to the tree-width. For instance, if H is a minor of G, then the tree-width of H is at most that of G. *minor*

The clique-width of H is at most that of G if H is an *induced subgraph* of G, see [CO00]. But the induced subgraph relation is not rich enough to yield theorems similar to those with the minor relation. For example, there is a list of infinitely many graphs of clique-width at most 4, none of them is an induced subgraph of another. *induced subgraph*

Here, we define a richer relation, vertex-minor, related to rank-width. In fact, the vertex-minor was called l-reduction by A. Bouchet [Bou94]. Note that for sets A and B, $A \Delta B = (A - B) \cup (B - A)$.

Definition 2.6. Let $G = (V, E)$ be a graph and $v \in V$. The graph obtained by *local complementation* at v is defined by $G * v = (V, E \Delta \{xy \mid xv, yv \in E, x \neq y\})$. H is *locally equivalent* to G iff G can be obtained by applying a sequence of local complementations to G. H is a *vertex-minor* of G if H can be obtained by applying a sequence of vertex deletions and local complementations to G. *local complementation* *locally equivalent* *vertex-minor*

Note that if H is locally equivalent to a vertex-minor of G, then it is also a vertex-minor of G.

In [Bou90], it was shown that cut-rank is preserved by local complementations. Therefore, rank-width is preserved too. So, it's easy to see the following proposition.

Proposition 2.7 ([Oum04a]). *If H is a vertex-minor of G, then* $\mathrm{rwd}(H) \leq \mathrm{rwd}(G)$.

A reflexive and transitive binary relation $(Q, \leq)$ is called *well-quasi-ordering*(wqo) if for any infinite sequence $a_1, a_2, a_3, \ldots$ from Q, there exists $i < j$ such that $a_i \leq a_j$. Or, we say Q is *well-quasi-ordered* by $\leq$. *well-quasi-ordering* *well-quasi-ordered*

Theorem 2.8 ([Oum04b]). *A set of graphs of bounded clique-width is well-quasi-ordered by the vertex-minor relation.*

A typical technique gave the following corollary. In fact, this corollary has an elementary proof in [Oum04a].

Corollary 2.9 ([Oum04a, Oum04b]). *There is a finite set $\mathcal{C}_k$ of graphs such that* $\operatorname{rwd}(G) \leq k$ *iff G does not have any vertex-minor isomorphic to a graph in $\mathcal{C}_k$.*

If $\mathcal{C}_k$ contains two graphs H and H' with H' locally equivalent to a graph isomorphic to H, then one can replace $\mathcal{C}_k$ by $\mathcal{C}'_k = C_k - \{H'\}$ and one still obtains a characterization of the graphs of rank-width at most k as those without any vertex-minor isomorphic to a graph in $\mathcal{C}'_k$. Hence, in Corollary 2.9, one can require that $\mathcal{C}_k$ contains no two isomorphic graphs and no two locally equivalent graphs (up to isomorphism).

3. Matroids

In this section, we review concept of a matroid, its connections with bipartite graphs, and the grid theorem for matroids.

matroid
independent

3.1. **Matroid and branch-width.** $\mathcal{M} = (E, \mathcal{I})$ is called a *matroid* if E is a finite set and $\mathcal{I}$ is a set of *independent* subsets of E such that

i) $\emptyset \in \mathcal{I}$,
ii) if $B \in \mathcal{I}$ and $A \subseteq B$, then $A \in \mathcal{I}$,
iii) for any $Z \subseteq E$, the maximal independent subsets of Z have the same size $r(Z)$.

rank

r is called the *rank* function of a matroid $\mathcal{M}$. For more about matroids, we refer to the book by Oxley [Oxl92].

binary matroid

$\mathcal{M} = (E, \mathcal{I})$ is called a *binary matroid* if there exists a matrix N over GF(2) such that E is a set of column vectors of N and

$$\mathcal{I} = \{X \subseteq E : X \text{ is independent as a set of vectors}\}.$$

dual matroid

For a matroid $\mathcal{M} = (E, \mathcal{I})$, $\mathcal{M}^* = (E, \mathcal{I}')$ is the *dual matroid* of $\mathcal{M}$ such that X is independent in $\mathcal{M}^*$ iff there is a maximally independent set B in $\mathcal{M}$ such that $B \cap X = \emptyset$.

deletion
contraction
minor

For $e \in E(\mathcal{M})$, $\mathcal{M} \setminus e$ is a matroid $(E - \{e\}, \mathcal{I}')$ such that X is independent in $\mathcal{M} \setminus e$ iff $X \subseteq E - \{e\}$ is independent in $\mathcal{M}$. This operation is called the *deletion* of e. $\mathcal{M}/e$ is defined by $(\mathcal{M}^* \setminus e)^*$. This operation is called the *contraction* of e. A matroid $\mathcal{N}$ is called a *minor* of $\mathcal{M}$ if $\mathcal{N}$ can be obtained from $\mathcal{M}$ by applying a sequence of deletions and contractions.

connectivity

The *connectivity* $\lambda_{\mathcal{M}}(X)$ of $\mathcal{M} = (E, \mathcal{I})$ is defined by $r(X) + r(E - X) - r(E) + 1$.

branch-decomposition
width
branch-width

A cubic tree T with a bijection from $E(\mathcal{M})$ onto a set of leaves of T is called a *branch-decomposition* of $\mathcal{M}$. For a pair of adjacent nodes u, v in T, let $V_u^v(T)$ be the set of elements, mapped into leaves linked to v by a path in T not using the edge uv. The *width* of a branch-decomposition T is $\max_{uv \in E(T)} \lambda_{\mathcal{M}}(V_u^v(T))$. The *branch-width* of a matroid $\mathcal{M}$ is the minimum k such that there is a branch-decomposition T of width k.

3.2. **Bipartite graphs and binary matroids.** Let $G = (V, E)$ be a bipartite graph with a bipartition $V = A \cup B$. Let $Bin(G, A, B)$ be the binary matroid on V, represented by a $A \times V$ matrix $\begin{pmatrix} I_A & M_A^B(G) \end{pmatrix}$, where I_A is a $A \times A$ identity matrix. (Since $M_A^B(G)$ is $A \times B$ matrix, $\begin{pmatrix} I_A & M_A^B(G) \end{pmatrix}$ is a $A \times V$ matrix.)

It's straightforward to prove the following.

Proposition 3.1 ([Oum04a]). *Let $G = (V, E)$ be a bipartite graph with a bipartition $V = A \cup B$. Let $\mathcal{M} = Bin(G, A, B)$. Then, for any $X \subseteq V$,*

$$\lambda_{\mathcal{M}}(X) = \operatorname{cutrk}_G(X) + 1,$$

and the branch-width of $\mathcal{M}$ is equal to the rank-width of G +1.

Recall that $G * u$ means local complementation defined in Definition 2.6.

Proposition 3.2 ([Oum04a]). *Let $G = (V, E)$ be a bipartite graph with a bipartition $V = A \cup B$. Let $\mathcal{M} = Bin(G, A, B)$. Then,*

(1) $\mathcal{M}^* = Bin(G, B, A)$,
(2) For $v \in B$, $\mathcal{M} \setminus v = Bin(G \setminus v, A, B - \{v\})$.
(3) For $u \in A$, $v \in B$, and $uv \in E(G)$, $\mathcal{M} = Bin(G * u * v * u, (A - \{u\}) \cup \{v\}, (B - \{v\}) \cup \{u\})$.
(4) If $\mathcal{N}$ is a minor of $\mathcal{M}$, then there is a bipartite graph H such that $\mathcal{N} = Bin(H, A', B')$ for a bipartition $V(H) = A' \cup B'$ and H is a vertex-minor of G.
(5) For any bipartite graph H with a bipartition $V(H) = A' \cup B'$, if $\mathcal{M} = Bin(H, A', B')$, then H is locally equivalent to G.

3.3. **Grid theorem.** From Proposition 3.1, theorems about the branch-width of binary matroids give corollaries about the rank-width of bipartite graphs. One of the recent theorems about branch-width of binary matroids was proved by Geelen, Gerards, and Whittle [GGW03]. Here is the restatement of their theorem in the context of binary matroids.

Theorem 3.3 (Grid theorem for matroids). *For any positive integer k, there is an integer l such that if $\mathcal{M}$ is a binary matroid with branch-width at least l, then $\mathcal{M}$ contains a minor isomorphic to the cycle matroid of the $k \times k$ grid.*

By using this grid theorem, the following corollary was shown in [Oum04a]. We define a graph S_k, for $k > 1$ as follows. Let $A = \{a_i \mid 1 \le i \le k^2 - 1\}$ and $B = \{b_i \mid 1 \le i \le k^2 - k\}$. The graph S_k is a bipartite graph with $V(S_k) = A \cup B$ such that a_i and b_j are adjacent iff $i \le j < i + k$.

Corollary 3.4 ([Oum04a]). *For any positive integer k, there is an integer l such that if a bipartite graph G has rank-width at least l, then it contains a vertex-minor isomorphic to S_k*

This corollary will be used in the Section 7.

4. Isotropic systems

The notion of isotropic system, developed by A. Bouchet in [Bou87] and subsequent papers, is an algebraic structure, which represents equivalence classes of graphs by local equivalence. It has been used up to now in very few circumstances, but it provides a really powerful tool to study locally equivalent graphs, vertex-minor, and related notions.

4.1. **Definition.** Let K be the 2-dimensional vector space over GF(2). We may write $K = \{0, \alpha, \beta, \gamma\}$ with $0 = \alpha + \alpha = \beta + \beta = \gamma + \gamma = \alpha + \beta + \gamma$. We define a bilinear form $\langle\, ,\, \rangle$ by

$$\langle x, y \rangle = \begin{cases} 1 & \text{if } x \ne y, x \ne 0 \text{ and } y \ne 0, \\ 0 & \text{otherwise.} \end{cases}$$

For a finite set V, let K^V be a vector space over GF(2) with a bilinear form $\langle\, ,\, \rangle$ defined by

$$\text{for } a, b \in K^V, \quad \langle a, b\rangle = \sum_{v\in V} \langle a(v), b(v)\rangle.$$

isotropic system

Definition 4.1. $S = (V, L)$ is an *isotropic system* if V is a finite set, and L is a subspace of the vector space K^V over GF(2) such that $\dim(L) = |V|$ and $\langle x, y\rangle = 0$ for all $x, y \in L$. Denote $V(S) = V$, $L(S) = L$.

Definition 4.2.

complete

supplementary

(1) $a \in K^V$ is *complete* if $a(v) \neq 0$ for all $v \in V$.
(2) a, $b \in K^V$ are *supplementary* if $\langle a(v), b(v)\rangle = 1$ for all $v \in V$.
(3) For $a \in K^V$ and $P \subseteq V$, we define the restriction $a[P] \in K_V$ of a to P such that
$$(a[P])\,(v) = \begin{cases} a(v) & \text{if } v \in P \\ 0 & \text{otherwise} \end{cases}.$$
(4) For $U \subseteq V$, let $p_U : K^V \to K^U$ be the canonical projection. In other words, for $a \in K^V$, $p_U(a) \in K^U$ such that $(p_U(a))(v) = a(v)$ for all $v \in U$.

Let's define a minor of an isotropic system.

Definition 4.3. For $x \in K - \{0\}$, denote $L|^v_x = \{p_{V-v}(a) \mid a \in L, a(v) = 0 \text{ or } x\}$.

elementary minor

For an isotropic system $S = (V, L)$, the *elementary minor* $S|^v_x$ of S is $(V - \{v\}, L|^v_x)$.

minor

An isotropic system S' is called a *minor* of S if S' is obtained by taking a sequence of elementary minor operations, in other words,

$$S' = S|^{v_1}_{x_1}|^{v_2}_{x_2}\cdots|^{v_k}_{x_k}.$$

4.2. **Fundamental base and fundamental graphs.** The connection between isotropic systems and graphs was revealed by A. Bouchet [Bou88a].

Eulerian vector

Definition 4.4. $x \in K^V$ is called an *Eulerian vector* of $S = (V, L)$ if x is complete and if $\emptyset \neq P \subseteq V$ then $x[P] \notin L$.

Proposition 4.5 ([Bou88a])**.** *For any complete vector c of K^V, there is an Eulerian vector a of S, supplementary to c.*

Proposition 4.6 ([Bou88a, (4.3)])**.** *Let a be an Eulerian vector of an isotropic system $S = (V, L)$. For every $v \in V$, there exists a unique vector $b_v \in L$ such that*

(1) $b_v(v) \neq 0$ for all v.
(2) $b_v(w) = 0$ or $a(w)$ for $v \neq w$.

fundamental base

Furthermore, b_v is uniquely determined by (1) *and* (2) *and the family $\{b_v\}_{v\in V}$ is a base of L. $\{b_v\}_{v\in V}$ is called a* fundamental base *of L with respect to a.*

Remark. In [Bou88a, (4.3)], A. Bouchet wrote a weaker statement, saying that the family $\{b_v\}$ is uniquely determined. But, in his proof, he proved stronger one, which is the above statement. For the uniqueness of b_v, one doesn't need conditions for b_w for $w \neq v$. This stronger statement is helpful for Proposition 6.5.

For a graph G, $n_G(v)$ is the set of neighbors of v.

graphic presentation

Definition 4.7. Let $S = (V, L)$ be an isotropic system. a, b are supplementary vectors in K^V. Let $G = (V, E)$ be a graph with $V(G) = V$. (G, a, b) is called the *graphic presentation* of S if $\{n_G(v) + b[\{v\}] \mid v \in V\}$ is a base of L.

Definition 4.8. Let $\{b_v\}$ be a fundamental base of an isotropic system $S = (V, L)$ with respect to an Eulerian vector a. A graph G with $V(G) = V$ is called a *fundamental graph* of S with respect to a if v and w are adjacent in $G \Leftrightarrow v \neq w$ and $b_v(w) \neq 0$. *fundamental graph*

Since $\langle b_v, b_w \rangle = 0$ implies $b_v(w) \neq 0 \Leftrightarrow b_w(v) \neq 0$, the fundamental graph G is undirected.

Proposition 4.9. *Suppose G is a fundamental graph of an isotropic system S. If we take b such that $b(v) = b_v(v)$, then (G, a, b) is a graphic presentation of S.*

Conversely, for a graph $G = (V, E)$ and any supplementary pair of vectors a and b of K^V, let L be a subspace of K^V spanned by $\{a[n_G(v)] + b[\{v\}] \mid v \in V\}$. Then, $S = (V, L)$ is an isotropic system such that (G, a, b) is a graphic presentation of S.

4.3. **Isomorphism and locally equivalent graphs.** It is proved in [Bou88a] that if $S = (V, L)$ is presented by (G, a, b) and if a' is an Eulerian vector of S, then there exists a unique b' and a unique graph G' such that (G', a', b') is a graphic presentation of S. The graph G' is locally equivalent to G, and conversely, for every G' locally equivalent to G, there exists a graphic presentation (G', a', b') of S.

Let us clarify the notion of isomorphism of isotropic systems. A permutation π of K is linear iff $\pi(0) = 0$. Let V be a finite set and $\Pi = (\pi_v)_{v \in V}$ be a family of linear permutations of K. For every vector a in K^V, we let $\Pi(a)$ be the vector defined by $(\Pi(a))(v) = \pi_v(a(v))$ for all $v \in V$. The mapping Π is a linear automorphism of K^V. If $S = (V, L)$ is an isotropic system, then $(V, \Pi(L))$ is an isotropic system, denoted by $\Pi(S)$ and said to be *strongly isomorphic to* S. *strongly isomorphic to*

Let G be a graph with set of vertices V, let a and b be supplementary complete vectors, and S be the isotropic system presented by (G, a, b). Then $\Pi(S)$ is the isotropic system presented by $(G, \Pi(a), \Pi(b))$. The following lemma states a converse.

Lemma 4.10. *Two isotropic systems with same fundamental graph are strongly isomorphic.*

Proof. We first prove the following fact. If x, x', y, y' belong to $K - \{0\}$, with $x \neq y, x' \neq y'$, there exists a unique linear permutation of K mapping x to x' and y to y'. Without loss of generality, we can assume that $x = \alpha$ and $y = \beta$. By applying if necessary a linear permutation, we can also assume that $x' = \alpha$. There are two cases to consider. Either $y' = \beta$ or $y' = \gamma$. In both cases we get a unique linear permutation.

Consider now $S = (V, L)$ and $S' = (V, L')$ presented by (G, a, b) and (G, a', b'). By applying the above observation to $a(v), a'(v), b(v), b'(v)$ for each v in V, we can find a unique Π such that $\Pi(a) = \Pi(a')$ and $\Pi(b) = \Pi(b')$. Hence $S' = \Pi(S)$. □

We can consider two strongly isomorphic isotropic systems as the same mathematical object, because the three elements of $K - \{0\}$ are indistinguishable.

Two isotropic systems $S = (V, L)$ and $S' = (V', L')$ are defined as *isomorphic* if there is a bijection h of V' onto V and a family $\Pi = (\pi_v)_{v \in V}$ of linear permutations of K such that L' is the set of vectors b in $K^{V'}$ such that $b(v') = \pi_{h(v')}(a(h(v')))$ for all $v' \in V'$. Intuitively, h induces a bijection between L' and $\Pi(L)$. Hence S and S' are isomorphic iff the fundamental graphs of S are isomorphic to the fundamental graphs of S'. Up to isomorphism, isotropic systems represent classes of locally equivalent graphs. *isomorphic*

5. Monadic Second-order logic

We review background results on Monadic Second-order (MS) logic, on transformations of structures expressed in this language and its extensions. We discuss the links between clique-width and MS logic, and we present Seese's Conjecture. For the main definitions and results on MS logic and some examples of formulas, the reader is referred to the book chapter [Cou97], or the preliminary sections of any of the articles [Cou94, Cou96, Cou03, Cou04b, CE95]. However all necessary definitions are given in full in the present section.

relation symbols
set predicates
arity
domains

5.1. **Relational structures and Monadic Second-order logic.** If $R = \{A, B, C, \dots\}$ is a finite set of *relation symbols* and *set predicates*, each of them given with a nonnegative integer $\rho(A)$ called its *arity*. We denote by $\mathcal{STR}(R)$ the set of R-structures $S = \langle D_S, (A_S)_{A\in R}\rangle$ where $A_S \subseteq D_S^{\rho(A)}$ if $A \in R$ is a relation symbol, and $A_S \subseteq \mathcal{P}(D_S)^{\rho(A)}$ if A is a set predicate. Unless otherwise specified, structures will be finite, i.e., their *domains* D_S will be finite.

A simple graph G can be defined as an $\{edg\}$-structure $G = \langle V, edg\rangle$ where V is the set of vertices of G and $edg \subseteq V \times V$ is a binary relation representing the edges. Since we consider undirected graphs, the relation edg will be symmetric.

Remark. We write $G = \langle V, edg\rangle$ and not $G = (V, E)$ to stress the fact that, in this logical representation, the edges are defined by a binary relation on V and not as a set of objects apart from V, as this is the case for the logic MS_2 mentioned in the introduction where quantified variables may denote sets of edges.

A matroid M can be represented by a structure $M = \langle E, Indep\rangle$ where $Indep(F)$ holds iff F is an independent subset of E. See Hliněny [Hli03, Hli04] about MS logic for matroids. An isotropic system $S = (V, L)$ can be represented by a structure $\langle V, Member\rangle$ where $Member(X, Y, Z)$ holds iff X, Y, Z are pairwise disjoint subsets of V and the vector $a \in K^V$ such that $a(v) = \alpha$ iff $v \in X$, $a(v) = \beta$ iff $v \in Y, a(v) = \gamma$ iff $v \in Z, a(v) = 0$ otherwise, is in L. We denote also by S the structure representing an isotropic system S.

We will use subscripts G, M, S in notation like $V_G, edg_G, Indep_M, Member_S$ if it is necessary to make precise the relevant graph, matroid or isotropic system.

Monadic Second-order logic
MS

We recall that *Monadic Second-order logic* (*MS* logic for short) is the extension of First-Order logic by variables denoting subsets of the domains of the considered structures, and new atomic formulas of the form $x \in X$ expressing the membership of x in a set X. (Uppercase letters will denote set variables, lowercase letters will denote ordinary first-order variables). If A is an n-ary set predicate, then we will use atomic formulas of the form $A(X_1, \dots, X_n)$. We will denote by $MS(R, W)$ the set of MS formulas written with the set R of relation and set predicate symbols and having their free variables in a set W consisting of individual as well as of set variables.

As a typical and useful example of MS formula, we give a formula with free variables x and y expressing that (x, y) belongs to the reflexive and transitive closure of a binary relation A:

$$\forall X(x \in X \wedge \forall u, v[(u \in X \wedge A(u, v)) \Longrightarrow v \in X] \Longrightarrow y \in X)$$

If the relation A is not given in the structure but defined by an MS formula, then one replaces $A(u, v)$ by this formula with appropriate substitutions of variables.

We will use an extension of MS logic, denoted by C_2MS and called *Modulo-2 Counting Monadic Second-order logic,* using the set predicate $Even(X)$ expressing that a set X is *even*, i.e., has even cardinality. Since we consider structures with finite domains, that a set X has odd cardinality can be expressed by the formula $\neg Even(X)$). An even larger extension called *Counting Monadic Second-order logic*, denoted by CMS, uses set predicates $Card_p(X)$ meaning: the cardinality of X is a multiple of p, where $p > 1$. We will denote by $C_2MS(R,W)$ and $CMS(R,W)$ instead of $MS(R,W)$ the corresponding sets of formulas that use possibly modulo 2 and modulo p cardinality predicates (for all p) respectively.

Modulo-2 Counting Monadic Second-order logic,
even

Counting Monadic Second-order logic

We have a strict inclusion of languages considered as sets of formulas: $MS \subset C_2MS \subset CMS$. The corresponding hierarchy of expressive powers is strict. It can be proved that no MS formula $\varphi(X)$ can express, in every structure, that a set X has even cardinality [Cou90], and similarly, that the property that the cardinality of X is a multiple of 3 cannot be expressed by a C_2MS formula. (The argument of [Cou90] can be adapted). However, for particular classes $\mathcal{C}$ of structures, if there exists an MS formula defining a linear ordering of each structure in $\mathcal{C}$ (the formal definition will be given in Section 7), then the $Card_p$ predicates can be expressed by MS formulas and so, CMS is no longer more expressive than MS. For instance $Even(X)$ can be expressed as follows: X is partitioned into two sets Y and Z such that the least element of X is in Y, the largest one is in Z and the successor of an element in Y (resp. in Z) is in Z (resp. in Y). The definition of linear orders by MS formulas is investigated in [Cou96].

Let $\mathcal{C}$ be a set of (finite) relational structures that represent graphs, matroids, isotropic systems, or other combinatorial objects like hypergraphs and partial orders. The *MS satisfiability problem for* $\mathcal{C}$ is the following decision problem:

MS satisfiability problem for

for every closed MS formula φ,
we ask whether there exists a structure in $\mathcal{C}$ that satisfies φ.

This decision problem does not concern particular properties like the planarity of a graph, but *all properties expressible in Monadic Second-order logic*. Note that $\mathcal{C}$ is here fixed, and that the input is any formula of MS logic. This problem is trivially decidable if $\mathcal{C}$ is finite, because relational structures are assumed finite and the validity of a formula in a single finite structure can be decided, simply by applying the definition. If $\mathcal{C}$ is the set of all finite trees, the MS satisfiability problem is decidable, as a consequence of deep results relating MS logic and tree-automata. A conjecture by D. Seese [See91], says roughly speaking, that if a set of graphs has a decidable MS satisfiability problem, then it is, in a precise sense, definable from finite trees by MS formulas. This conjecture can formulated for extensions of MS logic, like C_2MS or CMS logic. Note that the condition "the C_2MS satisfiability problem for $\mathcal{C}$ is decidable" is *a priori* stronger than "the MS satisfiability problem for $\mathcal{C}$ is decidable", because the intended algorithm must take more formulas as input in the former case. However, we presently do not know any class for which the MS satisfiability problem is decidable whereas the C_2MS one is not.

all properties expressible in Monadic Second-order logic

5.2. **Transductions of relational structures.** We now define some transformations of relational structures that can be formalized in MS logic (or its extensions). They are called *MS transductions*, because they generalize transformations of words and trees called transductions in formal language theory. They are similar to polynomial reductions which make it possible to compare algorithmic problems, because

MS transductions

if a set of structures has a decidable MS satisfiability problem, then so has its image under an MS transduction. They make it possible to transfer decidability results from a set of structures to another one.

parameters

The basic idea is to specify a structure T inside a given structure S in terms of subsets of D_S specified by set variables called *parameters*, and by means of a fixed sequence of MS (or CMS) formulas. In particular, we will be able to specify by C_2MS formulas, inside a graph G, and by means of appropriately chosen sets of vertices taken as values of parameters, all its vertex-minors.

Actually, the general definition of an MS transduction allows to define T inside a structure built from a fixed number of disjoint copies of the given structure S. For the most general definition, we refer the reader to [Cou94, Cou97]. We only define formally the special transductions that will be useful for the main proofs.

parameters
definition scheme

We let R and Q be two finite sets of relation symbols. Let W be a finite set of set variables, called *parameters*. A *definition scheme*, intended to specify a transformation of R-structures into Q-structures is a tuple of formulas of the form $\Delta = (\varphi, \psi, (\theta_A)_{A\in Q})$ where

(a) $\varphi \in MS(R, W)$,
(b) $\psi \in MS(R, W \cup \{x_1\})$,
(c) $\theta_A \in MS(R, W \cup \{x_1, \cdots, x_{\rho(A)}\})$ for each relation symbol A,
(d) $\theta_A \in MS(R, W \cup \{X_1, \cdots, X_{\rho(A)}\})$ for each set predicate A.

assignment in
defined in

Let $S \in \mathcal{STR}(R)$, let γ be a W-*assignment in* S, i.e. a mapping from the the variables in W to subsets of D_S. The Q-structure T with domain $D_T \subseteq D_S$ is *defined in* (S, γ) by Δ if

(i) $(S, \gamma) \models \varphi$
(ii) $D_T = \{d \mid d \in D_S, (S, \gamma, d) \models \psi\}$
(iii) for each A in Q, if A is a relation,

$$A_T = \{(d_1, \cdots, d_{\rho(A)}) \in D_T^{\rho(A)} \mid (S, \gamma, d_1, \cdots, d_{\rho(A)}) \models \theta_A\},$$

and if A is a set predicate,

$$A_T = \{(U_1, \cdots, U_{\rho(A)}) \in \mathcal{P}(D_T)^{\rho(A)} \mid (S, \gamma, U_1, \cdots, U_{\rho(A)}) \models \theta_A\}.$$

By $(S, \gamma, d_1, \cdots, d_{\rho(A)}) \models \theta_A$, we mean $(S, \gamma') \models \theta_A$, where γ' is the assignment extending γ, such that $\gamma'(x_i) = d_i$ for all $i = 1, \cdots, \rho(A)$; a similar convention is used for $(S, \gamma, d) \models \psi$ and $(S, \gamma, U_1, \cdots, U_{\rho(A)}) \models \theta_A$.

Let us describe in words the roles of the formulas of Δ. Condition (i) expresses that the values of the parameters specified by the assignment γ satisfy a condition specified by φ. Condition (ii) defines the domain of the output structure T as a subset of that of the input structure S. This restriction is specified by the formula $\psi(x_1)$. Since this formula may also have the parameters as free variables, the domain of T may depend on γ. Condition (iii) defines the relations A of T by means of the formulas θ_A evaluated in S ; they also depend on γ. It defines in a similar way the set predicates of T. An example will be given shortly.

Since T is associated in a unique way with S, γ and Δ whenever it is defined, i.e., whenever $(S, \gamma) \models \varphi$, we can use the functional notation $def_\Delta(S, \gamma)$ for T.

transduction defined by

The *transduction defined by* Δ is the mapping $\mathcal{STR}(R) \to \mathcal{P}(\mathcal{STR}(Q))$:

$$def_\Delta(S) = \{T \mid T = def_\Delta(S, \gamma) \text{ for some } W\text{-assignment } \gamma \text{ in } S\}.$$

MS transduction

A mapping $\mathcal{STR}(R) \to \mathcal{P}(\mathcal{STR}(Q))$ is an *MS transduction* if it is equal to def_Δ

for some definition scheme Δ. If the formulas in the considered definition scheme may be C_2MS or CMS formulas, then the associated mapping is called a C_2MS or a *CMS transduction* respectively. Hence, like for formulas, we have a hierarchy of classes of transductions: $MS \subset C_2MS \subset CMS$.

C_2MS CMS C_2MS CMS transduction MS C_2MS CMS isomorphic

A mapping $\tau : \mathcal{STR}(R) \to \mathcal{P}(\mathcal{STR}(Q))$ is *isomorphic* to def_Δ if, for each S in $\mathcal{STR}(R)$, every T in $def_\Delta(S)$ is isomorphic to some T' in $\tau(S)$ and vice-versa.

Example 5.1 (Local complementation)**.** If G is a graph and X is a set of independent vertices, then the local complementations associated with the vertices in X can be performed in any order. We denote by $G * X$ the graph obtained by these local complementations. The mapping $\mathcal{LC}$ that associates with G the set of graphs $G * X$ for all independent sets X of vertices is a C$_2$MS -transduction defined by the definition scheme $(\varphi, \psi, \theta_{edg})$ where

(i) φ is $\forall x, y(x \in X \wedge y \in X \implies \neg edg(x,y))$ (expressing that X is a set of independent vertices)
(ii) ψ is *true* (because the vertices are the same in G and in $G * X$, so there is no need to restrict the domain)
(iii) $\theta_{edg}(x,y)$ is the formula

$$\begin{aligned}&[(x \in X \vee y \in X) \wedge edg(x,y)]\\ &\qquad \vee [x \notin X \wedge y \notin X \wedge edg(x,y) \wedge Even(N(x,y))]\\ &\qquad\quad \vee [x \neq y \wedge x \notin X \wedge y \notin X \wedge \neg edg(x,y) \wedge \neg Even(N(x,y))].\end{aligned}$$

(Let $N(x,y)$ be the set of common neighbors of x and y. In this formula, $Even(N(x,y))$ stands for a formula that is easy to write. We may express $Even(N(x,y))$ as $\forall Z[\forall v(v \in Z \Leftrightarrow edg(x,v) \wedge edg(y,v)) \Rightarrow Even(Z)]$.

The mapping $\mathcal{LC}$ is thus a C$_2$MS transduction with one parameter, namely X. The set predicate *Even* is here necessary, because the mapping $\mathcal{LC}$ is provably not an MS transduction; consider the graphs G_n with vertices $1, 2, \ldots, n$ and edges 1-2, 1-i, 2-i for $i = 3, \ldots, n$. Let $X \subseteq \{3, \ldots, n\}$. Then $G * X = G$ iff X has even cardinality. And in the graphs G_n, evenness is not MS expressible (see [Cou97]).

5.3. **Fundamental property of CMS transductions.** The following proposition says that if $T = def_\Delta(S, \gamma)$, then the monadic second-order properties of T can be expressed as monadic second-order properties of (S, γ). The usefulness of definable transductions is based on this proposition.

Proposition 5.2. *1) Let $\Delta = (\varphi, \psi, (\theta_A)_{A \in Q})$ be a definition scheme, written with a set of parameters W. Let V be a set of variables disjoint from W. For every formula β in $MS(Q, V)$, one can construct a formula $\beta^\#$ in $MS(R, V \cup W)$ such that, for every S in $\mathcal{STR}(R)$, for every assignment $\gamma : W \longrightarrow S$, for every assignment $\eta : V \longrightarrow S$, we have*

$$(S, \eta \cup \gamma) \models \beta^\# \text{ if and only if } \begin{pmatrix} def_\Delta(S,\gamma) \text{ is defined,} \\ \eta \text{ is a } V\text{-assignment in } def_\Delta(S,\gamma), \\ (def_\Delta(S,\gamma), \eta) \models \beta. \end{pmatrix}$$

2) If Δ is a C_2MS (resp. CMS) definition scheme or β is a C_2MS (resp. CMS) formula, then the same holds for some C_2MS (resp. CMS) formula $\beta^\#$.

Note that, even if $T = def_\Delta(S, \gamma)$ is well-defined, the mapping η is not necessarily a V-assignment in T, because the domain of T can be a proper subset of D_S. We call $\beta^\#$ the *backwards translation* of β relative to the transduction def_Δ.

backwards translation

Proof sketch. The formula is $\beta^{\#}$ of the form $\varphi_1 \wedge \widehat{\beta}$ where φ_1 is independent of β and $\widehat{\beta}$ is defined inductively from β. We let $V = \{u_1, \ldots, u_m, U_1, \ldots, U_q\}$.

The formula φ_1 defined as

$$\varphi \wedge \psi[u_1] \wedge \cdots \wedge \psi[u_m] \wedge \forall u(u \in U_1 \Longrightarrow \psi[u]) \wedge \cdots \wedge \forall u(u \in U_q \Longrightarrow \psi[u])$$

expresses that $def_\Delta(S, \gamma)$ is well-defined and that η is a V-assignment in $def_\Delta(S, \gamma)$. (We denote by $\psi[u]$ the formula resulting from the substitution of u for x_1 in ψ).

We now define $\widehat{\beta}$.

If β is $\beta_1 \wedge \beta_2$, or $\beta_1 \vee \beta_2$ or $\neg\beta_1$, then $\widehat{\beta}$ is $\widehat{\beta_1} \wedge \widehat{\beta_2}$, or $\widehat{\beta_1} \vee \widehat{\beta_2}$ or $\neg\widehat{\beta_1}$ respectively.

If β is $\exists u.\beta_1$, then $\widehat{\beta}$ is $\exists u.(\psi[u] \wedge \widehat{\beta_1})$.

If β is $\exists X.\beta_1$, then $\widehat{\beta}$ is $\exists X.[\forall u(u \in X \Longrightarrow \psi[u]) \wedge \widehat{\beta_1}]$.

Universal quantifications are treated as negated existential quantifications.

If β is $x = y$ or $x \in X$ or $Even(X)$ or $Card_p(X)$, then $\widehat{\beta}$ is β.

If β is $A(u_1, \cdots, u_n)$ for some relation symbol A, then $\widehat{\beta}$ is $\theta_A[u_1, \cdots, u_n]$ (where $\theta_A[u_1, \cdots, u_n]$ is obtained by substituting $u_1, \cdots, u_n$ for $x_1, \cdots, x_n$ in θ_A; the free variables of θ_A are among $x_1, \cdots, x_n$ and the parameters).

If β is $A(U_1, \cdots, U_n)$ for some set predicate A, then $\widehat{\beta}$ is $\theta_A[U_1, \cdots, U_n]$ (where $\theta_A[U_1, \cdots, U_n]$ is obtained as above by substitution of variables).

The verification that $\widehat{\beta}$ has the desired property is straightforward by induction on the structure of β. ☐

Proposition 5.3 ([Cou94, Cou97]).

(1) If a set of structures has a decidable MS satisfiability problem (resp. C_2MS satisfiability problem), then so has its image under an MS transduction (resp. under a C_2MS transduction).

(2) The composition of two MS transductions (resp. of two C_2MS transductions) is an MS transduction (resp. a C_2MS transduction).

Proof. We only prove (1). Let $\mathcal{C}$ be a set of structures having a decidable MS satisfiability problem, and τ be an MS transduction with parameters $Y_1, \ldots, Y_p$. For a given closed MS formula β, we want to know if $T \vDash \beta$ for some $T \in \tau(\mathcal{C})$. Consider any $T = def_\Delta(S, \gamma)$ in $\tau(\mathcal{C})$ for S in $\mathcal{C}$. Then, by using Proposition 5.2, $T \vDash \beta$ iff $(S, \gamma) \models \beta^{\#}$ (since β is closed, the set V is empty). Hence $T \vDash \beta$ for some $T \in \tau(\mathcal{C})$ iff $(S, \gamma) \models \beta^{\#}$ for S in $\mathcal{C}$ and some γ iff $S \models \exists Y_1, \ldots, Y_p.\beta^{\#}$ for S in $\mathcal{C}$. Since $\mathcal{C}$ has a decidable MS satisfiability problem, one can decide the existence of such a structure S, hence the existence in $\tau(\mathcal{C})$ of T satisfying β. ☐

Since every MS transduction is a C_2MS transduction, the composition of an MS and a C_2MS transduction is a C_2MS transduction.

5.4. **Seese's Conjecture.** A recent article about Seese's Conjecture, with references to the particular cases proved in several other articles, is [Cou04b]. This conjecture has several equivalent formulations and we first review some results establishing the equivalences. There is an intimate relation between clique-width and MS transductions, as shown by the following proposition.

Proposition 5.4. *A set of graphs has bounded clique-width iff it is the image of a set of trees under an MS transduction iff it is the image of a set of trees under a C_2MS transduction.*

Proof. The first equivalence is proved in [CE95, EvO97]. One can also replace "trees" by "binary trees" and "is the image" by "is contained in the image". For the last equivalence, let us consider a set of graphs L that is the image of a set of trees T under a C_2MS transduction η. There exists a set of binary trees B and a bijective MS transduction β of B onto T. Hence $L = \eta \circ \beta(B)$, and $\eta \circ \beta$ is a C_2MS transduction. But on binary trees a linear order is definable by an MS formula. Hence the atomic formulas $Even(X)$ in the formulas of the definition scheme of $\eta \circ \beta$ can be replaced by MS formulas, and $\eta \circ \beta$ also has an MS definition scheme. Hence L is the image of a set of trees under an MS transduction. □

This proof also works for CMS instead of C_2MS. One important consequence of this result and of Proposition 5.3.2 is that the image of a set of graphs of bounded clique-width under a CMS transduction has bounded clique-width. This is not immediate from the definitions of clique-width operations on the one hand, and of CMS transductions on the other.

By the results of [OS04], clique-width can be replaced by rank-width in this statement. Clique-width is also defined for directed graphs and Proposition 5.4 is valid for them.

Using the terminology of the present article, the conjecture by Seese [See91] can be stated as follows.

Conjecture. *If a set of graphs has a decidable MS satisfiability problem, then it is contained in the image of a set of trees under an MS transduction, equivalently, it has bounded clique-width.*

Any two isomorphic graphs satisfy the same formulas, have the same clique-width and one is the image of a set of trees under an MS transduction iff the other is. Concrete constructions will handle graphs but this conjecture and the related statements actually concern isomorphism classes of graphs.

This conjecture is formulated in [See91, HS04] for sets of graphs (or matroids) having a *decidable monadic theory*. This means for a set of structures $\mathcal{C}$ that the problem of deciding whether a given MS formula is true in every structure of $\mathcal{C}$ is decidable. But a formula φ is true in every structure in $\mathcal{C}$ iff the formula $\neg\varphi$ is not satisfied in any structure of C. Hence, $\mathcal{C}$ has a decidable monadic theory iff it has a decidable MS satisfiability problem.

decidable monadic theory

The conjecture has been proved for a large variety of particular graph classes: planar graphs [See91], graphs of bounded degree, graphs without a fixed graph as a minor, *uniformly k-sparse graphs* (i.e., graphs for which every subgraph has a number of edges bounded by k times the number of vertices) [Cou03], interval graphs, line graphs, partial orders of dimension 2 [Cou04b]. Furthermore

uniformly k-sparse graphs

Proposition 5.5 ([Cou04b])**.** *The Conjecture is valid for graphs, iff it is valid for bipartite graphs, iff it is valid for directed graphs, iff it is valid for comparability graphs, iff it is valid for partial orders.*

One of the main results of this article is the proof of the following weakening of the Conjecture.

Theorem 5.6. *If a set of graphs has a decidable C_2MS satisfiability problem, it is contained in the image of a set of trees under an MS transduction, equivalently, it has bounded clique-width, equivalently bounded rank-width.*

The proof of Proposition 5.5 yields the corresponding results for directed graphs, partial orders, etc.

For all particular cases where the Conjecture has been proved, the proofs use, *via* some reductions based on MS transductions, the result of Robertson and Seymour saying that excluding a planar graph as a minor implies bounded tree-width. Theorem 5.6 uses the analogous result by Geelen, Gerards, and Whittle [GGW03] extended to graphs by Oum [Oum04a] which says that bipartite graphs not containing certain graphs, transformable by MS transductions into grids, as vertex-minors have bounded rank-width. We will also give another proof using binary matroids and results by Geelen, Gerards, and Whittle [GGW03] and Hliněny and Seese [HS04]. For both proofs, connection between bipartite graphs and binary matroids is essential.

5.5. **Evaluation of CMS formulas.** We explain why and how CMS formulas can be evaluated in linear time on graphs of clique-width at most k that are given by k-expressions.

quantifier-height The *quantifier-height* $qh(\varphi)$ of a CMS formula is defined as follows.

(i) $qh(\varphi) = 0$ if φ is atomic, i.e., is of the form $x = y$ or $x \in X$ or $Card_p(X)$ or $A(u_1, \cdots, u_n)$ or $A(U_1, \cdots, U_n)$.
(ii) $qh(\neg\varphi) = qh(\varphi)$.
(iii) $qh(\varphi_1 \wedge \varphi_2) = qh(\varphi_1 \vee \varphi_2) = \max\{qh(\varphi_1), qh(\varphi_2)\}$.
(iv) $qh(\exists u.\varphi) = qh(\forall u.\varphi) = qh(\exists U.\varphi) = qh(\forall U.\varphi) = 1 + qh(\varphi)$.

$C_pMS^h(R,\emptyset)$ We denote by $C_pMS^h(R,\emptyset)$ the set of CMS formulas of quantifier height at most h, written with the relation symbols in a finite set R and the set predicates $Card_q$ for q at most p. This set is infinite because if it contains a formula φ, it contains also all the formulas $\varphi \vee \varphi \vee \cdots \vee \varphi$. However all these formulas are equivalent. One can actually replace (by an algorithm) every formula φ in $\mathrm{C}_p\mathrm{MS}^h(R,\emptyset)$ by a canonical formula $Can(\varphi)$ in $\mathrm{C}_p\mathrm{MS}^h(R,\emptyset)$ which is equivalent to φ (i.e., has the same truth value in every structure). This can be done in such a way that $Can(\mathrm{C}_p\mathrm{MS}^h(R,\emptyset))$ is finite. This classical fact is described formally in [CW04]. The cardinality of $Can(\mathrm{C}_p\mathrm{MS}^h(R,\varnothing))$ is however a tower of exponentials of height proportional to h.

For every p, R, h as above, and for every R-structure S, we let

$$Th_{p,R,h}(S) = \{\varphi \in Can(\mathrm{C}_p\mathrm{MS}^h(R,\emptyset)) \mid\ S \models \varphi\ \}.$$

(p,R,h)-theory of S We call it the (p,R,h)-*theory of* S. There are thus finitely many (p,R,h)-theories, and each of them is a finite set of formulas.

A k-graph $G = (V_G, E_G, lab_G)$ is represented by the relational structure

$$(V_G, edg_G, p_{1G}, ..., p_{kG}),$$

also denoted by G, if edg_G is the edge relation and $p_{iG}(x)$ holds iff $lab(x) = i$. The following proposition summarizes well-known results of which similar forms have been published in e.g. [Cou90, Mak04].

Proposition 5.7 ([Cou97, Theorem 5.7.5])**.** *Let us fix a positive integer k.*

(1) Let $R = \{edg, p_1, ..., p_k\}$ with edg of arity 2 and p_i of arity 1. For all positive integers p, h, i, j (where $i, j \in [k]$ and $i \neq j$), there exist mappings

$f_{k,\oplus}$, $f_{k,\eta_{i,j}}$, $f_{k,\rho_{i\to j}}$ on subsets of $Can(\mathrm{C}_p\mathrm{MS}^h(R,\emptyset))$ such that for all k-graphs G and H,

$$Th_{p,R,h}(\eta_{i,j}(G)) = f_{k,\eta_{i,j}}(Th_{p,R,h}(G)),$$
$$Th_{p,R,h}(\rho_{i\to j}(G)) = f_{k,\rho_{i\to j}}(Th_{p,R,h}(G)),$$
$$Th_{p,R,h}(G\oplus H) = f_{k,\oplus}(Th_{p,R,h}(G), Th_{p,R,h}(H)).$$

(2) *If a graph G is given as $val(t)$ for some k-expression t, then $Th_{p,R,h}(G)$ can be computed in time proportional to the size of t.*

(3) *Every CMS graph property can be evaluated on a graph of clique-width at most k, given by a k-expression, in time proportional to the number of vertices.*

Proof. (1) Let us observe that the mapping $\eta_{i,j}$ is a *quantifier-free transduction*, i.e., a transduction defined by a definition scheme consisting of formulas without quantifiers and without parameters. From the proof of Proposition 5.2, it follows that the backwards translation (denoted by $^{\#}$) associated with $\eta_{i,j}$ does not increase quantifier height and does not add new counting modulo set predicates. Hence for every formula φ in $\mathrm{C}_p\mathrm{MS}^h(R,\emptyset)$, $\eta_{i,j}(G)\models\varphi$ iff $G\models\varphi^{\#}$ iff $G\models Can(\varphi^{\#})$ and $\varphi^{\#}$ belongs to $\mathrm{C}_p\mathrm{MS}^h(R,\emptyset)$. Hence, we can take, for every subset Φ of $Can(\mathrm{C}_p\mathrm{MS}^h(R,\emptyset))$, *quantifier-free transduction*

$$f_{k,\eta_{i,j}}(\Phi) = \{\varphi\in Can(\mathrm{C}_p\mathrm{MS}^h(R,\emptyset)) \mid Can(\varphi^{\#})\in\Phi\}.$$

The proof is similar for $\rho_{i\to j}$.

The case of $\oplus$ is a particular case of a result by Feferman, Vaught and Shelah. The proof is in [Cou90, Lemma (4.5)]. We also refer the reader to the survey by Makowsky [Mak04] for the history and the numerous consequences of this result.

(2) Consider a graph $G = val(t)$ where t is a k-expression.

Each set $Th_{p,R,h}(val(\mathbf{i}))$ can be computed from the definitions. Then, using (1) one can compute $Th_{p,R,h}(val(t))$ by induction on the structure of t.

For example, if $t = t_1\oplus t_2$, then we get

$$Th_{p,R,h}(val(t)) = f_{k,\oplus}(Th_{p,R,h}(val(t_1)), Th_{p,R,h}(val(t_2))).$$

(3) If we want to know whether $val(t)\models\varphi$, we compute by (2) the set $Th_{p,R,h}(val(t))$ where p and h are the smallest integers such that $\varphi\in\mathrm{C}_p\mathrm{MS}^h(R,\emptyset)$. Then one determines whether $Can(\varphi)$ belongs to $Th_{p,R,h}(val(t))$, which gives the answer. □

This method applies to optimization and enumeration (counting) problems formalized in monadic second-order logic. We refer the reader to [Mak04].

6. Logical expression of vertex-minors

6.1. **Vertex-minor through isotropic systems.** An isotropic system $S=(V,L)$ will be represented by the structure $\langle V, Member_S\rangle$ (also denoted by S) where the ternary set predicate $Member_S(X,Y,Z)$ holds iff X,Y,Z are pairwise disjoint subsets of V and the vector $a\in K^V$ is in L if

$$a(v) = \begin{cases}\alpha & \text{if } v\in X,\\ \beta & \text{if } v\in Y,\\ \gamma & \text{if } v\in Z,\\ 0 & \text{otherwise.}\end{cases}$$

Proposition 6.1. *There exists an MS transduction that maps an isotropic system to the set of isotropic systems strongly isomorphic to it.*

Proof. A strong isomorphism of isotropic systems with base set V is defined from a family $\Pi = (\pi_v)_{v\in V}$ of linear permutations of K. Since a linear permutation is nothing but a permutation of $\{\alpha, \beta, \gamma\}$, there are 6 such permutations, say $\pi^1, \dots, \pi^6$. Hence a family Π as above can be specified by 6 set variables $W_1, \dots, W_6$ forming a partition of V, with the condition that $\pi_v = \pi^i$ iff $v \in W_i$. With this assumption, it is then straightforward to write an MS formula expressing $Member_{\Pi(S)}$ in terms of $Member_S$ and $W_1, \dots, W_6$. □

Proposition 6.2. *There exists an MS transduction μ with parameters $V_\alpha, V_\beta, V_\gamma$ that maps an isotropic system to the set of its minors.*

Proof. From the Definition 4.3, an isotropic system $S' = (V', L')$ is a minor of $S = (V, L)$ iff there is 3 pairwise disjoint subsets $V_\alpha = \{x_1, x_2, \dots, x_a\}$, $V_\beta = \{y_1, y_2, \dots, y_b\}$, $V_\gamma = \{z_1, z_2, \dots, z_c\}$ of vertices, such that

$$S' = S|_\alpha^{x_1}|_\alpha^{x_2}\cdots|_\alpha^{x_a}|_\beta^{y_1}|_\beta^{y_2}\cdots|_\beta^{y_b}|_\gamma^{z_1}|_\gamma^{z_2}\cdots|_\gamma^{z_c}.$$

Then, $V' = V - (V_\alpha \cup V_\beta \cup V_\gamma)$ and

$$L' = \{p_{V'}(a) \mid a \in L \text{ and for all } v \in V, a(v) \neq 0 \Longrightarrow v \in V_{a(v)}\}. \tag{6.1}$$

(We denote by $p_{V'}(a)$ the vector in $K^{V'}$ obtained from a by deleting the components having an index not in V'; it should not be confused with $a[V']$ which is the vector in K^V obtained by setting to 0 the value of these components.)

We define $Member_{S'}(X, Y, Z)$ by an MS formula $\mu_1(V_\alpha, V_\beta, V_\gamma, X, Y, Z)$, to be evaluated in S.

If $a \in L$ is represented by (X_a, Y_a, Z_a), then a triple (X, Y, Z) represents the vector $p_{V'}(a)$ and $p_{V'}(a) \in L'$ iff the following conditions hold.

(i) X, Y, Z are pairwise disjoint,
(ii) $(X \cup Y \cup Z) \cap (V_\alpha \cup V_\beta \cup V_\gamma) = \emptyset$,
(iii) $X = X_a - V_\alpha, Y = Y_a - V_\beta, Z = Z_a - V_\gamma$,
(iv) $V_\alpha \subseteq X_a \cup (V - (Y_a \cup Z_a))$, $V_\beta \subseteq Y_a \cup (V - (X_a \cup Z_a))$, and $V_\gamma \subseteq Z_a \cup (V - (X_a \cup Y_a))$.

Conditions (i)-(iii) express that (X, Y, Z) represents $p_{V'}(a)$ where $V' = V - (V_\alpha \cup V_\beta \cup V_\gamma)$; condition (iv) translates condition (6.1) hence expresses that $p_{V'}(a) \in L'$. Hence, the desired formula $\mu_1(V_\alpha, V_\beta, V_\gamma, X, Y, Z)$ can be taken of the form

$$\mu_2 \wedge \exists X_a, Y_a, Z_a(Member(X_a, Y_a, Z_a) \wedge \mu_3)$$

where μ_2 with free variables $V_\alpha, V_\beta, V_\gamma, X, Y, Z$ expresses conditions (i) and (ii) and μ_3 with free variables $V_\alpha, V_\beta, V_\gamma, X, Y, Z, X_a, Y_a, Z_a$ expresses conditions (iii) and (iv). □

Corollary 6.3. *Let $T = (\{v_1, \dots, v_n\}, L_T)$ be an isotropic system. One can construct an MS formula $\psi_T(x_1, \dots, x_n)$ such that if $S = (V, L)$ is an isotropic system and $v_1, \dots, v_n \in V$, then $S \vDash \psi_T(v_1, \dots, v_n)$ iff S has a minor identical to T.*

Proof. Let T be defined by the structure $\langle \{v_1, \dots, v_n\}, Member_T \rangle$.

Note that $v_1, \dots, v_n$ is an enumeration of the vertex set of T, and that we wish to express a property of isotropic systems with a vertex set containing $v_1, \dots, v_n$. We will construct a formula $\psi_T(x_1, \dots, x_n)$ with free variables $x_1, \dots, x_n$ that somehow encodes the structure of T, and expresses (by assuming that v_i is taken as value of x_i) that T is the minor obtained from S by certain minor reductions performed on the vertices not in $\{v_1, \dots, v_n\}$.

By using the proof of the previous proposition, we write a formula μ_4 with free variables $x_1, \dots, x_n, V_\alpha, V_\beta, V_\gamma$ expressing the following.

$$\begin{aligned}&\text{“}x_1, \dots, x_n \text{ are pairwise distinct”} \wedge \text{“}V - (V_\alpha \cup V_\beta \cup V_\gamma) = \{x_1, \dots, x_n\}\text{”}\\ &\wedge \forall X, Y, Z(\mu_1(V_\alpha, V_\beta, V_\gamma, X, Y, Z) \Longleftrightarrow \\ &\qquad \bigvee\{\text{“}X \text{ is } A\text{”} \wedge \text{“}Y \text{ is } B\text{”} \wedge \text{“}Z \text{ is } C\text{”} \mid Member_T(A, B, C)\}),\end{aligned}$$

where the disjunction extends to the finitely many triples (A, B, C) in $Member_T$ and "X is A" stands for the conjunction of

the formulas $x_i \in X$ for all i such that $v_i \in A$,
and of the formulas $\neg x_i \in X$ for all i such that $v_i \notin A$.

The meanings of "Y is B" and "Z is C" are similar. The desired formula is thus $\exists V_\alpha, V_\beta, V_\gamma . \mu_4(x_1, \dots, x_n)$. □

It is proved by Bouchet that if S is an isotropic system with associated fundamental graph G (for some pair of vectors (a, b)), and S' is a minor of S, then the fundamental graphs of S' (which are all locally equivalent) are vertex-minors of G. And conversely, every vertex-minor of G is a fundamental graph of some minor of S. By expressing these constructions in C_2MS logic, we will be able to define by C_2MS formulas the vertex-minors of a graph

We recall first a construction from Proposition 4.9. If $G = \langle V, edg\rangle$ is a graph, then we denote by $S(G)$ the isotropic system (V, L) where L is the set of vectors of the form

$$l(U) = \sum_{x \in U} (a[n(x)] + b[\{x\}]) \tag{6.2}$$

for all $U \subseteq V$, where a is the vector in K^V such that $a(v) = \alpha$ for all $v \in V$, and b is the vector in K^V such that $b(v) = \beta$ for all $v \in V$. This definition of $S(G)$ corresponds to a particular choice of a pair (a, b) of supplementary complete vectors.

Proposition 6.4.

(1) The set predicate $Member_{S(G)}$ is expressible in $\langle V, edg\rangle$ by a C_2MS formula.
(2) The mapping associating with a graph G the isotropic systems $S(G)$ is a C_2MS transduction.
(3) There is a C_2MS transduction associating with G the set of isotropic systems strongly isomorphic to $S(G)$, i.e., the set of isotropic systems of which G is a fundamental graph.

Proof. (1) We first show how to define $S(G) = (V, L)$ in logical terms.

A triple (X, Y, Z) of subsets of V represents a vector in L iff its components are pairwise disjoint and there exists a subset U of V such that, the vector $l(U)$ defined by (6.2) corresponds to (X, Y, Z), i.e., for all v in V,

$$\sum_{x \in U} (a[n(x)](v) + b[\{x\}](v)) = \begin{cases} \alpha & \text{if } v \in X \\ \beta & \text{if } v \in Y \\ \gamma & \text{if } v \in Z \end{cases}$$

From the definitions, we have $a[n(x)](v) = \alpha$ if $edg(x,v)$ holds, otherwise $= 0$, and $b[\{x\}](v) = \beta$ if $x = v$, otherwise $= 0$. Thus

$$\sum_{x \in U} (a[n(x)](v) + b[\{x\}](v))$$
$$= \begin{cases} \beta & \text{if } v \in U \text{ and } n(x) \cap U \text{ is even (because } \alpha + \alpha = 0), \\ \gamma & \text{if } v \in U \text{ and } n(x) \cap U \text{ is odd (because } \alpha + \alpha = 0 \text{ and } \alpha + \beta = \gamma), \\ 0 & \text{if } v \notin U \text{ and } n(x) \cap U \text{ is even,} \\ \alpha & \text{if } v \notin U \text{ and } n(x) \cap U \text{ is odd.} \end{cases}$$

From these observations, it is easy to write a $\mathrm{C_2MS}$ formula expressing conditions (6.1).

(2) The mapping S from graphs to isotropic systems is thus a $\mathrm{C_2MS}$ transduction.

(3) One obtains from $S(G)$ all strongly isomorphic isotropic systems by applying the MS transduction of Proposition 6.1. The composition of these two transductions is a $\mathrm{C_2MS}$ transduction. □

Remark. In the definition of $S(G)$ we have chosen a particular pair (a, b) of supplementary complete vectors which is easy to encode by logical formulas because all components are the same. By taking any other pair, one obtains an isotropic system strongly isomorphic to $S(G)$. The transformation of $S(G)$ into the systems strongly isomorphic to it is done by using Proposition 6.1. Applying a family of permutations Π to $S(G)$ is exactly the same thing as changing (a, b) into an other pair of supplementary complete vectors.

We now consider the inverse transformation.

Proposition 6.5. *The mapping from an isotropic system to the sets of its fundamental graphs is an MS transduction ν.*

Eulerian

Proof. Let $S = (V, L)$ be an isotropic system. Let a be a complete vector, described by (X_a, Y_a, Z_a). That a is complete means that $V = X_a \cup Y_a \cup Z_a$. It is said to be *Eulerian* if $L \cap \{a[U] \mid U \subseteq V\} = \{0\}$. This is equivalent to

$$(X \subseteq X_a, Y \subseteq Y_a, Z \subseteq Z_a \text{ and } Member_S(X,Y,Z)) \Rightarrow (X,Y,Z) = (\emptyset, \emptyset, \emptyset)$$

One can thus "select" an Eulerian vector and express by an MS formula that it is actually Eulerian. In terms of transductions, this vector will be specified by a triple (X_a, Y_a, Z_a) of set variables that will be the parameters of the transduction we are defining. By Proposition 4.9, for every v in V, there exists a unique vector b_v in L such that

$$b_v(v) \neq 0 \text{ for all } v \quad \text{and} \quad b_v(w) \in \{0, a(w)\} \text{ for } v \neq w.$$

fundamental graph

These vectors satisfy the following properties: $a(v) \neq b_v(v) \neq 0$ for all v, and $b_v(w) \neq 0$ iff $b_w(v) \neq 0$ for $v \neq w$. The graph G with vertex set V and an edge between v and w iff $b_v(w) \neq 0$ and $v \neq w$ is the *fundamental graph* of S associated with the Eulerian vector a. (Different graphs are obtained from other Eulerian vectors, but the are all locally equivalent).

The translation in MS logic is easy. We let $\nu_1(X, Y, Z, X_a, Y_a, Z_a, v)$ be the formula:

$$Member(X,Y,Z) \wedge v \in X \cup Y \cup Z$$
$$\wedge \forall w[w \neq v \Rightarrow \{(w \in X \Rightarrow w \in X_a) \wedge (w \in Y \Rightarrow w \in Y_a) \wedge (w \in Z \Rightarrow w \in Z_a)\}].$$

It expresses that (X, Y, Z) represents b_v. Now the edge relation of the graph G can be defined by

$$edg(v,w) \Longleftrightarrow v \neq w \wedge \exists X, Y, Z[\nu_1(X, Y, Z, X_a, Y_a, Z_a, v) \wedge w \in X \cup Y \cup Z].$$

Hence we have constructed an MS transduction ν that transforms an isotropic system given with a triple of sets (X_a, Y_a, Z_a) representing an Eulerian vector into the corresponding fundamental graph. □

Theorem 6.6.

1. *There exist C_2MS transductions that associate with a graph the set of its vertex-minors, and the set of its locally equivalent graphs.*
2. *For every graph H, one can build a closed C_2MS formula which expresses that a given graph contains a vertex-minor isomorphic to H.*

Proof. (1) We have constructed (Proposition 6.4) a C_2MS transduction S that associates with a graph an isotropic system, an MS transduction μ with parameters $V_\alpha, V_\beta, V_\gamma$ that associates with an isotropic system the set of its minors, and finally, an MS transduction ν with parameters X_a, Y_a, Z_a that associates with an isotropic system the set of its fundamental graphs. By results recalled in Section 4, the composition $\overline{\mu} = \nu \circ \mu \circ S$ of these transductions is the desired one. It is a C_2MS transduction by Proposition 5.3.2, with parameters $V_\alpha, V_\beta, V_\gamma, X_a, Y_a, Z_a$.

For obtaining from G the set of its locally equivalent graphs, it suffices to use $\nu \circ S$.

(2) For every graph H with vertices $1, \ldots, n$, one can construct a closed MS formula $\varkappa_H$ that is true in a graph iff this graph is isomorphic to H. This formula is written as following.

$$\begin{aligned}\exists x_1, \ldots, x_n[&\text{“}x_1, \ldots, x_n \text{ are pairwise distinct”}\\ &\wedge \text{“every vertex is equal to } x_i \text{ for some } i\text{”}\\ &\wedge \text{“for all } i, j,\ edg(x_i, x_j) \text{ holds iff } i \text{ and } j \text{ are neighbors in } H\text{”}]\end{aligned}$$

This formula is actually a first-order formula, because no set quantification is used. Its backwards translation relative to the transduction $\overline{\mu}$ is a C_2MS formula $\varkappa_H^{\#}$ with free variables $V_\alpha, V_\beta, V_\gamma, X_a, Y_a, Z_a$. It is valid in a graph G iff its vertex-minor defined by the sets $V_\alpha, V_\beta, V_\gamma, X_a, Y_a, Z_a$ ("defined" in the sense of the first part of the corollary) is isomorphic to H. Hence G has a vertex-minor isomorphic to H iff it satisfies $\exists V_\alpha, V_\beta, V_\gamma, X_a, Y_a, Z_a.\varkappa_H^{\#}$. □

A *circle graph* is the intersection graph of a set of chords of a circle. *circle graph*

Corollary 6.7. *There exists C_2MS formulas expressing that a graph is a circle graph, a distance-hereditary graph, or a graph locally equivalent to a tree.*

Proof. A. Bouchet has proved [Bou94] that a graph is a circle graph iff it has no vertex-minor isomorphic to either W_5, W_7 or Y_6, where W_5 is the 5-wheel (W_n is an n-cycle with an additional universal vertex ; Y_6 is a 6-cycle with an additional vertex and an edge linking it to the even vertices of this cycle assumed to be 1-2-3-4-5-6-1). The result follows then from Theorem 6.6.2.

The articles by Bouchet [Bou88b, Bou94] show that the class of distance-hereditary graphs is characterized by C_5 as an excluded vertex-minor. We obtain thus the result in the same way.

For graphs locally equivalent to a tree, the result follows from the definition by Theorem 6.6.1 and the fact [Cou97] that the class of trees is characterized by an MS formula. □

Remark. 1) The case of distance-hereditary graphs is given as an example of a set of graphs characterized by known excluded vertex-minors. There are not so many yet. It is also characterized by an infinite set of excluded induced subgraphs, namely the cycles C_n for $n \geq 5$ and three particular graphs (Bandelt and Mulder [BM86]). A definition of this set by an MS formula is easily derivable from this characterization because the infinitely many cycles C_n for $n \geq 5$ can easily be characterized by a unique MS formula.

2) The set of graphs locally equivalent to a tree is not closed under taking vertex-minors. By using the characterization given in [Bou88b] one can prove that graph consisting of the union of the paths $1-2-3-4$, $2-5-6$ and the edge $5-3$ is not locally equivalent to a tree but it is a vertex-minor of a tree, the union of the paths $1-2-7-3-4$ and $7-5-6$. One might ask for a characterization of the set vertex-minors of trees. Since these graphs have rank-width at most 1, they are characterized by a finite set of excluded vertex-minors.

Example 6.8. Let G be the "house" with vertices 1, 2, 3, 4, 5 forming the cycle 1-2-3-4-5-1, augmented with the edge 2-5.

If we use the construction of Proposition 6.4, we obtain the isotropic system $S = (\{1,2,3,4,5\}, L)$ where L contains the following vectors among a total number of 32.

$$\begin{aligned}
l(\{1\}) &= (\beta, \alpha, 0, 0, \alpha)\\
l(\{2\}) &= (\alpha, \beta, \alpha, 0, \alpha)\\
l(\{3\}) &= (0, \alpha, \beta, \alpha, 0)\\
l(\{4\}) &= (0, 0, \alpha, \beta, \alpha)\\
l(\{5\}) &= (\alpha, \alpha, 0, \alpha, \beta)\\
l(\{1,2\}) &= (\gamma, \gamma, \alpha, 0, 0)\\
l(\{1,5\}) &= (\gamma, 0, 0, \alpha, \gamma)\\
l(\{1,3,4\}) &= (\beta, 0, \gamma, \gamma, 0)\\
l(\{2,3,5\}) &= (0, \beta, \gamma, 0, \gamma)\\
l(\{2,4,5\}) &= (0, \gamma, 0, \gamma, \beta)
\end{aligned}$$

The vectors $l(\{1,2\}), l(\{1,5\}), l(\{1,3,4\}), l(\{2,3,5\}), l(\{2,4,5\})$ are respectively the vectors b_3, b_4, b_1, b_2, b_5 of the fundamental base of S relative to the Eulerian vector $(\gamma, \gamma, \gamma, \gamma, \gamma)$. The corresponding graph is C, the cycle 1-3-2-5-4-1.

One can transform C into G by the following sequence of local complementations: 1, 4, 2, 5, 3. The successive Eulerian vectors are

$$\begin{aligned}&(\gamma,\gamma,\gamma,\gamma,\gamma) \text{ for } C,\\&(\alpha,\gamma,\gamma,\gamma,\gamma),\\&(\alpha,\gamma,\gamma,\alpha,\gamma),\\&(\alpha,\alpha,\gamma,\alpha,\gamma),\\&(\alpha,\alpha,\gamma,\alpha,\alpha),\\&(\alpha,\alpha,\alpha,\alpha,\alpha) \text{ for } G.\end{aligned}$$

We now examine the three minor reductions associated with vertex 5. It is proved in [Bou88a, Theorem 9.1], that if an isotropic system S has a graphic presentation (G, a, b), $S' = S|^v_\rho$ and

1) if $\rho = a(v)$ and then S' has the graphic presentation $(G \setminus v, p_{V'}(a), p_{V'}(b))$,
2) if $\rho = a(v) + b(v)$ and then S' has the graphic presentation

$$(G * v \setminus v, p_{V'}(a), p_{V'}(b + a[n(v)])),$$

3) if $\rho = b(v)$ and then S' has the graphic presentation

$$(G * v * w * v \setminus v, p_{V'}(a + a[\{w\}] + b[\{w\}]), p_{V'}(b + a[\{w\}] + b[\{w\}]))$$

where w is any vertex adjacent to v in G.

For a graph H containing a vertex v we denote by $H \setminus v$ its induced subgraph obtained by deleting v.

First case : $\rho = \alpha = a(5)$. We recall that $a = (\alpha,\alpha,\alpha,\alpha,\alpha)$ and $b = (\beta,\beta,\beta,\beta,\beta)$. By computing L' from the definition, we can see it is vector space generated by

$$\begin{aligned}l'(\{1\}) &= (\beta,\alpha,0,0),\\l'(\{2\}) &= (\alpha,\beta,\alpha,0),\\l'(\{3\}) &= (0,\alpha,\beta,\alpha),\\l'(\{4\}) &= (0,0,\alpha,\beta).\end{aligned}$$

Then $(\alpha,\alpha,\alpha,\alpha)$ is Eulerian, and using the facts recalled in Proposition 6.4, we find that the corresponding fundamental graph is $G \setminus v$.

Second case : $\rho = \gamma = a(5)$. By computing L' from the definition, we can see it contains the following vectors.

$$\begin{aligned}l'(\{3\}) &= (0,\alpha,\beta,\alpha),\\l'(\{1,2\}) &= (\gamma,\gamma,\alpha,0),\\l'(\{1,4\}) &= (\beta,\alpha,\alpha,\beta),\\l'(\{1,5\}) &= (\gamma,0,0,\alpha),\\l'(\{2,4\}) &= (\alpha,\beta,0,\beta),\\l'(\{2,5\}) &= (0,\gamma,\alpha,\alpha),\\l'(\{4,5\}) &= (\alpha,\alpha,\alpha,\gamma).\end{aligned}$$

Then $(\alpha,\alpha,\alpha,\alpha)$ is again Eulerian, and using Proposition 6.4, we find that the corresponding fundamental base is

$$b_1 = l'(\{1,5\}), b_2 = l^{'}(\{2,5\}), b_3 = l'(\{3\}), b_4 = l'(\{4,5\}).$$

The corresponding fundamental graph is the 3-cycle 2-3-4-2 augmented with the edge 1-4. This graph is actually $G * 5 \setminus 5$.

One can check that the vectors $(\alpha,\alpha,\alpha,\alpha)$ and $(\gamma,\gamma,\beta,\gamma) = p_{\{1,2,3,4\}}(b+a[n(5)])$ yield a graphic presentation of S'.

Third case : $\rho = \beta = a(5)$. By computing L' from the definition, we can see it contains the following vectors

$$\begin{aligned} l'(\{5\}) &= (\alpha,\alpha,0,\alpha),\\ l'(\{1,2\}) &= (\gamma,\gamma,\alpha,0),\\ l'(\{1,4\}) &= (\beta,\alpha,\alpha,\beta),\\ l'(\{2,4\}) &= (\alpha,\beta,0,\beta),\\ l'(\{3,5\}) &= (\alpha,0,\beta,0). \end{aligned}$$

Then $(\alpha,\alpha,\alpha,\alpha)$ is no longer Eulerian. By changing the last component into β one obtains an Eulerian vector for which the corresponding fundamental base is

$$b_1 = l'(\{1,4\}), b_2 = l'(\{2,4\}), b_3 = l'(\{3,5\}), b_4 = l'(\{5\}).$$

The corresponding fundamental graph is G', the 3-cycle 1–2-4-1 augmented with the edge 1-3. One can check that this graph is actually $G * 5 * 4 * 5 \setminus 5$.

One can check that the vectors $(\alpha,\alpha,\alpha,\beta) = p_{\{1,2,3,4\}}(a + a[\{4\}] + b[\{4\}])$ and $(\beta,\beta,\beta,\alpha) = p_{\{1,2,3,4\}}(b + a[\{4\}] + b[\{4\}])$ yield a graphic presentation of S'.

6.2. **Computing a set of excluded vertex-minors.** It has been recalled in Section 2 that the vertex-minor relation is a well-quasi-ordering of the set of graphs of rank-width at most k. It follows by standard arguments, that if a set of graphs L is closed under taking vertex-minors and has rank-width (or clique-width) at most k then it is characterized by a finite set of excluded vertex-minors.

How can one compute this finite set? Does there exist an algorithm that would use as input the bound k and a finite formal description of the set L, typically a logical formula.

This question is not trivial. In the case of minor inclusion, it is proved in Courcelle et al. [CDF97], that for a set of graphs L that is minor-closed, whence characterized by a finite set $O_M(L)$ of excluded minors, it is not enough to know a membership algorithm for L in order to be able to compute $O_M(L)$. Formally, there is no algorithm taking as input an MS formula or a Turing Machine characterizing L and producing within a finite time the finite set $O_M(L)$ whenever L is minor-closed.

The following proposition may help in particular cases to compute finite sets of excluded vertex-minors.

Proper

For every set of graphs L closed under isomorphism, we denote by $O_{VM}(L)$ the set of graphs not in L, every proper vertex-minor of which is in L. *Proper* means that at least one vertex is deleted. For every set of graphs K, we denote by $Forb_{VM}(K)$ the set of graphs that have no vertex-minor isomorphic to a graph in K. If L is closed under isomorphism and vertex-minor, then

$$L = Forb_{VM}(O_{VM}(L)). \tag{6.3}$$

We are interested in the computation of $O_{VM}(L)$ when this set is finite up to isomorphism, and in its replacement by a smallest possible set.

Lemma 6.9. *If $L = \{G \mid G \models \xi\}$ where ξ is a closed CMS formula, then we have $O_{VM}(L) = \{G \mid G \models \psi\}$ for some closed CMS formula ψ that one can construct from ξ by an algorithm.*

Proof. We will use the C_2MS-transduction $\overline{\mu}$ of Theorem 6.6.1 that associates with a graph G the set of its proper vertex-minors. The parameters of this transduction are $V_\alpha, V_\beta, V_\gamma, X_a, Y_a, Z_a$. Let φ be the MS formula with free variables $V_\alpha, V_\beta, V_\gamma, X_a, Y_a, Z_a$ expressing that the parameters are correctly chosen, i.e., that a vertex-minor is defined from them by $\overline{\mu}$. It is the first formula of the definition scheme for $\overline{\mu}$ resulting from the proof of Theorem 6.6.1. The defined vertex-minor is proper iff $V_\alpha \cup V_\beta \cup V_\gamma$ is not empty. We let $\xi^\#$ be the backwards translation of ξ with respect to $\overline{\mu}$.

The desired formula ψ is thus :

$$\neg\xi \wedge \forall V_\alpha, V_\beta, V_\gamma, X_a, Y_a, Z_a[\varphi \wedge (V_\alpha \cup V_\beta \cup V_\gamma \neq \emptyset) \Longrightarrow \xi^\#]. \qquad \square$$

Note that this construction is correct even if L is not closed under taking vertex-minors. When it is, then (6.3) holds. If in addition, L has bounded rank-width, then $O_{VM}(L)$ is finite up to isomorphism by Theorem 2.8. Our objective is to find a "small" finite set K such that $L = Forb_{VM}(K)$.

It is clear that we can take for K any subset of $O_{VM}(L)$ that contains exactly one graph isomorphic to each graph in $O_{VM}(L)$. Furthermore, for any graphs G, H, and H', if H is isomorphic to a vertex-minor of G and H' is locally equivalent to H, then H' is isomorphic to a vertex-minor of G; hence we can reduce K by taking a subset of $O_{VM}(L)$ such that for any graph G in $O_{VM}(L)$, K contains exactly one graph, that is isomorphic to a graph locally equivalent to G. $O_{VM}(L)$. We call such a set K a *minimal set of vertex-minor obstructions* of L.

minimal set of vertex-minor obstructions

Our effort will now be to do this by an algorithm.

Lemma 6.10. *For every integer k and every closed CMS formula φ, one can decide whether the set $L = \{G \mid \mathrm{cwd}(G) \le k, G \models \varphi\}$ is finite (up to isomorphism). There exists an algorithm enumerating L when it is finite. One can compute from k and φ an integer m such that, either all graphs in L have at most m vertices or L has arbitrarily large graphs.*

Proof sketch. For each k, the graphs of clique-width at most k are the values of the finite terms built with a finite set of binary operations and nullary symbols denoted by $\mathcal{F}_k$, where $1, \ldots, k$ are the labels. (See Section 2). The nullary symbols denote the graphs with a single vertex labeled by i, for each $i = 1, \ldots, k$. There are only finitely many inequivalent compositions of the unary operations with k labels that relabel vertices (denoted $\rho_{i \to j}$) and create edges (denoted $\eta_{i,j}$). (Two compositions are *equivalent* if they define the same function.) For each equivalence class of these compositions, we select a representative λ and we define a binary operation $\otimes_\lambda$ by $G \otimes_\lambda H = \lambda(G \oplus H)$. We obtain thus the desired finite signature $\mathcal{F}_k$ consisting of k nullary symbols and the binary operations $\otimes_\lambda$.

equivalent

The value of each term t in $T(\mathcal{F}_k)$ is a graph $val(t)$ of clique-width at most k, and the number of vertices of $val(t)$ is equal to the number of occurrences of nullary symbols in t. The height of t (i.e., the length of a longest branch from the root to a leaf where t is considered as a rooted tree) is between $\log_2(Card(V))$ and $Card(V)$, where V is the set of vertices of $val(t)$. Every graph of clique-width at most k is the value a term in $T(\mathcal{F}_k)$, and there are only finitely many terms denoting a graph.

The set of terms in $T(\mathcal{F}_k)$, the value of which satisfies a closed CMS formula φ, i.e. the value of which is in $L = \{G \mid \text{cwd}(G) \leq k, G \models \varphi\}$ is defined by a finite tree-automaton $A(k,\varphi)$ that one can construct from k and φ by an algorithm : this is the basic fact underlying the existence of algorithms which verify in linear time the graph properties specified in CMS logic, on graphs of clique-width at most k, given as values of terms in $T(\mathcal{F}_k)$. However, its number of states is a tower of exponentials of height proportional to the quantifier depth of φ. (See 5.5.)

The so-called "Pumping Lemma" for tree-automata states that, if a tree-automaton accepts a term of height more than the number of states, then it accepts infinitely many terms. (Terms are usually called "trees" in automata theory.) It follows that one can decide whether the set of terms accepted by a tree-automaton is finite, and when it is, an algorithm can enumerate the accepted terms. For definitions and results on tree-automata, the reader is referred to the book by Comon et al. [CDG+97], available on line.

The set of terms defined by $A(k,\varphi)$ is finite iff the set of graphs L is finite (up to isomorphism). This can be decided, and in case of finiteness, the terms accepted by $A(k,\varphi)$, can be enumerated. By evaluating these terms one obtains at least one graph isomorphic to each graph in L. It remains to remove graphs which have an isomorphic copy in the list (because two different terms may define isomorphic graphs).

Let $m = 2^N$ where N is the number of states of $A(k,\varphi)$. If a graph in L has more than 2^N vertices, it must be defined by a term in $T(\mathcal{F}_k)$ of height more than N accepted by $A(k,\varphi)$ which then accepts infinitely many terms. The values of these terms are graphs with an unbounded number of vertices, since the number of vertices of a graph is at least the height of a term $T(\mathcal{F}_k)$. This proves the last assertion. □

Proposition 6.11. *There exists an algorithm that takes as input an integer k and a closed CMS formula ξ, and that produces a minimal set of vertex-minor obstructions for $L = \{G \mid\ G \models \xi\}$ if this set is closed under taking vertex-minors and has rank-width at most k. If these conditions are not satisfied the algorithm stops but reports a failure or produces irrelevant output.*

Proof. Let us assume that $L = \{G \mid G \models \xi\}$ has rank-width at most k. Then the graphs in $O_{VM}(L)$ have rank-width at most $k+1$. Hence they have clique-width at most $f(k)$, where $f(k) = 2^{k+2} - 1$. We let ψ be obtained by Lemma 6.9. Then

$$O_{VM}(L) = \{G \mid\ G \models \psi\} = \{G \mid \text{cwd}(G) \leq f(k), G \models \psi\}.$$

If L is closed under taking vertex-minors, then $O_{VM}(L)$ is finite up to isomorphism and can be computed by the algorithm of Lemma 6.10, applied to the formula ψ and the integer $f(k)$. Computed means that one can construct a finite subset K of $O_{VM}(L)$ that contains exactly one graph in each isomorphism class. Then, This set can be reduced into a subset K' of K that for any graph G in $O_{VM}(L)$, K' contains exactly one graph isomorphic to a graph locally equivalent to G. It is clear that K' is a minimal set of vertex-minor obstructions for L.

If the conditions on L are not satisfied, the algorithm may report that $\{G \mid \text{cwd}(G) \leq f(k), G \models \psi\}$ is infinite or produce a finite set K which does not satisfy $L = Forb_{VM}(K)$. □

The algorithms of Lemma 6.10 and Proposition 6.11 are clearly not implementable. They are interesting as computability results.

6.3. **Recognizing graphs of rank-width at most k.** By Corollary 2.9, for a fixed k, there are only finitely many graphs, such that a graph does not contain any of them as a vertex-minor iff it has rank-width at most k. By Theorem 6.6.2, for any fixed graph H, there is a C_2MS formula expressing that H is isomorphic to a vertex-minor of an input graph. In Theorem 2.5, we have a polynomial-time algorithm that either confirms the input graph has rank-width at least $k+1$ or confirms the rank-width is at most $3k+1$ and outputs a rank-decomposition of width at most $3k+1$. In [OS04], an algorithm that converts the rank-decomposition into a k-expression is developed. In 5.5, we recall that any property specified by a CMS formula can be checked in linear time on graphs given by a k-expression.

By combining all of these, we get the following.

Theorem 6.12. *For any fixed k, there is a polynomial-time algorithm to check that the input graph has rank-width at most k.*

Even though the algorithm of Theorem 2.5 is fixed-parameter-tractable, this one may not be so, because the number of excluded vertex-minors for rank-width at most k may not be polynomial.

7. Proof of Seese's Conjecture via vertex-minors

The main result of this section is the following one :

Theorem 5.6. *If a set of graphs has a decidable C_2MS satisfiability problem, then it has bounded rank-width and bounded clique-width.*

The proof will use a family of bipartite graphs, S_k, for $k > 1$ from which $k \times (2k-2)$ rectangular grids can be built by a fixed MS transduction, and satisfying the following result.

Proposition 7.1. *Let L be a set of bipartite graphs of unbounded rank-width. Infinitely many graphs S_k are isomorphic to vertex-minors of graphs in L.*

Proof. Suppose not. Then, there is an integer k such that S_k is not isomorphic to any vertex-minor of every graph in L. By Corollary 3.4, there is an integer l such that every graph in L has rank-width at most $l-1$. Contradiction. □

Proposition 7.2. *There exists an MS transduction τ such that $G_{k\times(2k-2)}$ belongs to $\tau(S_k)$ for all $k > 1$.*

Proof. The transduction τ is the composition of several transductions. We do not give the explicit formulas but we explain how they can be obtained. We are given S_k as $\langle V, A, B, edg\rangle$. Our aim is to build from it a grid with $2k-2$ rows of length k , namely :

$(a_1, \ldots, a_k)$,
$(b_1, \ldots, b_k)$,
$(a_{k+1}, \ldots, a_{2k})$,
$(b_{k+1}, \ldots, b_{2k})$,
$\ldots$
$(a_{k^2-2k+1}, \ldots, a_{k^2-k})$,
$(b_{k^2-2k+1}, \ldots, b_{k^2-k})$.

Step 1 : Ordering A and B.

We first define by MS formulas the orderings of A and B defined by the indices. (The sets A and B are given in $\langle V, A, B, edg\rangle$ as unordered sets ; the indices are used to define S_k shortly, but are not expressed in the relational structure). We assume that $\{b_1\}$ is given by means of a parameter, say Y.

consecutive

Two elements b and b' of B are *consecutive*, i.e., $b = b_i$ and $b' = b_{i+1}$ or vice-versa iff $n(b)\Delta n(b')$ (the symmetric difference of their neighborhoods which are subsets of A) has exactly two elements. From this, and by knowing b_1 one can determine the order on B such that $b < b'$ iff $b = b_i$ and $b' = b_j$ for some $j > i$. We have actually $b < b'$ iff $b \neq b'$ and :

either $b = b_1$

or there exists a subset X of B that contains b_1 and b but not b', and is such that each of b and b_1 is consecutive to exactly one element of X, and each element of $X - \{b, b_1\}$ is consecutive to exactly two elements of X.

This characterization is expressible by an MS formula.

The analogous strict linear order $<$ on A is characterized by : $a < a'$ iff there exist $b, b' \in B$ with $b < b'$, and $a \in n(b) - n(b'), a' \in n(b')$ or $a \in n(b), a' \in n(b') - n(b)$. It is also expressible by an MS formula. We can thus transform S_k into the structure $S'_k = \langle V, A, B, <, edg\rangle$ by an MS transduction τ_1.

Step 2 : Some edge modifications.

minimal
maximal

A *minimal* (resp. *maximal*) edge is one between b_i and a_i (resp. a_{i+k-1}). Each b in B is incident to a unique minimal (resp. maximal) edge, the A-vertex of which is the least (greatest) neighbor of b, where "least" and "greatest" are relative to $<$. On the drawing of S_4 in Figure 1, the minimal edges are vertical. The maximal edges are oblique and drawn with a thick line.

These edges can be identified by MS formulas evaluated in S'_k. We build T_k from S'_k as follows:

1) We add edges between each b_i and a_{i+k} for $i = 1, \ldots, k^2 - 2k$. This is possible because MS formulas can identify b_{k^2-k} (as the maximal element of B), and thus a_{k^2-k} (linked to b_{k^2-k} by a minimal edge), whence also b_{k^2-2k+1} linked to a_{k^2-k} by a maximal edge. Hence an MS formula can identify b_{k^2-2k} as the predecessor of b_{k^2-2k+1}. An MS formula can identify for each b the corresponding a_{i+k-1} where $b = b_i$, $i \in \{1, \ldots, k^2 - 2k\}$. The new edges to be added between b_i and a_{i+k} can thus be defined by an MS formula, since one can determine a_{i+k} as the successor of a_{i+k-1} in A.

2) We delete all edges except the minimal edges and of course, the edges added in 1).

3) We delete the isolated vertices, which are the vertices a_i for $i > k^2 - k$.

We get thus by an MS transduction τ_2 , a graph T_k, equipped with the orderings $<$ of A and B.

Step 3 : Making T_k into a rectangular grid.

The graph T_k consists of k disjoint paths with $2k - 2$ vertices. To make T_k into the grid $G_{k\times(2k-2)}$, it suffices to add edges between a_i and a_{i+1}, and between b_i and b_{i+1} for each $i \in I$ defined as $I = \{1, \ldots, k^2 - k\} - \{pk \mid p = 1, \ldots, k - 1\}$. The edges added during this step are the horizontal lines in the grid $G_{4\times 6}$ of Figure 1.

This can be done from the set $U = \{a_i, b_i \mid i \in I\}$. This set can be "guessed", i.e., given as a parameter to the transduction τ_3 we are defining. This transduction also deletes the orderings $<$.

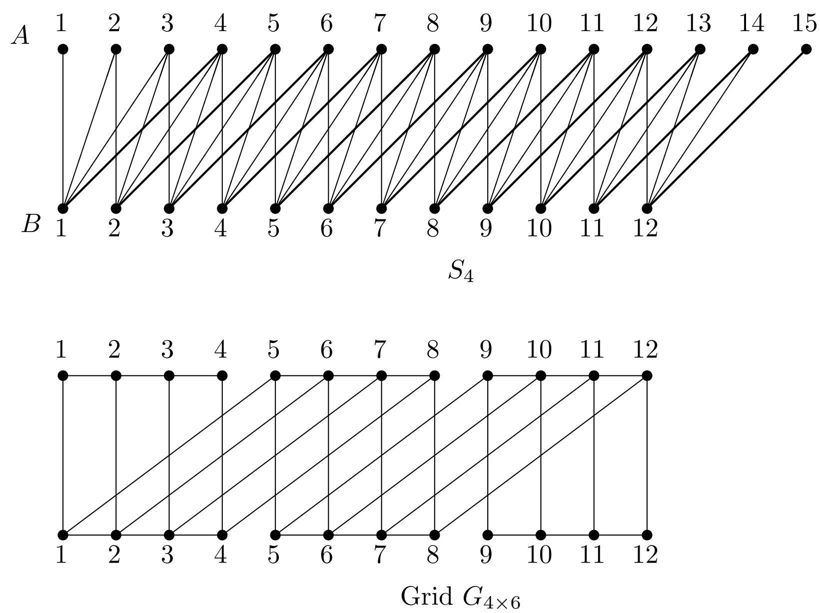

FIGURE 1. Getting the grid from S_k

We let τ be the transduction $\tau_3 \circ \tau_2 \circ \tau_1$. It uses actually two parameters, Y intended to specify b_1 (by $Y = \{b_1\}$) and the above set U. Whenever the sets Y and U are "correctly chosen" (so that the above construction works as described) for a graph isomorphic to S_k, then the structure $\tau(S_k, Y, U)$ is the grid $G_{k\times(2k-2)}$. If they are not correctly chosen, a graph that is not a grid may be produced. But we only demand that τ produces grids $G_{k\times(2k-2)}$ among other graphs we need not care about. Hence, we are done. □

The following lemma is a consequence of several lemmas proved in [Cou04b]. In order to facilitate the reading of the present article, we give a direct proof.

Lemma 7.3. *There a bijection γ of the set of graphs onto a set of 4-colored bipartite graphs, such that γ and its inverse are isomorphic to MS transductions. Under any of these two transformations, the image of a set of graphs of bounded clique-width (equivalently bounded rank-width) has bounded clique-width (equivalently bounded rank-width).*

Proof. Let G be a graph with set of vertices V.We let $B(G)$ be the bipartite graph with set of vertices $W = V \times \{1, 2, 3, 4\}$, colored by 1,2,3,4 in the obvious way, and undirected edges linking $(x, 1)$ and $(x, 2)$, $(x, 2)$ and $(x, 3)$, $(x, 3)$ and $(x, 4)$, $(x, 1)$ and $(y, 4)$ for every x in V and every y in $n_G(x)$. It is clear that $B(G)$ is bipartite. The graph G is obtained from $B(G)$ by the MS transduction that contracts the edges the ends of which are colored by 1 and 2, or by 2 and 3, or by 3 and 4. Parallel edges are fused : we only consider simple graphs, and actually, the relational structures representing graphs do not distinguish parallel edges. Colors 1,2,3,4 are deleted.

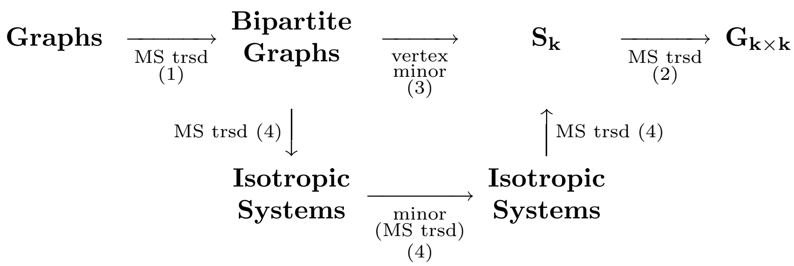

FIGURE 2. Sketch of the first proof

This transduction can be defined as follows. One defines an auxiliary first-order formula $path(u,v)$ saying that there is a path $u-w-z-v$ the vertices of which have colors 1,2,3,4 in this order. One deletes the vertices colored by 2,3,4. One defines edges by taking for $\theta_{edg}(x,y)$ the formula $\exists z((edg(x,z) \wedge path(y,z)) \vee (edg(y,z) \wedge path(x,z))$. One deletes the unary relation indicating the color 1.

The opposite transformation of G into $B(G)$ is an MS transduction that duplicates a fixed number of times (here 4 times) a given structure before defining the new structure inside it. (This technical notion is not defined in this paper. The reader is referred to [Cou94, Cou97, Cou04b].)

The images of a graph of clique-width at most k under an MS transduction τ, have clique-width at most $f(k)$, where f is a function depending on τ (see Proposition 5.4). Since for every graph G, we have $\mathrm{rwd}(G) \leq \mathrm{cwd}(G) \leq 2^{\mathrm{rwd}(G)+1} - 1$, a similar statement holds for rank-width and for some function g, effectively computable from the definition of τ. It follows that the mapping B and its inverse preserve bounded clique-width and bounded rank-width. □

Proof of Theorem 5.6. Let $\mathcal{C}$ be a set of graphs having a decidable C_2MS satisfiability problem and unbounded rank-width. We will get a contradiction.

For a graph G, we let $B'(G)$ be $B(G)$ without the colors 1,…,4. The set $B'(\mathcal{C})$ has unbounded rank-width by the above lemma (because colors do not matter). By applying to $B'(\mathcal{C})$ the C_2MS transduction $\overline{\mu}$ of Theorem 6.6, we obtain among its vertex-minors, an infinite set of graphs S_k, by Corollary 3.4 and by applying then the MS transduction τ of Proposition 7.2, one gets an infinite set of $k \times (2k-2)$ grids.

We now observe that these transformations preserve the decidability of C_2MS satisfiability. This is so because B' and τ are MS transductions, and $\overline{\mu}$ is a C_2MS transduction.

But a set of graphs containing infinitely many $k\times(2k-2)$ grids has an undecidable MS satisfiability problem. We have reached a contradiction.

Hence if $\mathcal{C}$ has a decidable C_2MS satisfiability problem, it must have bounded rank-width. It has also bounded clique-width. □

The proof is illustrated on Figure 2: (1) is the MS transduction of Lemma 7.3, (3) is vertex-minor reduction expressible by C_2MS formulas by means of isotropic systems, and (2) is the MS transduction constructed in Proposition 7.2. The transformation from bipartite graphs to isotropic systems is a C_2MS transduction (Proposition 6.4), those from isotropic systems to their minors and to their fundamental graphs are MS transductions (Propositions 6.2 and 6.5).

Corollary 7.4. *There exists an C_2MS transduction θ such that, if $\mathcal{C}$ is a set of graphs of unbounded clique-width or of unbounded rank-width, then $\theta(\mathcal{C})$ contains infinitely many square grids.*

Proof. We let $\theta = Ind \circ \tau \circ \overline{\mu} \circ B'$ where Ind is the MS transduction that associates with a graph the set of its induced subgraphs. It transforms $G_{k\times(2k-2)}$ into a set of graphs containing $G_{k\times k}$. □

By using an MS transduction encoding directed graphs into bipartite graphs [Cou04b], one obtains a similar statement for directed graphs.

Definition 7.5 (MS orderable classes of graphs (Courcelle [Cou96])**.** We say that a set of graphs $\mathcal{C}$ is *MS orderable* if there exists a pair *MS orderable*

$$(\delta(X_1, \ldots, X_n), \sigma(x, y, X_1, \ldots, X_n))$$

of MS formulas such that :

1) For ever G in $\mathcal{C}$, there exist sets of vertices $X_1, \ldots, X_n$ such that :

$$(G, X_1, \ldots, X_n) \models \delta,$$

2) For every n-tuple as above, the binary relation defined by

$$xRy \quad \text{iff} \quad (G, x, y, X_1, \ldots, X_n) \models \sigma$$

is a linear order of the set of vertices of G.

Theorem 7.6. *If a set of graphs (resp. of directed graphs) is MS orderable and has a decidable MS satisfiability problem, then it has bounded rank-width and bounded clique-width (resp. bounded clique-width).*

Proof. If $\mathcal{C}$ is MS orderable and has a decidable MS satisfiability problem, then its C_2MS satisfiability problem is decidable (and even the CMS one is), and then we can conclude using Theorem 5.6.

The proof of this claim is as follows. Let φ be a CMS formula for which we ask whether it is satisfied by some graph in $\mathcal{C}$. Then we can rewrite it into an MS formula φ' by expressing the cardinality predicates in term of the linear order defined by σ. The formula φ' has thus free variables $X_1, \ldots, X_n$. Then, $G \models \varphi$ for any G in $\mathcal{C}$ iff $G \models \exists X_1, \ldots, X_n(\delta(X_1, \ldots, X_n) \wedge \varphi')$. From the initial hypothesis and since the formula $\exists X_1, \ldots, X_n(\delta(X_1, \ldots, X_n) \wedge \varphi')$ is MS (and not CMS) one can decide whether there exists G in $\mathcal{C}$ such that $G \models \varphi$. □

Example 7.7. Consider the set $\mathcal{D}$ of directed graphs without circuits having a directed Hamiltonian path. The relation "$x = y$ or there exists a directed path from x to y" is a linear order and it is definable by an MS formula since MS formulas can express transitive closure. Hence $\mathcal{D}$ satisfies the conditions of Theorem 7.6, whence the Conjecture.

The validity of the Conjecture for $\mathcal{D}$ cannot be established with the methods of Courcelle [Cou04b], i.e., by reduction to the result of Robertson and Seymour on excluded planar minors, because these methods apply only to sets of graphs having at most $2^{O(n\log(n))}$ graphs with n vertices. But $\mathcal{D}$ has $2^{(n-2)(n-3)/2}$ graphs with n vertices.

Theorem 5.6 extends easily to countable graphs. We first adapt the logical language. The even cardinality predicate is only meaningful for finite sets. Hence, for countable structures, we will use the logical language $\mathrm{C}_2^f\mathrm{MS}$ containing the following set predicates : $Finite(X)$ which says that X is finite, and $Even(X)$ which says that X is finite and has even cardinality. That a set is odd can be expressed by the formula $Finite(X) \wedge \neg Even(X)$.

The extension of Theorem 5.6 to countable graphs rests on a "compactness" theorem by Courcelle [Cou04a] stating that a set of countable graphs has bounded clique-width iff the set of all its finite induced subgraphs has bounded clique-width. We refer the reader to this paper for the definition of the clique-width of countable graphs. The above characterization is enough for the following

Theorem 7.8. *If L a set of finite or countable graphs has a decidable $\mathrm{C}_2^f\mathrm{MS}$ satisfiability problem, then it has bounded clique-width.*

Proof. The mapping associating with a graph the set of its finite induced subgraphs is a $\mathrm{C}_2^f\mathrm{MS}$ transduction, because the finiteness set predicate makes it possible to restrict graphs to their finite induced subgraphs. Hence the set of finite induced subgraphs of the graphs in L also has a decidable $\mathrm{C}_2\mathrm{MS}$ satisfiability problem (by Proposition 5.3.1), hence bounded clique-width. So has L by the compactness theorem of [Cou04a]. □

8. Seese's Conjecture proved via matroids

We give another, somewhat shorter proof of Theorem 5.6 based on binary matroids instead of isotropic systems and using results by Geelen, Gerards, and Whittle [GGW03] and Hliněny and Seese [HS04]. The proof of Section 7 using vertex-minors is of independent interest because it deals with graphs, and the tools it uses yield the polynomial-time algorithm for recognizing graphs rank-width at most k. The set predicate $Even$ remains necessary in our alternative proof.

We let G be a bipartite graph given as $\langle V, A, B, edg\rangle$. The matroid $Bin(G, A, B)$ can be represented by the structure $\langle V, Indep\rangle$ where we recall that for each $U \subseteq V$, $Indep(U)$ holds iff U is an independent set. This property can be expressed by the $\mathrm{C}_2\mathrm{MS}$ formula $\iota(U)$:

$$\forall W[(\emptyset \neq W \subseteq U) \Longrightarrow \neg Zero(W)]$$

where $Zero(W)$ translates the following conditions :

(8.1a) $\forall x[x \in A \wedge x \in W \Longrightarrow \text{"the set } W \cap n(x) \text{ is odd"}]$

and

(8.1b) $\forall x[x \in A \wedge x \notin W \Longrightarrow \text{"the set } W \cap n(x) \text{ is even"}]$.

The set predicate $Zero(W)$ means that the sum of the vectors in W is the null vector.We check the correctness of its characterization by (8.1).

Let c be the vector $\sum W = \sum W \cap A + \sum W \cap B$. Then $c = (0, \ldots, 0)$ iff for each coordinate i, we have $c[i]= 0$. If $a_i \in A$, then $(\sum W \cap A)[i] = 1$, and $c[i]= 0$ iff $(\sum W \cap B)[i] = 1$ iff the number of vertices in B that are neighbors of a_i is odd. This is expressed by (8.1a). If $a_i \notin A$, then $(\sum W \cap A)[i] = 0$, and $c[i] = 0$

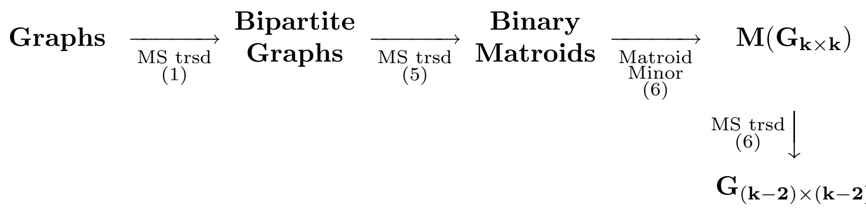

FIGURE 3. Sketch of the second proof

iff $(\sum W \cap B)[i] = 0$ iff the number of vertices in B that are neighbors of a_i is even. This is expressed by (8.1b). Conditions (8.1) are easily expressible by C_2MS formulas. We have thus proved the following proposition.

Proposition 8.1. *The mapping Bin from bipartite graphs to binary matroids is a C_2MS transduction.*

The following proposition is proved in [HS04] but stated in a different language. We recall the notation $G_{k\times k}$ for the $k \times k$ grid.

Proposition 8.2 ([HS04]). *1) The transduction associating with a matroid the set of its minors is an MS transduction.*

2) There exists an MS transduction ζ from matroids to graphs that associates $G_{(k-2)\times(k-2)}$ with $M(G_{k\times k})$, the cycle matroid of $G_{k\times k}$ for k even and at least 6.

Proof. Assertion 1) is the content of Lemmas 6.4 and 6.5, and Assertion 2) is that Lemmas 6.6 and 6.7 of [HS04]. □

Second proof of theorem 5.6. The method is similar to that of the first proof.

By Lemma 7.3, we need only consider a set of bipartite graphs $\mathcal{C}$ of unbounded rank-width having a decidable C_2MS satisfiability problem and derive a contradiction.

We will use the Proposition 3.1, which states that for a bipartite graph G with a bipartition $V(G) = A \cup B$, the branch-width of $Bin(G, A, B)$ is equal to the rank-width of G +1.

Let us apply to $\mathcal{C}$ the transduction $\kappa = \zeta \circ Bin$. Then the set of matroids $Bin(\mathcal{C})$ has unbounded branch-width, hence, by a result of Geelen, Gerards, and Whittle [GGW03], it contains cycle matroids $M(G_{k\times k})$ for infinitely many k. The transduction κ produces thus from $\mathcal{C}$ infinitely many square grids.

Since we assume that $\mathcal{C}$ has a decidable C_2MS satisfiability problem, and since κ is a C_2MS transduction, then so has $\kappa(\mathcal{C})$. But it cannot contain infinitely many square grids. This is the desired contradiction. □

The schema of the proof is illustrated on Figure 3: (1) is the MS transduction of Lemma 7.3, (5) is the C_2MS transduction Bin of Proposition 8.1, the MS transductions of (6) are from [HS04].

9. CONCLUSION

We have shown how isotropic systems can be handled in C_2MS logic. Together with other results, we could prove a slight weakening of Seese's Conjecture and obtain polynomial-time algorithms for recognizing graphs of rank-width at most k, for each k. Some questions remain open.

Question 1. Is the original Conjecture valid?

Question 2. Is it true that if a set of relational structures without set predicates has a decidable MS (or C_2MS) satisfiability problem, then it is contained in the image of a set of trees under an MS transduction (or a C_2MS transduction).

Even though the graphs of rank-width at most k are recognizable in polynomial time and $\mathrm{rwd}(G) \leq \mathrm{cwd}(G) \leq 2^{\mathrm{rwd}(G)+1} - 1$, it does not answer the following question for $k > 3$.

Question 3. Is there a polynomial-time algorithm recognizing graphs of clique-width at most k?

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