Scattered classes of graphs

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Abstract

For a class $\mathcal{C}$ of graphs G equipped with functions f_G defined on subsets of $E(G)$ or $V(G)$, we say that $\mathcal{C}$ is *k-scattered* with respect to f_G if there exists a constant ℓ such that for every graph $G \in \mathcal{C}$, the domain of f_G can be partitioned into subsets of size at most k so that the union of every collection of the subsets has f_G value at most ℓ. We present structural characterizations of graph classes that are k-scattered with respect to several graph connectivity functions.

In particular, our theorem for cut-rank functions provides a rough structural characterization of graphs having no $mK_{1,n}$ vertex-minors, which allows us to prove that such graphs have bounded linear rank-width.

1 Introduction

All graphs in this paper are undirected and simple. For a graph G, we write $V(G)$ and $E(G)$ to denote vertex set and edge set of G, respectively.

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In the theory of split decompositions, Cunningham [5] introduced the concept of a *brittle graph* that is a connected graph whose every non-trivial vertex bipartition is a split, which is a partition (A, B) of the vertex set such that $|A|, |B| \geqslant 2$, and there exist $A' \subseteq A$, $B' \subseteq B$ such that the set of edges incident with both A and B is exactly $\{vw : v \in A', w \in B'\}$. In undirected graphs, only complete graphs and stars are such graphs. Brittle graphs form basic classes of graphs in canonical split decompositions.

Motivated by brittle graphs, we introduce general concept of a partition $(X_1, X_2, \ldots, X_m)$ of the vertex set or the edge set of a graph such that each X_i has at most k elements, and for every $I \subseteq \{1, 2, \ldots, m\}$, some complexity measurement between $\bigcup_{i\in I} X_i$ and the rest is at most ℓ, for given integers k and ℓ. Brittle graphs then can be seen as graphs that admit a partition $(X_1, X_2, \ldots, X_m)$, where $X_1, X_2, \ldots, X_m$ consist of distinct individual vertices, and for every $I \subseteq \{1, 2, \ldots, m\}$, the cut-rank function of $\bigcup_{i\in I} X_i$ is at most 1. This concept trades off between the allowed sizes of parts in a partition and the allowed values for a selected complexity measurement.

We formally define this concept and provide examples. Let X be a finite set and $f : 2^X \to \mathbb{Z}$. The *f-width* of a partition $(X_1, X_2, \ldots, X_m)$ of X, for some m, is

$$\max \left\{ f\Big(\bigcup_{i\in I} X_i\Big) : I \subseteq \{1, 2, \ldots, m\} \right\}.$$

The *k-brittleness* of f is the minimum f-width of all partitions of X into parts of size at most k.

We are mainly interested in the following four functions arising from graphs naturally.

- For a subset F of $E(G)$, let $\kappa_G(F)$ be the number of vertices incident with both an edge in F and an edge not in F.
- For a subset S of $V(G)$, let $\eta_G(S)$ be the number of edges incident with both a vertex in S and a vertex not in S.
- For a subset S of $V(G)$, let $\nu_G(S)$ be the size of a maximum matching of a bipartite subgraph of G obtained by taking edges joining S and $V(G)\backslash S$.
- For a subset S of $V(G)$, let $\rho_G(S)$ be the rank of the $S \times (V(G)\backslash S)$ 0-1 matrix over the binary field whose (a, b)-entry for $a \in S$, $b \notin S$ is 1 if a, b are adjacent and 0 otherwise. This function is called the *cut-rank* function of G.

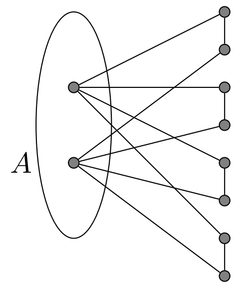

Figure 1: The graph $4P_4/A$ for a path $P_4 = abcd$ with $A = \{a, d\}$.

The k-brittleness of κ_G, η_G, ν_G, ρ_G are called the *vertex k-brittleness* $\beta_k^\kappa(G)$, the *edge k-brittleness* $\beta_k^\eta(G)$, the *matching k-brittleness* $\beta_k^\nu(G)$, the *rank k-brittleness* $\beta_k^\rho(G)$ of G, respectively. We say that a class $\mathcal{C}$ of graphs is *vertex k-scattered* if vertex k-brittleness of graphs in $\mathcal{C}$ are bounded, *edge k-scattered* if edge k-brittleness of graphs in $\mathcal{C}$ are bounded, *matching k-scattered* if matching k-brittleness of graphs in $\mathcal{C}$ are bounded, and *rank k-scattered* if rank k-brittleness of graphs in $\mathcal{C}$ are bounded.

A class $\mathcal{C}$ of graphs is called a *subgraph ideal* if it contains every graph isomorphic to a subgraph of a graph in $\mathcal{C}$. We characterize subgraph ideals which are vertex k-scattered, or edge k-scattered, or matching k-scattered. Our first theorem characterizes a vertex k-scattered subgraph ideal. For a graph H, we write mH to denote the disjoint union of m copies of H. For a graph H and an independent set $A \subsetneq V(H)$, we write mH/A to the graph obtained from mH by identifying all m copies of each vertex in A into one vertex. Note that the number of vertices of mH/A is $m(|V(H)| - |A|) + |A|$ and $1H/A = H$. See Figure 1 for an illustration.

Theorem 1.1. *Let k be a positive integer. A subgraph ideal $\mathcal{C}$ is vertex k-scattered if and only if*

$$\{1H/A, 2H/A, 3H/A, 4H/A, \ldots\} \nsubseteq \mathcal{C}$$

for every connected graph H with $k+1$ edges and each of its independent subset $A \subsetneq V(H)$.

Our second theorem characterizes an edge k-scattered subgraph ideal.

Theorem 1.2. *Let k be a positive integer. A subgraph ideal $\mathcal{C}$ is edge k-scattered if and only if*

$$\{K_{1,1}, K_{1,2}, K_{1,3}, \ldots\} \nsubseteq \mathcal{C}$$

and

$$\{T, 2T, 3T, 4T, \ldots\} \nsubseteq \mathcal{C}$$

for every tree T on $k+1$ vertices.

Our third theorem characterizes a matching k-scattered subgraph ideal.

Theorem 1.3. *Let k be a positive integer. A subgraph ideal $\mathcal{C}$ is matching k-scattered if and only if*

$$\{T, 2T, 3T, \ldots\} \nsubseteq \mathcal{C}$$

for every tree T on $k+1$ vertices.

Finally we characterize rank k-scattered graph classes. As the cut-rank function may increase when we take a subgraph, subgraph ideals are not suitable for the study of rank k-scattered graph classes. For instance, complete graphs are rank 1-scattered and yet an arbitrary graph is a subgraph of a complete graph.

Instead of subgraphs, the containment relation called *vertex-minors* is more suitable for the study of rank k-scattered graph classes. A vertex-minor of a graph G is an induced subgraph of a graph that can be obtained from G by a sequence of *local complementations* [1, 2, 3, 14], where local complementation at a vertex v is an operation to flip the adjacency relations between every pair of neighbors of v. The precise definition will be presented in Section 2. The cut-rank function is preserved when applying local complementations [2, 14] and therefore, the rank k-brittleness of a graph does not increase when taking vertex-minors.

A class $\mathcal{C}$ of graphs is called a *vertex-minor ideal* if it contains every graph isomorphic to a vertex-minor of a graph in $\mathcal{C}$. Here is our last theorem on the characterization of rank k-scattered vertex-minor ideals.

Theorem 1.4. *Let k be a positive integer. A vertex-minor ideal $\mathcal{C}$ is rank k-scattered if and only if*

$$\{H, 2H, 3H, 4H, \ldots\} \nsubseteq \mathcal{C}$$

for every connected graph H on $k+1$ vertices.

There are lots of interesting open problems on vertex-minors. In particular, the conjecture of Oum [15] implies that for every circle graph H, every graph G with sufficiently large rank-width has a vertex-minor isomorphic to H. This conjecture is known to be true when G is a bipartite graph, a circle graph, or the line graph [14, 15]. Kanté and Kwon [11] proposed the following analogous conjecture for linear rank-width.

Conjecture 1.5 (Kanté and Kwon [11]). *For every fixed tree T, there is an integer $f(T)$ such that every graph of linear rank-width at least $f(T)$ contains a vertex-minor isomorphic to T.*

By the Ramsey theorem, every sufficiently large connected graph contains one of $K_{1,n}$, K_n, or P_n as an induced subgraph and if n is huge, then each of these graphs contains a large star graph as a vertex-minor. Therefore for each fixed n, each component of a graph having no $K_{1,n}$ vertex-minor has bounded number of vertices and thus it has bounded linear rank-width. Thus, Conjecture 1.5 is true when T is a star.

We can strengthen this observation by Theorem 1.4 and verify Conjecture 1.5 when T is the disjoint union of stars.

Theorem 1.6. *For positive integers m and n, the class of graphs having no vertex-minor isomorphic to $mK_{1,n}$ has bounded linear rank-width.*

By Theorem 1.6, we can recognize whether a graph contains a vertex-minor isomorphic to the fixed disjoint union of stars and complete graphs in polynomial time. This works as follows. By Theorem 1.6 if the input graph has large linear rank-width, then trivially it has a vertex-minor isomorphic to $mK_{1,n}$ for some large m and n where $mK_{1,n}$ contains the disjoint union of stars and complete graphs as a vertex-minor. Otherwise, the input graph has bounded rank-width and so the theorem of Courcelle and Oum [4] provides a polynomial-time algorithm.

This paper is organized as follows. In Section 2, we present necessary definitions and notations. Section 3 proves Theorem 1.1 for vertex k-scatted subgraph ideals, Section 4 proves Theorem 1.2 for edge k-scatted subgraph ideals, Section 5 proves Theorem 1.3 for matching k-scatted subgraph ideals, and Section 6 proves Theorem 1.4 for rank k-scatted vertex-minor ideals. Section 7 discusses the application of Theorem 1.4 for linear rank-width and proves Theorem 1.6.

2 Preliminaries

For a graph G and a vertex set S of G, we write $G[S]$ to denote the subgraph of G induced by S. For $v \in V(G)$ and $S \subseteq V(G)$, $G-v$ is the graph obtained from G by removing v and all edges incident with v, and $G-S$ is the graph obtained by removing all vertices in S. For $F \subseteq E(G)$, $G-F$ is the subgraph of G with the vertex set $V(G)$ and the edge set $E(G)\backslash F$. For a vertex v of a graph G, $N_G(v)$ is the set of *neighbors* of v in G, and the *degree* of v is the number of edges incident with v. For two disjoint vertex subsets

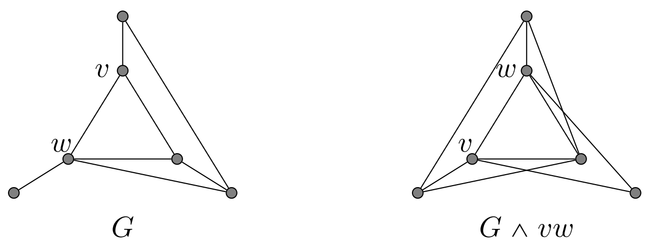

Figure 2: An example of pivoting.

A and B of G, we write $G[A, B]$ to denote the bipartite subgraph on the bipartition (A, B) consisting of all edges of G having one end in A and the other end in B. For two graphs G and H, let $G \cup H$ be the graph $(V(G) \cup V(H), E(G) \cup E(H))$.

A *matching* of a graph is a set of edges of which no two edges share an end. For a matching M, we write $V(M)$ to denote the set of all vertices incident with an edge in M. A *clique* in a graph is a set of pairwise adjacent vertices, and an *independent set* in a graph is a set of pairwise non-adjacent vertices.

The *adjacency matrix* of a graph $G = (V, E)$, denoted by $A(G)$, is a $V \times V$ 0-1 matrix whose (v, w) entry is 1 if and only if v and w are adjacent.

We write P_n and K_n to denote a path on n vertices and a complete graph on n vertices respectively. We write $K_{m,n}$ to denote a complete bipartite graph with bipartition (A, B) where $|A| = m$ and $|B| = n$. For a graph G, we denote by $\overline{G}$ the *complement* of G.

We write $R(n; k)$ to denote the minimum number N such that every coloring of the edges of K_N into k colors induces a monochromatic complete subgraph on n vertices. The classical theorem of Ramsey implies that $R(n; k)$ exists.

Vertex-minors For a vertex v in a graph G, performing a *local complementation* at v is to replace the subgraph of G induced on $N_G(v)$ by its complement graph. We write $G * v$ to denote the graph obtained from G by applying a local complementation at v. Two graphs G and H are *locally equivalent* if G can be obtained from H by a sequence of local complementations. A graph H is a *vertex-minor* of a graph G if H is an induced subgraph of a graph locally equivalent to G.

For an edge uv of a graph G, *pivoting* the edge uv in G is to take a series of three local complementations at u, v, and u. We write $G \wedge uv$ to denote the graph obtained by pivoting uv. In other words, $G \wedge uv = G * u * v * u$.

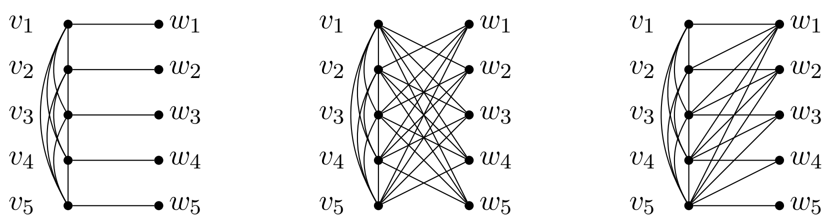

Figure 3: $K_5 \boxminus \overline{K_5}$, $K_5 \boxtimes \overline{K_5}$, and $K_5 \boxslash \overline{K_5}$.

Note that $G \wedge uv$ is identical to the graph obtained from G by flipping the adjacency relation between every pair of vertices x and y where x and y are contained in distinct sets of $N_G(u)\backslash(N_G(v) \cup \{v\})$, $N_G(v)\backslash(N_G(u) \cup \{v\})$, and $N_G(u) \cap N_G(v)$, and finally swapping the labels of u and v [14]. To *flip* the adjacency relation between two vertices, we delete the edge if it exists and add it otherwise. See Figure 2 for an example. For more details, see [14].

Graph operations For two graphs G and H on the disjoint vertex sets, each having n vertices, we would like to introduce operations to construct graphs on $2n$ vertices by making the disjoint union of them and adding some edges between two graphs. Roughly speaking, $G \boxminus H$ will add a perfect matching, $G \boxtimes H$ will add the complement of a perfect matching, and $G \boxslash H$ will add a bipartite chain graph. Formally, for two n-vertex graphs G and H with fixed ordering on the vertex sets $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_n\}$ respectively, let $G \boxminus H$, $G \boxtimes H$, $G \boxslash H$ be graphs on the vertex set $V(G) \cup V(H)$ whose subgraph induced by $V(G)$ or $V(H)$ is G or H, respectively such that for all $i, j \in \{1, 2, \ldots, n\}$,

(i) $v_i w_j \in E(G \boxminus H)$ if and only if $i = j$,

(ii) $v_i w_j \in E(G \boxtimes H)$ if and only if $i \neq j$,

(iii) $v_i w_j \in E(G \boxslash H)$ if and only if $i \geqslant j$.

See Figure 3 for illustrations of $K_5 \boxminus \overline{K_5}$, $K_5 \boxtimes \overline{K_5}$, and $K_5 \boxslash \overline{K_5}$. In each of constructed graphs, we say that v_i is *matched with* w_j when $i = j$.

3 Vertex k-scattered subgraph ideals

In this section, we characterize vertex k-scattered subgraph ideals.

Theorem 1.1. *Let k be a positive integer. A subgraph ideal $\mathcal{C}$ is vertex k-scattered if and only if*

$$\{1H/A, 2H/A, 3H/A, 4H/A, \ldots\} \nsubseteq \mathcal{C}$$

for every connected graph H with $k+1$ edges and each of its independent subset $A \subsetneq V(H)$.

For the forward part, we show the following.

Lemma 3.1. *Let k, ℓ be positive integers. Let H be a connected graph with $k+1$ edges and let A be an independent subset of vertices of H. Then the vertex k-brittleness of $(2\ell+1)H/A$ is at least $\ell+1$.*

Proof. Suppose not. Let $G = (2\ell+1)H/A$. Let $(X_1, X_2, \ldots, X_t)$ be a partition of $E(G)$ such that its κ_G-width is at most ℓ and $|X_i| \leqslant k$ for all $1 \leqslant i \leqslant t$.

For each component C of $G - A$, there are at least two $i \neq j$ such that vertices in C are incident with an edge in X_i and an edge X_j, because $|X_1|, |X_2|, \ldots, |X_t| \leqslant k$, vertices in C are incident with more than k edges in total and C is connected.

Let us pick a random subset I of $\{1, 2, \ldots, t\}$. The probability that a fixed component C of $G-A$ has a vertex incident with both an edge in X_i for some $i \in I$ and an edge in X_j for some $j \in \{1, 2, \ldots, t\} \backslash I$ is at least $1/2$. By the linearity of the expectation, there exists a subset I' of $\{1, 2, \ldots, t\}$ such that at least half of the components of $G-A$ has a vertex incident with both an edge in X_i for some $i \in I$ and an edge in X_j for some $j \in \{1, 2, \ldots, t\} \backslash I$. This means that $\kappa_G(\bigcup_{i \in I} X_i) \geqslant \ell + 1$, contradicting our assumption. □

For the converse direction of Theorem 1.1, we prove that for positive integers k and n, every graph with sufficiently large vertex k-brittleness must contain a subgraph isomorphic to nH/A for some connected graph H with $k+1$ edges and some independent subset $A \subsetneq V(H)$. We prove this statement by induction on k. The following lemma will be used in the induction step.

Lemma 3.2. *Let H be a graph with k edges and let $A \subsetneq V(H)$ be an independent set. Let m, n be integers such that $m > k(k+1)(n-1)(4n+1)$. Let G be a graph containing mH/A as a subgraph. If for each component C of $(mH/A) - A$, G has an edge not in mH/A but incident with vertices in C, then G contains a subgraph isomorphic to nH'/A' for some connected graph H' with $k+1$ edges and an independent subset $A' \subsetneq V(H')$.*

Proof. It is trivial if $n = 1$. Thus we may assume that $n > 1$. Let us choose a minimal subgraph G' of G such that $V(G) = V(G')$, $E(G') \cap E(mH/A) = \varnothing$ and each component C of $(mH/A) - A$ has an edge in G' incident with a vertex of C. Then G' is a forest and $(V(G) \backslash V(mH/A)) \cup A$ is independent in G' by the minimality. Moreover, between two components of $(mH/A) - A$, G' has at most one edge and for each component C of $(mH/A) - A$, the graph $G'[A \cup V(C)]$ has at most one edge. As one edge of G' is incident with at most two components, G' has at least $m/2$ edges.

If there are more than $\binom{k+1}{2}(n-1)$ components C of $(mH/A) - A$ having an edge in $G'[A \cup V(C)]$, then by the pigeon-hole principle, there are at least n components C of $(mH/A) - A$ such that the (unique) edge in $G'[A \cup V(C)]$ is incident with the corresponding vertices by the isomorphism. Then let H' be the graph obtained from H by adding that edge incident with the corresponding vertices. Then G has nH'/A as a subgraph. So, we may assume that at most $\binom{k+1}{2}(n-1)$ components C of $(mH/A) - A$ have an edge in $G'[A \cup V(C)]$. Let G'' be the subgraph of G' obtained by deleting edges in $G'[A \cup V(C)]$ for all components C of $(mH/A) - A$. Then $|E(G'')| \geqslant m/2 - \binom{k+1}{2}(n-1)$.

If G'' has a vertex v of degree more than $(k+1)(n-1)$, then more than $(k+1)(n-1)$ components of $(mH/A) - A$ have vertices adjacent to v in G''. Then, for at least n components of $(mH/A) - A$, their vertices adjacent to v in G'' are the copies of the same vertex w of H. Let H' be the graph obtained from H by adding a new vertex v of degree 1 adjacent to w. Let $A' = A \cup \{v\}$. Then G has nH'/A' as a subgraph and H' is connected. So we may assume that the maximum degree of G'' is at most $(k+1)(n-1)$.

As G'' is a forest, G'' is bipartite. By the theorem of König, G'' is $(k+1)(n-1)$-edge-colorable. So G'' has a matching M with

$$|M| \geqslant \frac{|E(G'')|}{(k+1)(n-1)} \geqslant \frac{m - k(k+1)(n-1)}{2(k+1)(n-1)} > 2kn.$$

Let $C_1, C_2, \ldots, C_m$ be the components of $(mH/A) - A$. Let I be a random subset of $\{1, 2, \ldots, m\}$ and $X = \bigcup_{i \in I} V(C_i)$ and $Y = \bigcup_{j \in \{1,\ldots,m\} \backslash I} V(C_j)$. The probability that $e \in M$ has one end in X and the other end in Y is $1/2$. Thus, there exist I and $M' \subseteq M$ such that $|M'| \geqslant |M|/2 > kn$ and every edge of M' has one end in X and the other end in Y.

By the pigeon-hole principle, there exists a vertex w of H such that at least n edges e of M' have the property that the end in X is a copy of w. Then let H' be the graph obtained from H by adding a new vertex v and an edge from v to w and let $A' = A$. Then G has nH'/A' as a subgraph and H' is connected. □

Lemma 3.3. *Every graph with vertex* 1*-brittleness more than* $64n^3(n-1)$ *contains* nP_3/A *as a subgraph for some independent set* $A \subsetneq V(P_3)$.

Proof. Let G be a graph with vertex 1-brittleness more than $64n^3(n-1)$. We may assume that G has no components with at most 2 vertices. If G has at least n components, then each component has P_3 as a subgraph and therefore $nP_3/\varnothing$ is a subgraph of G. So we may assume that G has less than n components.

Let G' be the induced subgraph of G obtained by deleting all degree-1 vertices. Then if a vertex of G' has degree less than 2, then it has its private neighbor in $V(G')\backslash V(G)$ of degree 1 in G.

If G' has a vertex v of degree more than $2(n-1)(4n+1)$, then by Lemma 3.2, G contains nP_3/A for some independent set $A \subsetneq V(P_3)$. So we may assume that every vertex of G' has degree at most $2(n-1)(4n+1)$.

If G' has a matching M of size more than $2(n-1)(4n+1)$, then by Lemma 3.2, G contains nP_3/A for some independent set $A \subsetneq V(P_3)$. So we may assume that every matching of G' has at most $2(n-1)(4n+1)$ edges.

Then by the theorem of Vizing, $|E(G')| \leqslant (2(n-1)(4n+1)+1)(2(n-1)(4n+1))$. As G' has at most $n-1$ components, $|V(G')| \leqslant (2(n-1)(4n+1)+1)(2(n-1)(4n+1))+n-1$. Then the vertex 1-brittleness of G is at most $(2(n-1)(4n+1)+1)(2(n-1)(4n+1))+n-1 = 64n^4-96n^3+12n^2+19n+1 \leqslant 64n^3(n-1)$, which is a contradiction. $\square$

For a set A of vertices of a graph G, a *Tutte bridge* of A in G is either an edge joining two vertices in A or a subgraph of G consisting of one component C of $G-A$ and all edges joining C and A and all vertices of A incident with those edges. For a Tutte bridge B of A in G, *deleting* B from G is to remove all edges in B and remove all vertices in $V(B)\backslash A$. (For an edge e of G, $V(e)$ denotes the set of ends of e.)

Lemma 3.4. *Let G be a graph and A be a set of vertices of G. If G' is the subgraph of G obtained by deleting all edges in each Tutte bridge of A with at most k edges, then* $\beta_k^\kappa(G') \geqslant \beta_k^\kappa(G) - |A|$.

Proof. Let $P' = (X_1, X_2, \ldots, X_t)$ be a partition of $E(G')$ whose $\kappa_{G'}$-width is equal to $\beta_k^\kappa(G')$. We extend P' to a partition P of $E(G)$ by adding $E(B)$ as one part for each Tutte bridge B of A in G with at most k edges. Then the κ_G-width of P is at most $\beta_k^\kappa(G') + |A|$ and therefore $\beta_k^\kappa(G) \leqslant \beta_k^\kappa(G') + |A|$. $\square$

Lemma 3.5. *Let m, n, k be positive integers. Let H be a connected graph with k edges and let $A \subsetneq V(H)$ be a non-empty independent subset. Let G*

be a graph having mH/A as a subgraph and let X be a set of vertices of G. If no subgraph of G is isomorphic to nH'/A' for some connected graph with $k+1$ edges and an independent set $A' \subsetneq V(H')$ and

$$m > 4n^2k(k+1) + |X|,$$

then G has two distinct Tutte bridges B_1, B_2 of A, satisfying the following.

(i) *Each B_i has exactly k edges.*

(ii) *$V(B_1) \cap A = V(B_2) \cap A$.*

(iii) *neither $B_1 - A$ nor $B_2 - A$ has a vertex in X.*

Proof. By Lemma 3.2, we may assume that no more than $k(k+1)(n-1)(4n+1)$ components C of mH/A are incident with an edge in $E(G) \backslash E(mH/A)$. As $k(k+1)(n-1)(4n+1)+1 \leqslant 4n^2k(k+1)$, there are at least $|X|+2$ components of mH/A that form Tutte bridges of A in G with exactly k edges. Among them, at least two will not intersect X. □

We need the sunflower lemma. Let $\mathcal{F}$ be a family of sets. A subset $\{M_1, M_2, \ldots, M_p\}$ of $\mathcal{F}$ is a *sunflower* with *core* A (possibly an empty set) and p *petals* if for all distinct $i, j \in \{1, 2, \ldots p\}$, $M_i \cap M_j = A$.

Theorem 3.6 (Sunflower Lemma [8, Erdős and Rado])**.** *Let k and p be positive integers, and $\mathcal{F}$ be a family of sets each of cardinality k. If $|\mathcal{F}| > k!(p-1)^k$ then $\mathcal{F}$ contains a sunflower with p petals.*

Lemma 3.7. *Let m, n, k, t be positive integers. Let G be a graph. Let F_1, F_2, $\ldots$, F_m be a connected subgraph with t edges and for each $i \in \{1, 2, \ldots, m\}$, let S_i be an independent subset of $V(F_i)$ such that $|S_i| \leqslant k$ and $F_i - X$ is connected for all $X \subsetneq S_i$, and $V(F_i) \cap V(F_j) \subseteq S_i \cap S_j$ for all $1 \leqslant i < j \leqslant m$. If $m > k \cdot k!\binom{(t+1)t/2}{t}^k (n-1)^k$, then G has a subgraph isomorphic to nH/A for some connected graph H with t edges and an independent set $A \subsetneq V(H)$.*

Proof. Let $p = \binom{(t+1)t/2}{t}(n-1)+1$. We may assume that S_i is non-empty because every connected graph H with at least two vertices has a vertex v such that $H - v$ is connected. By Theorem 3.6, there exist $i_1 < i_2 < \cdots < i_p$ such that $\{S_{i_1}, S_{i_2}, \ldots, S_{i_p}\}$ is a sunflower with p petals. Let A be the core, that is $\bigcap_{j=1}^{p} S_{i_j}$. Since $V(F_i) \cap V(F_j) \subseteq S_i \cap S_j$ for all $1 \leqslant i < j \leqslant m$, we deduce that $F_{i_1} - A$, $F_{i_2} - A$, $\ldots$, $F_{i_p} - A$ are vertex-disjoint. There are at most $\binom{(t+1)t/2}{t}$ graphs having t edges and so at least n of $F_{i_1} - A$, $F_{i_2} - A$,

..., $F_{i_p} - A$ are pairwise isomorphic with isomorphisms fixing A, because $p > \binom{(t+1)t/2}{t}(n-1)$. This proves the lemma. $\square$

Proposition 3.8. *For positive integers k and n, there exists an integer $\ell = \ell(k,n)$ such that every graph with vertex k-brittleness more than ℓ contains nH/A for some connected graph H with $k+1$ edges and an independent set $A \subsetneq V(H)$.*

Proof. We define that

$$\ell(1,n) := 64n^3(n-1),$$

and for $k \geqslant 2$,

$$\ell(k,n) := \ell\left(k-1, 4n^2k(k+1) + k\cdot k!\binom{(2k+1)k}{2k}^k(n-1)^k\right) + k^2\cdot k!\binom{(2k+1)k}{2k}^k(n-1)^k.$$

We prove the statement by induction on k. If $k=1$, then it is true by Lemma 3.3. Now, we prove for $k \geqslant 2$. Suppose G has vertex k-brittleness more than $\ell = \ell(k,n)$ and no subgraph of G is isomorphic to nH'/A' for a connected graph H' with $k+1$ edges and an independent set $A' \subsetneq V(H')$. Let $m = 4n^2k(k+1) + k\cdot k!\binom{(2k+1)k}{2k}^k(n-1)^k$. Let G_1 be the subgraph of G obtained by deleting all components with at most k edges. By Lemma 3.4, $\beta_k^\kappa(G_1) = \beta_k^\kappa(G)$. By the induction hypothesis, G_1 has mH_1/A_1 as a subgraph for some connected graph H_1 with k edges and an independent subset $A_1 \subsetneq V(H_1)$. By Lemma 3.5, G_1 has two Tutte bridges $B_{1,1}$ and $B_{1,2}$ of A_1, each having exactly k edges such that $V(B_{1,1}) \cap A_1 = V(B_{1,2}) \cap A_1$. Note that A_1 is non-empty because every component of G_1 has at least $k+1$ edges. Let $F_1 = B_{1,1} \cup B_{1,2}$ and $S_1 = V(F_1) \cap A_1$. Then for all $X \subsetneq S_1$, $F_1 - X$ is connected.

For $i = \{2, \ldots, k\cdot k!\binom{(2k+1)k}{2k}^k(n-1)^k\}$, we define G_i as the subgraph of G_{i-1} obtained by deleting all Tutte bridges of A_{i-1} having at most k edges. By Lemma 3.4, $\beta_k^\kappa(G_i) \geqslant \beta_k^\kappa(G_{i-1}) - |A_{i-1}| \geqslant \beta_k^\kappa(G_{i-1}) - k$. By induction,

$$\beta_k^\kappa(G_i) > \ell(k-1,m) + k^2\cdot k!\binom{(2k+1)k}{2k}^k(n-1)^k - (i-1)k \geqslant \ell(k-1,m)$$

and so G_i has a subgraph isomorphic to mH_i/A_i for some connected graph H_i with k edges and an independent subset $A_i \subsetneq V(H_i)$. Again by Lemma 3.5,

G_i has two Tutte bridges $B_{i,1}$ and $B_{i,2}$ of A_i, each having exactly k edges such that $V(B_{i,1}) \cap A_i = V(B_{i,2}) \cap A_i$ and neither $B_{i,1} - A_i$ nor $B_{i,2} - A_i$ has a vertex in $A_1 \cup A_2 \cup \cdots \cup A_{i-1}$. Let $F_i = B_{i,1} \cup B_{i,2}$ and $S_i = V(F_i) \cap A_i$. Then for all $X \subsetneq S_i$, $F_i - X$ is connected.

We claim that for $i < j$, $V(F_i) \cap V(F_j) \subseteq S_i \cap S_j$. Suppose not. Let $x \in V(F_i) \cap V(F_j)$. If $x \notin S_i$, then $x \notin A_i$ and so $x \notin V(G_j)$ because when we construct G_{i+1} from G_i, we remove all Tutte bridges of A_i with at most k edges. Since F_j is a subgraph of G_j, we deduce that $x \notin V(F_j)$, contradicting the assumption. Thus $x \in S_i$. If $x \notin S_j$, then $x \notin A_j$ and so by the construction, $x \notin S_1 \cup S_2 \cup \cdots \cup S_{j-1}$, contradicting the assumption that $x \in S_i$. Thus, $x \in S_i \cap S_j$. This proves the claim.

By applying Lemma 3.7 to F_i and A_i for all i, we deduce that G has a subgraph isomorphic to mH/A for some connected graph H with $2k$ edges and an independent set $A \subsetneq V(H)$. This contradicts our assumption because H contains a connected subgraph H' with $k+1$ edges and G contains mH'/A' as a subgraph where $A' = A \cap V(H')$. $\square$

Lemma 3.1 and Proposition 3.8 imply Theorem 1.1.

4 Edge k-scattered subgraph ideals

In this section, we characterize edge k-scattered subgraph ideals.

Theorem 1.2. *Let k be a positive integer. A subgraph ideal $\mathcal{C}$ is edge k-scattered if and only if*

$$\{K_{1,1}, K_{1,2}, K_{1,3}, \ldots\} \nsubseteq \mathcal{C}$$

and

$$\{T, 2T, 3T, 4T, \ldots\} \nsubseteq \mathcal{C}$$

for every tree T on $k+1$ vertices.

We prove that for some connected graph H on $k+1$ vertices, the disjoint union of sufficiently many copies of H should have large edge k-brittleness. In fact, this is same for matching k-brittleness and rank k-brittleness, we prove at the same time.

Lemma 4.1. *Let m, n, k be positive integers with $n > 2m$ and H be a connected graph on $k+1$ vertices. Then the following hold.*

(i) nH has edge k-brittleness at least $m+1$.

(ii) nH has matching k-brittleness at least $m+1$.

(iii) nH has rank k-brittleness at least $m+1$.

Proof. Let $G := nH$. Let $(X_1, X_2, \ldots, X_t)$ be a partition of $V(G)$ such that $|X_i| \leqslant k$. Let $C_1, C_2, \ldots, C_n$ be the components of G. Note that each C_i intersects at least two of X_1, X_2, …, X_t. Let I be a random subset of $\{1, 2, \ldots, t\}$. For each ℓ, the probability that C_ℓ contains both a vertex in $\bigcup_{i\in I} X_i$ and a vertex in $\bigcup_{j\in\{1,2,\ldots,t\}\setminus I} X_j$ is at least $1/2$. Thus, by the linearity of expectation, more than m components of G have both a vertex in $\bigcup_{i\in I} X_i$ and a vertex in $\bigcup_{j\in\{1,2,\ldots,t\}\setminus I} X_j$. This implies that $\eta_G(\bigcup_{i\in I} X_i) > m$, $\nu_G(\bigcup_{i\in I} X_i) > m$, and $\rho_G(\bigcup_{i\in I} X_i) > m$. $\square$

For edge k-brittleness, a large star is also an obstruction.

Lemma 4.2. *For positive integers k and m, $K_{1,k+m}$ has edge k-brittleness at least $m+1$.*

Proof. Let $(X_1, X_2, \ldots, X_t)$ be a partition of $V(K_{1,k+m})$ such that $|X_i| \leqslant k$. We may assume that X_1 contains the center of $K_{1,k+m}$. Then $\eta_{K_{1,k+m}}(X_1) \geqslant (k+m) - (k-1)$. $\square$

Now, we show the converse direction of Theorem 1.2.

Proposition 4.3. *For every positive integers k and n, there exists an integer $\ell = \ell(k,n)$ such that every graph with edge k-brittleness more than ℓ contains a subgraph isomorphic to either $K_{1,n}$ or nT for some tree T on $k+1$ vertices.*

Proof. Let $\ell(1,n) = n(n-1)$ and $\ell(k,n) = \ell(k-1, 4k(n-1)^2+1)$ for $k \geqslant 2$.

We proceed by induction on k. We may assume that every vertex has degree at most $n-1$. If $k=1$, then by the theorem of Vizing, G has a matching of size at least $|E(G)|/n$. Since the edge 1-brittleness is less than or equal to $|E(G)|$, we have a matching of size more than $\ell(1,n)/n = n-1$. Thus, we may assume that $k > 1$.

We may assume that every component of G has more than k vertices, because otherwise removing them does not decrease the edge k-brittleness. By the induction hypothesis, G has a subgraph isomorphic to mT for a tree T on k vertices where $m = 4k(n-1)^2+1$. Let C_1, C_2, …, C_m be the disjoint copies of T in G.

Let G' be a minimal subgraph of G such that for all $1 \leqslant i \leqslant m$, G' has at least one edge joining C_i with a vertex not in C_i. Since each edge of G' is incident with at most two of C_1, C_2, …, C_m, we have $|E(G')| \geqslant \lceil m/2 \rceil > 2k(n-1)^2$. Note that G' is a forest. So by the theorem of König, G' is

$(n-1)$-edge-colorable and so it has a matching M with $|M| > 2k(n-1)$. Each edge of M is incident with a copy of some vertex of T in mT.

Let I be a random subset of $\{1, 2, \ldots, m\}$. Let $X = \bigcup_{i \in I} V(C_i)$ and $Y = \bigcup_{j \in \{1,2,\ldots,m\} \setminus I} V(C_j)$. The probability that an edge in M has one end in X and the other end in Y is $1/2$ and therefore there exist I and $M' \subseteq M$ such that $|M'| \geqslant |M|/2 > k(n-1)$ and each edge of M' has one end in X and the other end in Y.

Now M' has a subset M'' with $|M''| > n-1$ such that there exists a vertex w of T with the property that for every edge of M'', its end in X is a copy of w in mT. Let T' be the tree obtained from T by adding a new vertex adjacent to w only. Then G has nT as a subgraph. □

Proposition 4.3 and Lemmas 4.1 and 4.2 imply Theorem 1.2.

5 Matching k-scattered subgraph ideals

In this section, we characterize matching k-scattered subgraph ideals. We already proved in Lemma 4.1 that for a connected graph H on $k+1$ vertices, the disjoint union of sufficiently many copies of H has large matching k-brittleness. Such obstructions exactly characterize matching k-scattered subgraph ideals.

Theorem 1.3. *Let k be a positive integer. A subgraph ideal $\mathcal{C}$ is matching k-scattered if and only if*

$$\{T, 2T, 3T, \ldots\} \nsubseteq \mathcal{C}$$

for every tree T on $k+1$ vertices.

First let us prove that deleting a vertex does not decrease the matching k-brittleness a lot.

Lemma 5.1. *Let k be a positive integer. For a vertex v of a graph G,*

$$\beta_k^\nu(G) \leqslant \beta_k^\nu(G - v) + 1.$$

Proof. Let $P' = (X_1, X_2, \ldots, X_t)$ be a partition of $V(G-v)$ such that $|X_i| \leqslant k$ and the ν_{G-v}-width of P' is minimum, that is $\beta_k^\nu(G-v)$. Let $P = (X_1, X_2, \ldots, X_t, \{v\})$. Then the ν_G-width of P is at most $\beta_k^\nu(G-v)+1$. □

The following proposition with Lemma 4.1 proves Theorem 1.3.

Proposition 5.2. *For every positive integers k and n, there exists $\ell = \ell(k,n)$ such that every graph with matching k-brittleness more than ℓ contains a subgraph isomorphic to nT for some tree T on $k+1$ vertices.*

Proof. Let $\ell(k,n) = (k+1)^k(n-1)$. Let G be a graph with matching k-brittleness more than $\ell(k,n)$. Let $G_0 = G$ and $S_0 = \varnothing$. We claim that there exist disjoint subsets S_1, S_2, …, $S_{(k+1)^{k-1}(n-1)}$, $S_{(k+1)^{k-1}(n-1)+1}$ such that each S_i induces a connected subgraph of G with $k+1$ vertices. For $i = 1, 2, \ldots, (k+1)^{k-1}(n-1)+1$, let G_i be the induced subgraph of $G_{i-1} - S_{i-1}$ obtained by deleting all components with at most k vertices. Notice that by Lemma 5.1, $\beta_k^\nu(G_i) \geqslant \beta_k^\nu(G_{i-1}) - |S_{i-1}| = \beta_k^\nu(G_{i-1}) - (k+1)$. By induction, we deduce that $\beta_k^\nu(G_i) \geqslant \beta_k^\nu(G) - (k+1)(i-1) > 0$. Thus G_i contains a component with more than k vertices and therefore it has a vertex set S_i of size $k+1$ inducing a connected subgraph. This proves the claim.

Let T_i be a spanning tree of $G[S_i]$ for each i. Since the number of labeled trees on $k+1$ vertices is $(k+1)^{k-1}$, there exist more than $n-1$ of these spanning trees that are pairwise isomorphic. □

6 Rank k-scattered vertex-minor ideals

We characterize rank k-scattered vertex-minor ideals. As we mentioned, the rank k-brittleness of a graph may increase when taking a subgraph. Instead we use vertex-minors because of the following lemma.

Lemma 6.1 (See Oum [14, Proposition 2.6])**.** *If G is locally equivalent to G', then for every subset X of vertices of G, $\rho_G(X) = \rho_{G'}(X)$.*

Here is our main theorem for rank k-scattered vertex-minor ideals.

Theorem 1.4. *Let k be a positive integer. A vertex-minor ideal $\mathcal{C}$ is rank k-scattered if and only if for every connected graph H on $k+1$ vertices,*

$$\{H, 2H, 3H, 4H, \ldots\} \nsubseteq \mathcal{C}.$$

First, it is easy to observe the following.

Proposition 6.2. *If H is a vertex-minor of G, then*

$$\beta_k^\rho(G) \leqslant \beta_k^\rho(H) + |V(G)| - |V(H)|.$$

Proof. Let G' be a graph locally equivalent to G such that H is an induced subgraph of G. Note that applying local complementation does not change the rank k-brittleness of a graph by Lemma 6.1. Therefore, we have $\beta_k^\rho(G') = \beta_k^\rho(G)$. It is easy to observe that removing a vertex may decrease the rank k-brittleness by at most 1 by a proof analogous to the proof of Lemma 5.1. Therefore, $\beta_k^\rho(H) \geqslant \beta_k^\rho(G') - (|(V(G')| - |V(H)|) = \beta_k^\rho(G) - (|(V(G)| - |V(H)|)$, as required. $\square$

Lemma 4.1 states that for a connected graph H on $k+1$ vertices, the disjoint union of sufficiently many copies of H has large rank k-brittleness. It means that if $\{H, 2H, 3H, 4H, \ldots\} \subseteq \mathcal{C}$ for some connected graph H on $k+1$ vertices, then $\mathcal{C}$ is not rank k-scattered. So we focus on the other direction of Theorem 1.4. We need the following Ramsey-type theorem for bipartite graphs without twins.

Theorem 6.3 (Ding, Oporowski, Oxley, Vertigan [7]). *For every positive integer n, there exists an integer $f(n)$ such that for every bipartite graph G with a bipartition (S,T), if no two vertices in S have the same set of neighbors and $|S| \geqslant f(n)$, then S and T have n-element subsets S' and T', respectively, such that $G[S',T']$ is isomorphic to $\overline{K_n} \boxminus \overline{K_n}$, $\overline{K_n} \boxslash \overline{K_n}$, or $\overline{K_n} \boxtimes \overline{K_n}$.*

In the several places of the proof, when we obtain $H_1 \boxminus H_2$ or $H_1 \boxtimes H_2$ where $H_1, H_2 \in \{\overline{K_n}, K_n\}$, we want to make each part an independent set. The following lemma describes how to reduce each of them to $\overline{K_{n'}} \boxminus \overline{K_{n'}}$ for some n'.

Lemma 6.4. *Let n be an integer.*

(1) If $n \geqslant 2$, then $K_n \boxminus \overline{K_n}$ has a vertex-minor isomorphic to $\overline{K_{n-1}} \boxminus \overline{K_{n-1}}$.

(2) If $n \geqslant 3$, then $K_n \boxminus K_n$ has a vertex-minor isomorphic to $\overline{K_{n-2}} \boxminus \overline{K_{n-2}}$.

(3) If $n \geqslant 3$, then $\overline{K_n} \boxtimes \overline{K_n}$ has a vertex-minor isomorphic to $\overline{K_{n-2}} \boxminus \overline{K_{n-2}}$.

(4) If $n \geqslant 3$, then $K_n \boxtimes \overline{K_n}$ has a vertex-minor isomorphic to $\overline{K_{n-2}} \boxminus \overline{K_{n-2}}$.

(5) If $n \geqslant 2$, then $K_n \boxtimes K_n$ has a vertex-minor isomorphic to $\overline{K_{n-1}} \boxminus \overline{K_{n-1}}$.

Proof. (1) Let $V(K_n) = \{v_i : 1 \leqslant i \leqslant n\}$ and $V(\overline{K_n}) = \{w_i : 1 \leqslant i \leqslant n\}$. The graph $(K_n \boxminus \overline{K_n} - w_1) * v_1 - v_1$ is isomorphic to $\overline{K_{n-1}} \boxminus \overline{K_{n-1}}$.

(2) Let $\{v_i : 1 \leqslant i \leqslant n\}$ and $\{w_i : 1 \leqslant i \leqslant n\}$ be the vertex sets of two copies of K_n. The graph $((K_n \boxminus K_n - \{v_1, w_2\}) * v_2 * w_1) - \{v_2, w_1\}$ is isomorphic to $\overline{K_{n-2}} \boxminus \overline{K_{n-2}}$.

(3) Let $\{v_i : 1 \leqslant i \leqslant n\}$ and $\{w_i : 1 \leqslant i \leqslant n\}$ be the vertex sets of two copies of $\overline{K_n}$. The graph $((\overline{K_n} \boxtimes \overline{K_n} - \{v_1, w_2\}) \wedge v_2w_1) - \{v_2, w_1\}$ is isomorphic to $\overline{K_{n-2}} \boxminus \overline{K_{n-2}}$.

(4) Let $V(K_n) = \{v_i : 1 \leqslant i \leqslant n\}$ and $V(\overline{K_n}) = \{w_i : 1 \leqslant i \leqslant n\}$. The graph $(K_n \boxtimes \overline{K_n} - w_1) * v_1 - v_1$ is isomorphic to $\overline{K_{n-1}} \boxminus K_{n-1}$. Thus, by (1), it contains a vertex-minor isomorphic to $\overline{K_{n-2}} \boxminus \overline{K_{n-2}}$.

(5) Let $\{v_i : 1 \leqslant i \leqslant n\}$ and $\{w_i : 1 \leqslant i \leqslant n\}$ be the vertex sets of two copies of K_n. The graph $(K_n \boxtimes K_n - w_1) * v_1 - v_1$ is isomorphic to $\overline{K_{n-1}} \boxminus \overline{K_{n-1}}$. $\square$

From $H_1 \boxslash H_2$ with $H_1, H_2 \in \{\overline{K_n}, K_n\}$, we can obtain a long induced path as a vertex-minor. So, if n is sufficiently large, then this directly gives us mP_{k+1} for some large m.

Lemma 6.5 (Kwon and Oum [12]). *Let n be a positive integer.*

(1) $\overline{K_n} \boxslash \overline{K_n}$ is locally equivalent to P_{2n}.

(2) $K_n \boxslash \overline{K_n}$ is locally equivalent to P_{2n}.

(3) If $n \geqslant 2$, then $K_n \boxslash K_n$ has a vertex-minor isomorphic to P_{2n-2}.

Proof. (1) and (2) are proved in [12]. To prove (3), let $\{v_i : 1 \leqslant i \leqslant n\}$ and $\{w_i : 1 \leqslant i \leqslant n\}$ be the vertex sets of two copies of K_n, where v_i is adjacent to w_j if and only if $i \geqslant j$. Then $(K_n \boxslash K_n - w_1) * v_1 - v_1$ is isomorphic to $\overline{K_{n-1}} \boxslash K_{n-1}$. Thus, the result follows from (2). $\square$

We will prove the converse direction of Theorem 1.4 by induction on k. In the procedure, we find a vertex-minor containing a vertex set S which induces a subgraph isomorphic to mH for some connected graph H on k vertices. Generally, we meet two situations: the cut-rank of S is large or small. In the next lemma, we prove that if the cut-rank of S is large, then we can directly find a vertex-minor isomorphic to the disjoint union of many copies of some connected graph on $k + 1$ vertices. If the cut-rank is small, then we will recursively find another such set after excluding S.

Lemma 6.6. *For positive integers k and n, there exists a positive integer $m = f_1(k, n)$ such that if a graph G admits a set $W = \{w_1, \ldots, w_m\}$ that is a clique or an independent set satisfying the following two properties, then G has a vertex-minor isomorphic to nH' for some connected graph H' on $k + 1$ vertices.*

(i) $G - W = mH$ for some connected graph H on k vertices.

(ii) For some vertex v of H and its copies v_1, v_2, $\ldots$, v_m in mH, v_i is adjacent to w_j if and only if $i = j$.

Proof. Let H_i be the i-th copy of H in $G - W$. We fix an isomorphism from H to H_i and isomorphisms between copies of H so that these isomorphisms are compatible.

Assume that $m > 2^{k-1}(m_1 - 1)$. For each w_i, there are at most 2^{k-1} possible sets of neighbors in H_i. So there exists a subset W_1 of W with $|W_1| = m_1$ the set of all neighbors of each $w_i \in W_1$ in H_i are identical up to isomorphisms between copies of H.

Assume that $m_1 \geqslant R(m_2; (2^{k-1})^2)$. For a vertex w_i and $j \neq i$, there are 2^{k-1} possible ways of having edges between the j-th copy of $H - v$ and w_i. By applying the theorem of Ramsey, we deduce that there exists a subset $W_2 \subseteq W_1$ of size m_2 such that for all $i < j$ with $w_i, w_j \in W_2$, the set of all neighbors of w_i in H_j are identical up to isomorphisms between copies of H and the set of all neighbors of w_j in H_i are identical up to isomorphisms between copies of H.

Assume that $m_2 \geqslant ((k+2)n + 1)/2 + 1$. Suppose that there exist $i_1 < i_2 < i_3$ such that $w_{i_1}, w_{i_2}, w_{i_3} \in W_2$ and there exists a vertex u of H so that exactly one of the copies of u in H_{i_1} and H_{i_3} is adjacent to w_{i_2}. Then G contains $\overline{K_{m_2-1}} \boxslash \overline{K_{m_2-1}}$ or $\overline{K_{m_2-1}} \boxslash K_{m_2-1}$ as an induced subgraph. By Lemma 6.5, G has a vertex-minor isomorphic to $P_{(k+2)n-1}$ and therefore G has nP_{k+1} as a vertex-minor.

Thus, we may assume that for all $i \neq j$ with $w_i, w_j \in W_2$, the set of all neighbors of w_i in H_j are identical up to isomorphisms between copies of H.

Assume that $m_2 \geqslant n + 3$. Suppose that $w_i \in W_2$ has no neighbors in H_j when $j \neq i$ and $w_j \in W_2$. If W_2 is an independent set, then clearly G has an induced subgraph isomorphic to $m_2 H'$ for some connected graph H' on $k+1$ vertices. If W_2 is a clique, then for some $w_i \in W_2$, $G * w_i$ contains an induced subgraph isomorphic to $(m_2 - 1)H'$ for some connected graph H' on $k+1$ vertices.

Thus, we may assume that $w_i \in W_2$ has at least one neighbor u_j in H_j for some $j \neq i$ with $w_j \in W_2$. Let $G' = G \wedge w_i u_j - V(H_i) - V(H_j) - w_i - w_j$. If W_2 is an independent set, then G' has an induced subgraph isomorphic to $(m_2 - 2)H'$ for some connected graph H' on $k+1$ vertices. If W_2 is a clique, then let $w_{i_1} \in W_2 \backslash \{w_i, w_j\}$ and $G'' = G' * w_{i_1} - V(H_{i_1})$. Then G'' contains an induced subgraph isomorphic to $(m_2 - 3)H'$ for some connected graph H' on $k+1$ vertices.

So we can take $f_1(k, n) := 2^{k-1}(R(\max(((k+2)n+1)/2+1, n+3); (2^{k-1})^2) - 1) + 1$. $\square$

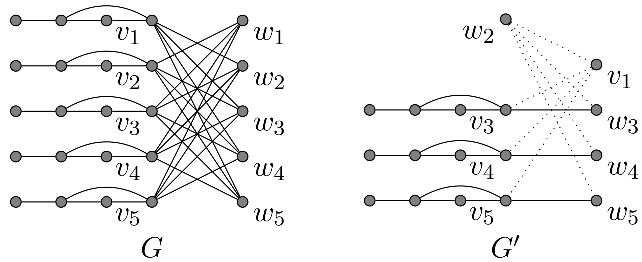

Figure 4: Obtaining $G' = (G \wedge v_1w_2) - V(H_1) - V(H_2) - w_1 - w_2$ from G in the proof of Lemma 6.7.

Lemma 6.7. *For positive integers k and n, there exists a positive integer $m = f_2(k, n)$ such that if a graph G admits a set $W = \{w_1, \ldots, w_m\}$ that is a clique or an independent set satisfying the following two properties, then G has a vertex-minor isomorphic to nH' for some connected graph H' on $k + 1$ vertices.*

(i) $G - W = mH$ for some connected graph H on k vertices.

(ii) For some vertex v of H and its copies v_1, v_2, $\ldots$, v_m in mH, v_i is adjacent to w_j if and only if $i \neq j$.

Proof. Let $f_2(k, n) := f_1(k, n) + 2$ for the function f_1 in Lemma 6.6. Let $G' = G \wedge v_1w_2 - V(H_1) - V(H_2) - w_1 - w_2$ where H_1, H_2 are the first and second copies of H. Then $G' - (W \backslash \{w_1, w_2\})$ is isomorphic to $(m-2)H$ and G' satisfies the condition for Lemma 6.6. See Figure 4 for an illustration. $\square$

Lemma 6.8. *For positive integers k and n, there exists an integer $N := N(k, n)$ with the following property. Let H be a connected graph on k vertices, and G be a graph and $S \subseteq V(G)$ such that $G[S]$ is isomorphic to qH for some integer q and $\rho_G(S) \geqslant N$. Then G contains a vertex-minor isomorphic to nH' for some connected graph H' on $k + 1$ vertices.*

Proof. Let f be the function defined in Theorem 6.3. Let f_1, f_2 be the functions defined in Lemmas 6.6 and 6.7. We define that

$$
\begin{aligned}
n_3(k, n) &:= \max(f_1(k, n), f_2(k, n)), \\
n_2(k, n) &:= \begin{cases} (k-1)n_3(k, n) + 1 & \text{if } k > 1, \\ \max(n + 2, (3n + 1)/2) & \text{if } k = 1, \end{cases} \\
n_1(k, n) &:= R\left(n_2(k, n); 2\right), \\
N(k, n) &:= f(n_1(k, n)).
\end{aligned}
$$

We shortly denote $n_1(k,n)$, $n_2(k,n)$, $n_3(k,n)$ as n_1, n_2, n_3 respectively.

Let $v_1, v_2, \ldots, v_k$ be the vertices of H, and for each component C of $G[S]$, let v_i^C be the copy of v_i in C. For each $i \in \{1,2,\ldots,k\}$, let R_i be the union of all copies of v_i in the components of $G[S]$. Choose $B \subseteq V(G)\backslash S$ such that $|B| = N$ and $\operatorname{rank}(A(G)[S,B]) = N$.

Observe that two distinct vertices in B have distinct sets of neighbors in S. Since $N = f(n_1)$, by Theorem 6.3, there exist $A_1 \subseteq S$ and $B_1 \subseteq B$ with $|A_1| = |B_1| = n_1$ such that $G[A_1, B_1]$ is isomorphic to $\overline{K_{n_1}} \boxminus \overline{K_{n_1}}$, $\overline{K_{n_1}} \boxslash \overline{K_{n_1}}$, or $\overline{K_{n_1}} \boxtimes \overline{K_{n_1}}$.

Since $n_1 = R(n_2; 2)$, by Ramsey's theorem, there exists $B_2 \subseteq B_1$ such that $|B_2| = n_2$ and B_2 is a clique or an independent set. Let $A_2 \subseteq A_1$ be the set of vertices matched with vertices in B_2 in the subgraph $G[A_1, B_1]$. Thus, $G[A_2, B_2]$ is isomorphic to $\overline{K_{n_2}} \boxminus \overline{K_{n_2}}$, $\overline{K_{n_2}} \boxslash \overline{K_{n_2}}$, or $\overline{K_{n_2}} \boxtimes \overline{K_{n_2}}$.

If $k = 1$, then by Lemma 6.4 or 6.5, $G[A_2 \cup B_2]$ contains a vertex-minor isomorphic to $\overline{K_n} \boxminus \overline{K_n}$, because $n_2 \geqslant n+2$, $n_2 \geqslant (3n+1)/2$, and P_{3n-1} has $\overline{K_n} \boxminus \overline{K_n}$ as an induced subgraph. So, we may assume that $k \geqslant 2$.

Observe that H has a vertex v' such that A_2 has at least $\lceil n_2/k \rceil = n_3$ copies of v'. Let A_3 be a set of n_3 copies of v' in A_2, and $B_3 \subseteq B_2$ be the set of vertices matched with vertices in A_3 in the subgraph $G[A_2, B_2]$. Let $\mathcal{C}$ be the set of components of $G[S]$ containing a vertex in A_3. Clearly, we have

- $|\mathcal{C}| = n_3$,
- $G[A_3, B_3]$ is isomorphic to $\overline{K_{n_3}} \boxminus \overline{K_{n_3}}$, $\overline{K_{n_3}} \boxslash \overline{K_{n_3}}$, or $\overline{K_{n_3}} \boxtimes \overline{K_{n_3}}$,
- A_3 is an independent set,
- B_3 is a clique or an independent set.

If $G[A_3, B_3]$ is isomorphic to $\overline{K_{n_3}} \boxslash \overline{K_{n_3}}$, then $G[A_3 \cup B_3]$ is isomorphic to $\overline{K_{n_3}} \boxslash \overline{K_{n_3}}$ or $\overline{K_{n_3}} \boxslash K_{n_3}$, and thus by Lemma 6.5, it is locally equivalent to P_{2n_3}. As $2n_3 \geqslant (k+2)n$, P_{2n_3} contains an induced subgraph isomorphic to nP_{k+1}. Therefore, we may assume $G[A_3, B_3]$ is isomorphic to $\overline{K_{n_3}} \boxminus \overline{K_{n_3}}$ or $\overline{K_{n_3}} \boxtimes \overline{K_{n_3}}$. By Lemmas 6.6 and 6.7, we deduce that G has a vertex-minor isomorphic to nH' for some connected graph H' on $k+1$ vertices. $\square$

Lemma 6.9. *Let k and n be positive integers and let $\ell = k2^{k(N(k,n)-1)} + 1$ for the function N in Lemma 6.8. Let F be a connected graph on k vertices. If G has an induced subgraph isomorphic to ℓF, then at least one of the following holds.*

(i) *G has a vertex-minor isomorphic to nH for some connected graph H on $k+1$ vertices.*

(ii) *There exists $A \subseteq V(G)$ such that $G[A]$ is isomorphic to $(k+1)F$ and for each vertex of F, its copies in $G[A]$ have the same set of neighbors not in A.*

Proof. Let $S \subseteq V(G)$ be a vertex set such that $G[S]$ is isomorphic to ℓF.

If $\rho_G(S) \geqslant N(k,n)$, then by Lemma 6.8, G contains a vertex-minor isomorphic to nH for some connected graph H on $k+1$ vertices. Therefore, we may assume that $\rho_G(S) < N(k,n)$.

Let $\mathcal{C} := \{C_1, C_2, \ldots, C_\ell\}$ be the set of components of $G[S]$, and let $V(F) = \{z_1, z_2, \ldots, z_k\}$. For each $i \in \{1, 2, \ldots, k\}$, let Z_i be the set of all copies of z_i in $\bigcup_{C \in \mathcal{C}} V(C)$. Since $\rho_G(S) < N(k,n)$,

$$\operatorname{rank} A(G)[Z_i, V(G) \backslash S] \leqslant N(k,n) - 1$$

for each $i \in \{1, 2, \ldots, k\}$ and so $A(G)[Z_i, V(G) \backslash S]$ has at most $2^{N(k,n)-1}$ distinct rows because it is a 0-1 matrix. In other words,

$$|\{N_G(v) \cap (V(G) \backslash S) : v \in Z_i\}| \leqslant 2^{N(k,n)-1}$$

for each $1 \leqslant i \leqslant k$.

Thus, by the pigeon-hole principle, there exists $I \subseteq \{1, 2, \ldots, \ell\}$ with $|I| \geqslant \lceil \frac{\ell}{2^{k(N(k,n)-1)}} \rceil \geqslant k+1$ such that for each $i \in \{1, 2, \ldots, k\}$, vertices in $Z_i \cap (\bigcup_{j \in I} V(C_j))$ have the same set of neighbors in $V(G) \backslash S$. It implies (ii). $\square$

Lemma 6.10. *Let k, n be positive integers. If a graph has more than $2^{\binom{k+1}{2}}(n-1)$ components having $k+1$ vertices, then it contains an induced subgraph isomorphic to nH for some connected graph H on $k+1$ vertices.*

Proof. The number of non-isomorphic graphs on $k+1$ vertices is at most $2^{\binom{k+1}{2}}$. By the pigeon-hole principle, at least n components are pairwise isomorphic. $\square$

Lemma 6.11. *Let k, t be integers such that $1 \leqslant t \leqslant k$. Let F be a connected graph on k vertices. Let G be a graph such that every component has more than k vertices and it contains $(t+1)F$ as an induced subgraph. If*

- *for each vertex of F, their copies in $(t+1)F$ have the same set of neighbors in $V(G) \backslash V((t+1)F)$ and*

- *each component of* $(t+1)F$ *has at most* t *vertices having a neighbor in* $V(G)\backslash V((t+1)F)$,

then there exist a graph G' *locally equivalent to* G*, disjoint subsets* S*,* T *of* $V(G')$ *and a vertex* v *in* S *such that*

(i) $G'[S]$ *is a connected graph on* $k+1$ *vertices,*

(ii) $|T| \leqslant t(k+1)$*, and*

(iii) $G'[S\backslash\{v\}]$ *is a component of* $G' - (T \cup \{v\})$*.*

Proof. Let $A \subseteq V(G)$ such that $G[A]$ is isomorphic to $(t+1)F$. Let $\mathcal{C} := \{C_1, C_2, \dots, C_{t+1}\}$ be the set of components of $G[A]$, and let $V(F) = \{z_1, z_2, \dots, z_k\}$. For each $i \in \{1, 2, \dots, k\}$, let Z_i be the set of all copies of z_i in A. Let U_i be the set of neighbors of vertices of Z_i on $V(G)\backslash A$ in G, that is, $U_i = N_G(r) \cap (V(G)\backslash A)$ for $r \in Z_i$. Let $X \subseteq \{1, 2, \dots, k\}$ be the set of integers i such that U_i is non-empty. By the assumption $|X| \leqslant t$. Since each component of G has more than k vertices, we have $|X| > 0$. Without loss of generality, we may assume $X = \{1, \dots, |X|\}$.

We proceed by induction on t.

If $t = 1$, then let $x \in Z_1 \cap V(C_1)$ and $y \in U_1$. We obtain a new graph from G by removing vertices of $V(C_1)\backslash\{x\}$ and pivoting xy. Note that the set of neighbors of x in $G - (V(C_1)\backslash\{x\})$ is exactly U_1. Thus, after pivoting xy, all edges between a vertex z in $Z_1 \cap V(C_2)$ and $U_1\backslash\{y\}$ are removed and z has exactly one neighbor x on $V(G)\backslash V(C_2)$. Therefore, $(G', S, T, v) = (G \wedge xy, V(C_2) \cup \{x\}, (V(C_1)\backslash\{x\}) \cup \{y\}, x)$ is a required tuple.

Now we may assume that $t \geqslant 2$. We may assume that $|X| = t$ by the induction hypothesis.

Let $x \in Z_1 \cap V(C_1)$ and $y \in U_1$. We obtain G_1 from G by removing vertices of $V(C_1)\backslash\{x\}$ and pivoting xy. Let $A_1 = A\backslash V(C_1)$. Note that in G, the set of neighbors of x in $V(G)\backslash V(C_1)$ is exactly U_1. Thus,

- the adjacency relations between two vertices in A_1 do not change by pivoting xy,
- all edges between $Z_1\backslash\{x\}$ and $U_1\backslash\{y\}$ are removed by pivoting xy.

Furthermore, as vertices in each Z_i have the same set of neighbors on $V(G)\backslash A$ in G, G_1 has the following properties.

- For all $i' \in \{2, \dots, t\}$, two vertices in $Z_{i'} \cap A_1$ have the same set of neighbors in $V(G_1)\backslash A_1$.

- If $t < k$, then for $i' \in \{t+1, \ldots, k\}$, vertices in $Z_{i'} \cap A_1$ have no neighbors in $V(G_1) \backslash A_1$.

If vertices in $Z_j \cap A_1$ have no neighbors on $V(G_1) \backslash A_1$ for all $2 \leqslant j \leqslant k$ in G_1, then $(G', S, T, v) = (G \wedge xy, V(C_2) \cup \{x\}, (V(C_1) \backslash \{x\}) \cup \{y\}, x)$ is a required tuple. Thus, we may assume that there is $j \in \{2, \ldots, k\}$ such that vertices in $Z_j \cap A_1$ have a neighbor on $V(G_1) \backslash A_1$ in G_1.

Note that $G_1 - \{x, y\}$ contains an induced subgraph isomorphic to tF on the vertex set A_1 such that

- for each vertex of F, their copies in tF have the same set of neighbors in $V(G_1 - \{x, y\}) \backslash A_1$,
- each component of tF has at least one and less than t vertices having a neighbor in $V(G_1 - \{x, y\}) \backslash A_1$.

By the induction hypothesis, $G_1 - x - y$ has the tuple (G', S, T, v). Let G'' be the graph locally equivalent to G such that $G'' - V(C_1) - y = G'$. Then $(G'', S, T \cup V(C_1) \cup \{y\}, v)$ is a required tuple for G. $\square$

We prove the main proposition.

Proposition 6.12. *For positive integers k and n, there exists an integer $\ell = \ell(k, n)$ such that every graph with rank k-brittleness more than ℓ contains a vertex-minor isomorphic to nH for some connected graph H on $k+1$ vertices.*

Proof. Let f, N be the functions defined in Theorem 6.3 and Lemma 6.8, respectively. We define

- $\ell_2(1, n) := \max(n + 2, \lceil (3n+1)/2 \rceil)$,
- $\ell_1(1, n) := R(\ell_2(1, n); 4)$,
- $\ell(1, n) := f(\ell_1(1, n)) - 1$,

and for $k \geqslant 2$, let

- $\ell_3(k, n) := k 2^{k(N(k,n)-1)} + 1$,
- $\ell_2(k, n) := 2^{\binom{k+1}{2}}(n-1) + 2$,
- $\ell_1(k, n) := R(\ell_2(k, n); 2^{k+1})$,
- $\ell(k, n) := \ell(k-1, \ell_3(k, n)) + (k+1)^2(\ell_1(k, n) - 1)$.

We will prove the statement by induction on k. We shortly denote $\ell_1(k,n)$, $\ell_2(k,n)$, $\ell_3(k,n)$ as ℓ_1, ℓ_2, ℓ_3, respectively.

Let us first consider the case that $k = 1$. Suppose G has rank 1-brittleness more than ℓ. Then, there exists a vertex set A such that $\rho_G(A) > \ell$. Choose $A_1 \subseteq A$ and $B_1 \subseteq V(G)\backslash A$ such that $|A_1| = |B_1| = \ell + 1$ and $\operatorname{rank}(A(G)[A_1, B_1]) = \ell + 1$. Note that two vertices in B_1 have distinct neighbors on A_1. Since $\ell + 1 = f(\ell_1)$, by Theorem 6.3 and there exist $A_2 \subseteq A_1$ and $B_2 \subseteq B_1$ with $|A_2| = |B_2| = \ell_1$ such that $G[A_2, B_2]$ is isomorphic to $\overline{K_{\ell_1}} \boxminus \overline{K_{\ell_1}}$, $\overline{K_{\ell_1}} \boxslash \overline{K_{\ell_1}}$, or $\overline{K_{\ell_1}} \boxtimes \overline{K_{\ell_1}}$.

As $\ell_1 = R(\ell_2; 4)$, by the theorem of Ramsey, there exist $A_3 \subseteq A_2$ and $B_3 \subseteq B_2$ such that

- $G[A_3, B_3]$ is isomorphic to $\overline{K_{\ell_2}} \boxminus \overline{K_{\ell_2}}$, $\overline{K_{\ell_2}} \boxslash \overline{K_{\ell_2}}$, or $\overline{K_{\ell_2}} \boxtimes \overline{K_{\ell_2}}$, and
- each of A_3 and B_3 is a clique or an independent set.

If $G[A_3, B_3]$ is isomorphic to $\overline{K_{\ell_2}} \boxslash \overline{K_{\ell_2}}$, then by Lemma 6.5, $G[A_3 \cup B_3]$ contains a vertex-minor isomorphic to $P_{2\ell_2-2}$. As $2\ell_2 - 2 \geqslant 2(\frac{3n+1}{2}) - 2 \geqslant 3n - 1$, $P_{2\ell_2-2}$ contains an induced subgraph isomorphic to nK_2. Therefore we may assume that $G[A_3, B_3]$ is isomorphic to $\overline{K_{\ell_2}} \boxminus \overline{K_{\ell_2}}$ or $\overline{K_{\ell_2}} \boxtimes \overline{K_{\ell_2}}$. Because $\ell_2 \geqslant n + 2$, by Lemma 6.4, G contains a vertex-minor isomorphic to $\overline{K_n} \boxminus \overline{K_n}$, which is isomorphic to nK_2, as required.

Now, we prove for $k \geqslant 2$. Suppose G has rank k-brittleness more than ℓ. Among all graphs G' locally equivalent to G, choose G' admitting a sequence of $m + 1$ tuples with the maximum m

$$(S_0, T_0), (S_1, T_1, v_1), (S_2, T_2, v_2), \ldots, (S_m, T_m, v_m)$$

such that

- $S_0 = T_0 = \varnothing$,
- $S_1, S_2, \ldots, S_m, T_1, T_2, \ldots, T_m$ are pairwise disjoint vertex subsets of G',
- for each $i \in \{1, 2, \ldots, m\}$,
  - $|S_i| = k + 1$ and $G'[S_i]$ is connected,
  - $|T_i| \leqslant k(k + 1)$,
  - $v_i \in S_i$,
  - no vertex in $S_i \backslash \{v_i\}$ has a neighbor in $V(G')\backslash(\bigcup_{0 \leqslant j \leqslant i}(S_j \cup T_j))$.

Such a graph G' exists trivially because (S_0, T_0) is a valid sequence for G and so $m \geqslant 0$.

Suppose that $m < \ell_1$. Let $G_1 := G' - (\bigcup_{0\leqslant j\leqslant m}(S_j \cup T_j))$. Since G' is locally equivalent to G, $\beta_k^\rho(G') = \beta_k^\rho(G)$, and therefore,

$$\beta_k^\rho(G') = \beta_k^\rho(G) > \ell(k-1, \ell_3) + (k+1)^2(\ell_1 - 1).$$

As $|\bigcup_{0\leqslant j\leqslant m}(S_j \cup T_j)| \leqslant (k+1)^2 m \leqslant (k+1)^2(\ell_1 - 1)$, by Proposition 6.2, we have that $\beta_k^\rho(G_1) > \ell(k-1, \ell_3)$. Let G_2 be the graph obtained from G_1 by removing all components of G_2 having at most k vertices. It is not difficult to observe that $\beta_k^\rho(G_2) = \beta_k^\rho(G_1)$.

As $\beta_{k-1}^\rho(G_2) \geqslant \beta_k^\rho(G_2)$, by the induction hypothesis, G_2 contains a vertex-minor isomorphic to $\ell_3 F$ for some connected graph F on k vertices. Thus, there exist a graph G_3 locally equivalent to G_2 and a vertex subset A of G_3 such that $G_3[A]$ is isomorphic to $\ell_3 F$.

Note that $\ell_3 = k2^{k(N(k,n)-1)} + 1$. So, by Lemma 6.9,

(1) G_3 contains a vertex-minor isomorphic to nH for some connected graph H on $k+1$ vertices or

(2) there exists $A' \subseteq V(G_3)$ such that $G_3[A']$ is isomorphic to $(k+1)F$ and for each vertex of F, its copies in $G_3[A']$ have the same set of neighbors not in A'.

We may assume that (2) holds. Since every component of G_3 has more than k vertices, there is at least one edge between A' and $V(G_3)\backslash A'$ in G_3. By Lemma 6.11 (with $t := k$), there exist a graph G_4 locally equivalent to G_3, disjoint subsets S, T of $V(G_4)$ and a vertex v in S such that

(i) $G_4[S]$ is a connected graph on $k+1$ vertices,

(ii) $|T| \leqslant k(k+1)$, and

(iii) $G_4[S\backslash\{v\}]$ is a component of $G_4 - (T \cup \{v\})$.

In G', no vertex in $S_i\backslash\{v_i\}$ has a neighbor in $V(G')\backslash(\bigcup_{0\leqslant j\leqslant m}(S_j \cup T_j))$. Let G'' be the graph obtained from G' by applying the same sequence of local complementations needed to obtain G_4 from G_2. Since G_2 has no vertex in $\bigcup_{0\leqslant j\leqslant m}(S_j \cup T_j)$, $G''[S_i] = G'[S_i]$. Therefore, G'' admits the sequence (S_0, T_0), (S_1, T_1, v_1), $\ldots$, (S_m, T_m, v_m), (S, T, v), contradicting the assumption on the choice of G' with the maximum m. Thus we may assume that $m \geqslant \ell_1$.

In G', for $i, j \in \{1, 2, \dots, \ell_1\}$ with $i < j$, v_i may have neighbors on S_j, but v_j has no neighbors on $S_i \backslash \{v_i\}$. Let $S_i = \{v_i, s_{i,1}, s_{i,2}, \dots, s_{i,k}\}$ for each i.

We construct a complete graph on the vertex set $\{w_1, w_2, \dots, w_{\ell_1}\}$, and for $i, j \in \{1, 2, \dots, \ell_1\}$ with $i < j$, we color the edge $w_i w_j$ by one of 2^{k+1} colors, depending on the adjacency relation between v_i and $s_{j,j'}$ for all $1 \leqslant j' \leqslant k$. As $\ell_1 = R(\ell_2; 2^{k+1})$, there exists a subset $I \subseteq \{1, 2, \dots, \ell_1\}$ such that $|I| = \ell_2$ and edges between two vertices in $\{w_i : i \in I\}$ are monochromatic. This also implies that $\{v_i : i \in I\}$ is a clique or an independent set.

For some $i, j \in I$ with $i < j$, if v_i is adjacent to $s_{j,j'}$ for some j', then for all $i, j \in I$ with $i \neq j$, v_i is adjacent to $s_{j,j'}$ if and only if $i < j$. By taking vertices v_1, v_3, $\dots$, $v_{2\lfloor \ell_2/2 \rfloor - 1}$ and $s_{2,j'}$, $s_{4,j'}$, $\dots$, $s_{2\lfloor \ell_2/2 \rfloor, j'}$, we obtain an induced subgraph of G' isomorphic to either $\overline{K_{\lfloor \ell_2/2 \rfloor}} \boxslash \overline{K_{\lfloor \ell_2/2 \rfloor}}$ or $\overline{K_{\lfloor \ell_2/2 \rfloor}} \boxslash K_{\lfloor \ell_2/2 \rfloor}$. By Lemma 6.5, G' contains a vertex-minor isomorphic to $P_{\ell_2 - 1}$. As $\ell_2 - 1 \geqslant (k+2)n - 1$, $P_{\ell_2 - 1}$ contains an induced subgraph isomorphic to nP_{k+1}. Thus, G contains a vertex-minor isomorphic to nP_{k+1}. Therefore we may assume that for $i, j \in I$ with $i < j$, v_i has no neighbors in $S_j \backslash \{v_j\}$.

If $\{v_i : i \in I\}$ is independent in G', then $G'[\bigcup_{i \in I} V(S_i)]$ is the disjoint union of ℓ_2 connected graphs, each having exactly $k+1$ vertices. Since $\ell_2 > 2^{\binom{k+1}{2}}(n-1)$, by Lemma 6.10, G contains a vertex-minor isomorphic to nH for some connected graph H on $k+1$ vertices.

If $\{v_i : i \in I\}$ is a clique in G', then let $i' \in I$ and let $G'' = G * v_{i'}$. Then $G''[\bigcup_{i \in I, i \neq i'} V(S_i)]$ is the disjoint union of ℓ_2 connected graphs, each having exactly $k+1$ vertices. Since $\ell_2 - 1 > 2^{\binom{k+1}{2}}(n-1)$, by Lemma 6.10, G contains a vertex-minor isomorphic to nH for some connected graph H on $k+1$ vertices. $\square$

Here is the proof of Theorem 1.3. Let $\mathcal{C}$ be a vertex-minor ideal. Suppose $\mathcal{C}$ is rank k-scattered, that is, there exists an integer ℓ such that every graph $G \in \mathcal{C}$ has rank k-brittleness at most ℓ. Then by (3) of Lemma 4.1, for every connected graph H on $k+1$ vertices, $\mathcal{C}$ does not contain $(k+1)(\ell+1)H$.

For the converse, suppose that for every connected graph H on $k+1$ vertices, there exists n_H such that $n_H H \notin \mathcal{C}$. Since there are only finitely many non-isomorphic graphs on $k+1$ vertices, there exists the maximum n among all n_H. Then $nH \notin \mathcal{C}$ for all connected graphs H on $k+1$ vertices. By Proposition 6.12, all graphs in $\mathcal{C}$ have rank k-brittleness at most $\ell(k, n)$.

7 An application

As an application of Theorem 1.4, we prove that for fixed positive integers m and n, $mK_{1,n}$-vertex-minor free graphs have bounded linear rank-width.

First let us present the definition of linear rank-width [9, 10, 16]. For a graph G, an ordering $(x_1, \ldots, x_n)$ of the vertex set $V(G)$ is called a *linear layout* of G. If $|V(G)| \geqslant 2$, then the *width* of a linear layout $(x_1, \ldots, x_n)$ of G is defined as $\max_{1 \leqslant i \leqslant n-1} \rho_G(\{x_1, \ldots, x_i\})$, and if $|V(G)| = 1$, then the width is defined to be 0. The *linear rank-width* of G is defined as the minimum width over all linear layouts of G. For two orderings $(x_1, \ldots, x_n)$, $(y_1, \ldots, y_m)$, we write $(x_1, \ldots, x_n) \oplus (y_1, \ldots, y_m) := (x_1, \ldots, x_n, y_1, \ldots, y_m)$ to denote the concatenation of two orderings.

We obtain a relation between rank k-brittleness and linear rank-width. We use the submodularity of the matrix rank function.

Proposition 7.1 (See [13, Proposition 2.1.9])**.** *Let M be a matrix over a field F. Let C be the set of column indexes of M, and R be the set of row indexes of M. Then for all $X_1, X_2 \subseteq R$ and $Y_1, Y_2 \subseteq C$,*

$$\begin{aligned}\operatorname{rank}(M[X_1, Y_1]) + \operatorname{rank}(M[X_2, Y_2]) \geqslant \\ \operatorname{rank}(M[X_1 \cap X_2, Y_1 \cup Y_2]) + \operatorname{rank}(M[X_1 \cup X_2, Y_1 \cap Y_2]).\end{aligned}$$

Proposition 7.2. *For an integer $k > 0$, the linear rank-width of a graph G is at most $\beta_k^{\rho}(G) + \lfloor k/2 \rfloor$.*

Proof. Let $x := \beta_k^{\rho}(G)$. Suppose G has rank k-brittleness x. By the definition of rank k-brittleness, there exists a partition $(X_1, X_2, \ldots, X_t)$ of $V(G)$ such that for each $i \in \{1, 2, \ldots, t\}$, $|X_i| \leqslant k$, and for every $I \subseteq \{1, 2, \ldots, t\}$, $\rho_G(\bigcup_{i \in I} X_i) \leqslant x$. For each $i \in \{1, 2, \ldots, t\}$, let L_i be any ordering of X_i.

We claim that the ordering $L = L_1 \oplus L_2 \oplus \cdots \oplus L_t$ is a linear layout of G having width at most $x + \lfloor k/2 \rfloor$. It suffices to prove that for each $i \in \{1, 2, \ldots, t\}$ and a partition (A, B) of X_i, $\rho_G(A \cup \bigcup_{j<i} X_j) \leqslant x + \lfloor k/2 \rfloor$. By symmetry, we may assume that $|A| \leqslant \lfloor k/2 \rfloor$. Let $X = \bigcup_{j<i} X_j$ and $Y = V(G) \backslash X$. Let M be the adjacency matrix of G. By the submodularity of the matrix rank function in Proposition 7.1, we have

$$\begin{aligned}\rho_G(A \cup X) &= \operatorname{rank} M[A \cup X, Y \backslash A] + \operatorname{rank} M[\varnothing, Y] \\ &\leqslant \operatorname{rank} M[X, Y] + \operatorname{rank} M[A, Y \backslash A] \leqslant x + \lfloor k/2 \rfloor.\end{aligned}$$

This proves the proposition. $\square$

For the corollary, we will use the fact that every sufficiently large connected graph contains either a vertex of large degree or a long induced path.

Theorem 7.3 (folklore; see Diestel [6])**.** *For $k \geqslant 1$ and $\ell \geqslant 3$, every connected graph on at least $k^{\ell-2}+1$ vertices contains a vertex of degree at least k or an induced path on ℓ vertices.*

Here is a corollary of Theorem 1.4 and Proposition 7.2.

Theorem 1.6. *For positive integers m and n, the class of graphs having no vertex-minor isomorphic to $mK_{1,n}$ has bounded linear rank-width.*

Proof. We may assume that $n \geqslant 3$. Trivially $K_{1,n}$ is locally equivalent to K_{n+1}. By Lemma 6.5, P_{2n} is locally equivalent to $\overline{K_n} \boxslash \overline{K_n}$, and a vertex of degree n in $\overline{K_n} \boxslash \overline{K_n}$ gives a vertex-minor isomorphic to $K_{1,n}$. Therefore, by Theorem 7.3, every connected graph on $(2n)^{R(n;2)-2}+1$ vertices has a vertex-minor isomorphic to $K_{1,n}$.

Let $k := (2n)^{R(n;2)-2}$. Let $\mathcal{C}$ be the class of graphs having no $mK_{1,n}$ as a vertex-minor. Then for every connected graph H on $k+1$ vertices, $mH \notin \mathcal{C}$. Therefore by Theorem 1.4, $\mathcal{C}$ is rank k-scattered. By Proposition 6.12, $\mathcal{C}$ has bounded linear rank-width. $\square$

References

[1] A. Bouchet. Isotropic systems. *European J. Combin.*, 8(3):231–244, 1987.

[2] A. Bouchet. Connectivity of isotropic systems. In *Combinatorial Mathematics: Proceedings of the Third International Conference (New York, 1985)*, volume 555 of *Ann. New York Acad. Sci.*, pages 81–93, New York, 1989. New York Acad. Sci.

[3] A. Bouchet. κ-transformations, local complementations and switching. In *Cycles and rays (Montreal, PQ, 1987)*, volume 301 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 41–50. Kluwer Acad. Publ., Dordrecht, 1990.

[4] B. Courcelle and S. Oum. Vertex-minors, monadic second-order logic, and a conjecture by Seese. *J. Combin. Theory Ser. B*, 97(1):91–126, 2007.

[5] W. H. Cunningham. Decomposition of directed graphs. *SIAM J. Algebraic Discrete Methods*, 3(2):214–228, 1982.

[6] R. Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Heidelberg, fourth edition, 2010.

[7] G. Ding, B. Oporowski, J. Oxley, and D. Vertigan. Unavoidable minors of large 3-connected binary matroids. *J. Combin. Theory Ser. B*, 66(2):334–360, 1996.

[8] P. Erdős and R. Rado. Intersection theorems for systems of sets. *J. London Math. Soc.*, 35:85–90, 1960.

[9] R. Ganian. Thread graphs, linear rank-width and their algorithmic applications. In *Combinatorial algorithms*, volume 6460 of *Lecture Notes in Comput. Sci.*, pages 38–42. Springer, Heidelberg, 2011.

[10] J. Jeong, O. Kwon, and S. Oum. Excluded vertex-minors for graphs of linear rank-width at most k. *European J. Combin.*, 41:242–257, 2014.

[11] M. M. Kanté and O. Kwon. Linear rank-width of distance-hereditary graphs II. Vertex-minor obstructions. arXiv:1508.04718, 2015.

[12] O. Kwon and S. Oum. Unavoidable vertex-minors in large prime graphs. *European J. Combin.*, 41:100–127, 2014.

[13] K. Murota. *Matrices and matroids for systems analysis*, volume 20 of *Algorithms and Combinatorics*. Springer-Verlag, Berlin, 2000.

[14] S. Oum. Rank-width and vertex-minors. *J. Combin. Theory Ser. B*, 95(1):79–100, 2005.

[15] S. Oum. Excluding a bipartite circle graph from line graphs. *J. Graph Theory*, 60(3):183–203, 2009.

[16] S. Oum. Rank-width: algorithmic and structural results. *Discrete Appl. Math.*, 231:15–24, 2017.