Classes of graphs with no long cycle as a vertex-minor are polynomially χ-bounded

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Abstract

A class $\mathcal{G}$ of graphs is χ-*bounded* if there is a function f such that for every graph $G \in \mathcal{G}$ and every induced subgraph H of G, $\chi(H) \leqslant f(\omega(H))$. In addition, we say that $\mathcal{G}$ is *polynomially* χ-*bounded* if f can be taken as a polynomial function. We prove that for every integer $n \geqslant 3$, there exists a polynomial f such that $\chi(G) \leqslant f(\omega(G))$ for all graphs with no vertex-minor isomorphic to the cycle graph C_n. To prove this, we show that if $\mathcal{G}$ is polynomially χ-bounded, then so is the closure of $\mathcal{G}$ under taking the 1-join operation.

1 Introduction

A class $\mathcal{G}$ of graphs is said to be *hereditary* if for every $G \in \mathcal{G}$, every graph isomorphic to an induced subgraph of G belongs to $\mathcal{G}$. A class $\mathcal{G}$ of graphs is χ-*bounded* if there is a function f such that for every graph $G \in \mathcal{G}$ and

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every induced subgraph H of G, $\chi(H) \leqslant f(\omega(H))$. The function f is called a *χ-bounding function*. This concept was first formulated by Gyárfás [9]. In particular, we say that $\mathcal{G}$ is *polynomially χ-bounded* if f can be taken as a polynomial function.

Recently, many open problems on χ-boundedness have been resolved; see a recent survey by Scott and Seymour [15]. Yet we do not have much information on graph classes that are polynomially χ-bounded. For instance, Gyárfás [9] showed that the class of P_n-free graphs is χ-bounded but it is still open [8, 14] whether it is polynomially χ-bounded for $n \geqslant 5$. Regarding polynomially χ-boundedness, Esperet proposed the following question, which remains open.

Question 1.1 (Esperet; see [10])**.** *Is every χ-bounded class of graphs polynomially χ-bounded?*

Towards answering this question, it is interesting to know some graph operations that preserve the property of polynomial χ-boundedness. If we have such graph operations, then we can use them to generate polynomially χ-bounded graph classes.

In this direction, Chudnovsky, Penev, Scott, and Trotignon [3] showed that if a hereditary class $\mathcal{C}$ is polynomially χ-bounded, then its closure under taking the disjoint union and substitution operations is again polynomially χ-bounded.

We prove the analog of their result for the 1-join. For graphs G_1 and G_2 with $|V(G_1)|, |V(G_2)| \geqslant 3$ and $V(G_1) \cap V(G_2) = \varnothing$, we say that a graph G is obtained from G_1 and G_2 by *1-join*, if there are vertices $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ such that G is obtained from the disjoint union of G_1 and G_2 by deleting v_1 and v_2 and adding all edges between every neighbor of v_1 in G_1 and every neighbor of v_2 in G_2. If so, then we say that G is the *1-join* of (G_1, v_1) and (G_2, v_2). For a class $\mathcal{G}$, let $\mathcal{G}^{\&}$ be its closure under taking the disjoint union and 1-join. Note that $\mathcal{G}^{\&}$ is also a class of graphs, that has to be closed under taking isomorphisms. We will see in Section 2 that $\mathcal{G}^{\&}$ is hereditary if $\mathcal{G}$ is hereditary.

Theorem 1.2. *If $\mathcal{G}$ is a polynomially χ-bounded class of graphs, then so is $\mathcal{G}^{\&}$.*

Dvořák and Král [7] and Kim [11] independently showed that for every hereditary class $\mathcal{G}$ of graphs that is χ-bounded, its closure under taking the 1-joins is again χ-bounded. However, in both papers, the χ-bounding function g for the new class is recursively defined as $g(n) = O(f(n)g(n-1))$ for a χ-bounding function f for $\mathcal{G}$. So, $g(n)$ is exponential under their constructions.

We shall see that if f is a polynomial, then $g(n-1)$ in the recurrence relation can be replaced by some polynomial f^*. This technique allows us to prove Theorem 1.2.

As an application, we investigate the following conjecture of Geelen proposed in 2009. The definition of vertex-minors will be reviewed in Section 4.

Conjecture 1.3 (Geelen; see [7])**.** *For every graph H, the class of graphs with no vertex-minor isomorphic to H is χ-bounded.*

Conjecture 1.3 is known to be true when H is a wheel graph, shown by Choi, Kwon, Oum, and Wollan [1]. Motivated by the question of Esperet, we may ask the following.

Question 1.4. *Is it true that for every graph H, the class of graphs with no vertex-minor isomorphic to H is polynomially χ-bounded?*

If this holds for H, then the class of H-vertex-minor free graphs satisfies the *Erdős-Hajnal property*, which means that there is a constant $c > 0$ such that every graph G in this class has an independent set or a clique of size at least $|V(G)|^c$. Recently, Chudnovsky and Oum [2] proved that the Erdős-Hajnal property holds for the class of H-vertex-minor free graphs for all H.

In Section 4, we prove the following theorem. We write P_n to denote the path graph on n vertices and C_n to denote the cycle graph on n vertices.

Proposition 1.5. *The class of graphs with no vertex-minor isomorphic to P_n is polynomially χ-bounded.*

Kwon and Oum [12] proved the following theorem, stating that a prime graph with long induced path must contain a long induced cycle as a vertex-minor. A graph is *prime* if it is not isomorphic to the 1-join of (G_1, v_1) and (G_2, v_2) for some graphs G_1, G_2 with $|V(G_1)|, |V(G_2)| \geqslant 3$.

Theorem 1.6 (Kwon and Oum [12])**.** *If a prime graph has an induced path of length $\lceil 6.75n^7 \rceil$, then it has a cycle of length n as a vertex-minor.*

We deduce the following stronger theorem from Proposition 1.5 by using Theorems 1.2 and 1.6. This answers Question 1.4 for a long cycle.

Theorem 1.7. *The class of graphs with no vertex-minor isomorphic to C_n is polynomially χ-bounded.*

Proof. Let $\mathcal{G}$ be the class of graphs having no vertex-minor isomorphic to P_m for $m = \lceil 6.75n^7 \rceil$. By Proposition 1.5, $\mathcal{G}$ is polynomially χ-bounded. By Theorem 1.2, $\mathcal{G}^{\&}$ is polynomially χ-bounded.

Let $\mathcal{H}$ be the class of graphs having no vertex-minor isomorphic to C_n. Let $G \in \mathcal{H}$. We claim that $G \in \mathcal{G}^{\&}$. We may assume that G is connected and has at least 4 vertices. Every connected prime induced subgraph of G is in $\mathcal{G}$ by Theorem 1.6. Then G can be obtained from copies of $K_{1,2}$, copies of K_3 and connected prime induced subgraphs of G on at least 4 vertices by taking 1-join repeatedly. (Such a decomposition is called a *split decomposition* [6, 13].) Since $m > 3$, $\mathcal{G}$ contains both $K_{1,2}$ and K_3. Thus, $G \in \mathcal{G}^{\&}$. This proves that $\mathcal{H} \subseteq \mathcal{G}^{\&}$. □

Because C_m contains C_n as a vertex-minor whenever $m \geqslant n$, we may ask a stronger question on whether or not the class of graphs with no induced subgraph isomorphic to C_m for some $m \geqslant n$ is polynomially χ-bounded. It is not known. The following theorem of Chudnovsky, Scott, and Seymour [5] was initially a conjecture of Gyárfás [9] in 1985.

Theorem 1.8 (Chudnovsky, Scott, and Seymour [5])**.** *The class of graphs with no induced subgraph isomorphic to a graph in $\{C_m : m \geqslant n\}$ is χ-bounded.*

We remark that as far as we know, it is not known whether the class of graphs with no P_5 induced subgraph is polynomially χ-bounded.

This paper is organized as follows. We will review necessary definitions in Section 2. In Section 3, we will present a proof of Theorem 1.2. In Section 4, we will prove Proposition 1.5. In Section 5, we focus on the special case of Proposition 1.5 for $n = 5$ and find the best possible bound.

2 Preliminaries

All graphs in this paper are simple and undirected. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. A *clique* of a graph is a set of pairwise adjacent vertices. For a graph G, let $\omega(G)$ be the maximum number of vertices in a clique of G and $\chi(G)$ be the chromatic number of G.

Let G be a graph. For a vertex subset S of G, we denote by $G[S]$ the subgraph of G induced by S. For a vertex v of G, we denote by $G\backslash v$ the graph obtained from G by removing v. For an edge e of G, we write $G\backslash e$ to denote the subgraph obtained from G by deleting e. For $v \in V(G)$, let $N_G(v)$ be the set of neighbors of v in G. For a set X of vertices, let $N_G(X) = (\bigcup_{v\in X} N_G(v))\backslash X$.

For two graphs G_1 and G_2, the *disjoint union* of G_1 and G_2 is a graph $(V(G_1')\cup V(G_2'), E(G_1')\cup E(G_2'))$ where G_1' is an isomorphic copy of G_1 and G_2' is an isomorphic copy of G_2 such that $V(G_1')\cap V(G_2') = \varnothing$. If $V(G_1)\cap V(G_2) = \varnothing$, then we take $G_1' = G_1$ and $G_2' = G_2$ for convenience.

For two graphs G_1 and G_2 on disjoint vertex sets and a vertex $v \in V(G_1)$, we say that a graph G is obtained from G_1 by *substituting* G_2 for v in G_1, if

- $V(G) = (V(G_1)\backslash\{v\}) \cup V(G_2)$,
- $E(G) = E(G_1\backslash v) \cup E(G_2) \cup \{xy : x \in N_{G_1}(v), y \in V(G_2)\}$.

For two sets A, A' with $A \subseteq A'$, we say that a function $f' : A' \to B$ *extends* a function $f : A \to B$ if $f'(a) = f(a)$ for all $a \in A$. For a positive integer m, we denote $[m] := \{1, 2, \dots, m\}$.

Lemma 2.1. *If $\mathcal{G}$ is a hereditary class of graphs, then so is $\mathcal{G}^{\&}$.*

Proof. We show that for every $G \in \mathcal{G}^{\&}$ and $v \in V(G)$, $G\backslash v \in \mathcal{G}^{\&}$. We proceed by induction on $|V(G)|$. If $G \in \mathcal{G}$, then we are done since $\mathcal{G}$ is hereditary.

So, we may assume that $G \notin \mathcal{G}$. Then, G is the disjoint union of G_1 and G_2 for some $G_1, G_2 \in \mathcal{G}^{\&}$ or the 1-join of (G_1, v_1) and (G_2, v_2) for some $G_1, G_2 \in \mathcal{G}^{\&}$ and $v_i \in V(G_i)$ for $i = 1, 2$ where $|V(G_1)|, |V(G_2)| \geqslant 3$.

In the first case, we may assume that $v \in G_1$. Then, by the induction hypothesis, $G_1\backslash v \in \mathcal{G}^{\&}$, and since $G\backslash v$ is the disjoint union of $G_1\backslash v$ and G_2, it follows that $G\backslash v$ is contained in $\mathcal{G}^{\&}$.

In the second case, by symmetry, we may assume that $v \in V(G_1)\backslash\{v_1\}$. By the induction hypothesis, $G_1\backslash v \in \mathcal{G}^{\&}$. If $|V(G_1)| > 3$, then $G\backslash v$ is in $\mathcal{G}^{\&}$ since it is the 1-join of $(G_1\backslash v, v_1)$ and (G_2, v_2). If $|V(G_1)| = 3$, then $G\backslash v$ is isomorphic to either G_2 or the disjoint union of K_1 and $G_2 \backslash v_2$. In either case $G \backslash v$ is contained in $\mathcal{G}^{\&}$. $\square$

3 Polynomially χ-boundedness for 1-join

For a class $\mathcal{G}$ of graphs, let $\mathcal{G}^*$ be the closure of $\mathcal{G}$ under taking the disjoint union and the substitution. We will use the following result due to Chudnovsky, Penev, Scott, and Trotignon [3].

Theorem 3.1 (Chudnovsky, Penev, Scott, and Trotignon [3])**.** *If $\mathcal{G}$ is a polynomially χ-bounded class of graphs, then $\mathcal{G}^*$ is polynomially χ-bounded.*

The following observation relates two graph classes $\mathcal{G}^{\&}$ and $\mathcal{G}^*$.

Lemma 3.2. *Let $\mathcal{G}$ be a hereditary class of graphs. If $G \in \mathcal{G}^{\&}$ and $v \in V(G)$, then $G[N_G(v)] \in \mathcal{G}^*$.*

Proof. We prove by induction on $|V(G)|$.

If $G \in \mathcal{G}$, then we are done, since $\mathcal{G}$ is closed under induced subgraphs and $\mathcal{G} \subseteq \mathcal{G}^*$. If G is the disjoint union of two graphs G_1, G_2 from $\mathcal{G}^{\&}$ and $v \in V(G_1)$, then by the induction hypothesis on the graph G_1, the claim follows.

Suppose that G is the 1-join of two graphs (G_1, v_1) and (G_2, v_2) where $G_1, G_2 \in \mathcal{G}^{\&}$ and $|V(G_1)|, |V(G_2)| \geqslant 3$. Without loss of generality, we may assume that $v \in V(G_1 \backslash v_1)$. Let $G_1[N_{G_1}(v)] = G_v$, and $G_2[N_{G_2}(v_2)] = G_2'$. Then, by the induction hypothesis, $G_v \in \mathcal{G}^*$, and $G_2' \in \mathcal{G}^*$ because $|V(G_1)|, |V(G_2)| < |V(G)|$.

We may assume that G_2' has at least one vertex because otherwise $G[N_G(v)] = G_v \backslash v_1 \in \mathcal{G}^*$. We may also assume that v is adjacent to v_1 in G_1 because otherwise $G[N_G(v)] = G_v \in \mathcal{G}^*$. Then $G[N_G(v)]$ can be obtained from G_v by substituting G_2' for v_1 and therefore $G[N_G(v)]$ belongs to $\mathcal{G}^*$. This completes the proof. $\square$

Let us now define a structure to describe how a connected graph in $\mathcal{G}^{\&}$ is composed from graphs in $\mathcal{G}$. A *composition tree* is a triple (T, ϕ, ψ) of a tree T, a map ϕ defined on $V(T)$ and a map ψ defined on $E(T)$ such that

- for $t \in V(T)$, $\phi(t)$ is a connected graph, say G_t, on at least 3 vertices where graphs in $\{G_t : t \in V(T)\}$ are vertex-disjoint,
- for $st \in E(T)$, $\psi(st) = \{u, v\}$ for some $u \in V(G_s)$ and $v \in V(G_t)$, and
- for distinct $e_1 \neq e_2 \in E(T)$, $\psi(e_1)$ and $\psi(e_2)$ are disjoint.

If a composition tree (T, ϕ, ψ) is given, then one can construct a connected graph G from (T, ϕ, ψ) by taking 1-joins repeatedly as follows:

- if $|V(T)| = 1$, say $V(T) = \{t\}$, then $G = \phi(t)$.

- if $|V(T)| > 1$, let $e = t_1t_2 \in E(T)$ and T_i be the subtree of $T\backslash e$ containing t_i for each $i = 1, 2$. Let ϕ_i be the restriction of ϕ on $V(T_i)$ and ψ_i be the restriction of ψ on $E(T_i)$ for each $i = 1, 2$. Let G_i be a graph constructed from (T_i, ϕ_i, ψ_i) for $i = 1, 2$. Then, G is the 1-join of (G_1, v_1) and (G_2, v_2) where $v_i \in V(G_i) \cap \psi(e)$. It is straightforward to see that the choice of e does not make any difference to the obtained graph G.

If a vertex v of $\phi(t)$ for some node t of T is in $\psi(e)$ for some edge e of T, then v is called a *marker vertex*. After applying all 1-joins, marker vertices will disappear.

Lemma 3.3. *Let $\mathcal{G}$ be a class of graphs. Let G be a connected graph in $\mathcal{G}^{\&}$ with at least three vertices. Then there exists a composition tree (T, ϕ, ψ) that constructs G such that $\phi(t) \in \mathcal{G}$ for every node t of T.*

Proof. We proceed by induction on $|V(G)|$. We may assume that $G \notin \mathcal{G}$. Since G is connected, G is the 1-join of (G_1, v_1) and (G_2, v_2) for some graphs G_1, G_2 in $\mathcal{G}^{\&}$ and $v_1 \in V(G_1)$, $v_2 \in V(G_2)$ where $|V(G_1)|, |V(G_2)| \geqslant 3$. Since G is connected, both G_1 and G_2 are connected. By the induction hypothesis, we obtain two composition trees. We can combine them to obtain a composition tree (T, ϕ, ψ) constructing G. $\square$

Lemma 3.4. *Let c_1, c_2 be positive integers. Let G be a connected graph constructed by a composition tree (T, ϕ, ψ) such that $\phi(t)$ is c_1-colorable for each node t of T and $G[N_G(w)]$ is c_2-colorable for each vertex w of G. Let v be a vertex of G. Then for every proper c_2-coloring β of $G[N_G(v)]$, there exist functions $\alpha' : V(G)\backslash\{v\} \to \{0, 1, 2, \ldots, c_1\}$ and $\beta' : V(G)\backslash\{v\} \to \{1, 2, \ldots, c_2\}$ extending β such that*

(1) $\alpha'(w) = 0$ for every neighbor w of v and

(2) $c = \alpha' \times \beta'$ is a proper $(c_1 + 1)c_2$-coloring of $G\backslash v$.

Proof. We proceed by induction on $|V(G)|$.

If $|V(T)| = 1$, then $G = \phi(t)$ for the unique node t of T and so G has a proper c_1-coloring $h : V(G\backslash v) \to \{1, 2, \ldots, c_1\}$. We define α' and β' on $V(G\backslash v)$ as follows:

$$\alpha'(w) = \begin{cases} 0 & \text{if } w \in N_G(v), \\ h(w) & \text{otherwise}, \end{cases} \quad \text{and} \quad \beta'(w) = \begin{cases} \beta(w) & \text{if } w \in N_G(v), \\ 1 & \text{otherwise}. \end{cases}$$

Clearly, $\alpha' \times \beta'$ is a proper $(c_1 + 1)c_2$-coloring of $G\backslash v$.

Thus we may assume $|V(T)| > 1$. Let t_0 be the unique node of T such that $v \in V(\phi(t_0))$. Let $G_0 := \phi(t_0)$. Let $t_1, t_2, \ldots, t_m$ be the neighbors of t_0 in T. For each $i \in \{1, 2, \ldots, m\}$, let $v_i \in V(G_0)$ and $u_i \in V(\phi(t_i))$ be vertices such that $\psi(t_0t_i) = \{v_i, u_i\}$ and let T_i be the connected component of $T\backslash t_0$ containing t_i. For each $i \in \{1, 2, \ldots, m\}$, let ϕ_i be the restriction of ϕ on $V(T_i)$ and ψ_i be the restriction of ψ on $E(T_i)$ and G_i be the graph constructed from a composition tree (T_i, ϕ_i, ψ_i).

Let $h : V(G_0) \to \{1, 2, \ldots, c_1\}$ be a proper c_1-coloring of G_0. Let α'_0, β'_0 be maps defined on $V(G_0)\backslash\{v, v_1, \ldots, v_m\}$ such that for $w \in V(G_0)\backslash\{v, v_1, \ldots, v_m\}$,

$$\alpha'_0(w) = \begin{cases} 0 & \text{if } w \in N_{G_0}(v), \\ h(w) & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \beta'_0(w) = \begin{cases} \beta(w) & \text{if } w \in N_G(v), \\ 1 & \text{otherwise.} \end{cases}$$

Now we are going to define, for each $i \in \{1, 2, \ldots, m\}$, a proper c_2-coloring β_i of $G_i[N_{G_i}(u_i)]$. If v_i is adjacent to v in G_0, then $N_{G_i}(u_i)$ is a subset of $N_G(v)$ and so let us define β_i to be a proper c_2-coloring of $G[N_{G_i}(u_i)]$ induced by β.

If v_i is non-adjacent to v in G_0, then we claim that there exists a vertex y of G such that $N_{G_i}(u_i) \subseteq N_G(y)$. Since G_0 is connected, v_i has a neighbor x in G_0. If x is not a marker vertex, then $G_i[N_{G_i}(u_i)] \subseteq G[N_G(x)]$. If x is a marker vertex, say $x = v_j$ for some $j \neq i$, then there exists a neighbor y of u_j in G_j because G_j is connected and $|V(G_j)| \neq 1$. Now we observe that $G_i[N_{G_i}(u_i)] \subseteq G[N_G(y)]$. This proves the claim. By the claim, we can define β_i as a proper c_2-coloring of $G_i[N_{G_i}(u_i)]$ induced by a proper c_2-coloring of $G[N_G(y)]$.

Observe that $|V(G_i)| < |V(G)|$ for all $i \in \{1, 2, \ldots, m\}$ because G_0 has at least three vertices. Now, by the induction hypothesis, for each $i \in \{1, 2, \ldots, m\}$, there exist maps $\alpha'_i : V(G_i\backslash u_i) \to \{0, 1, \ldots, c_1\}$ and $\beta'_i : V(G_i\backslash u_i) \to \{1, 2, \ldots, c_2\}$ extending β_i satisfying (1) and (2). If v_i is non-adjacent to v in G_0, then we assume $\alpha'_i(w) = h(v_i)$ for all $w \in N_{G_i}(u_i)$ by swapping colors 0 and $h(v_i)$ in α'_i.

Now we define maps α' and β' on $V(G)\backslash\{v\}$ such that for $w \in V(G)\backslash\{v\}$,

$$\alpha'(w) = \alpha_i(w) \text{ and } \beta'(w) = \beta'_i(w) \text{ if } w \in V(G_i) \text{ for some } i \in \{0, 1, 2, \ldots, m\}.$$

Clearly, β' extends β. In addition, $\alpha'(w) = 0$ for all neighbors w of v in G.

We claim that $c = \alpha' \times \beta'$ is a proper coloring of $G\backslash v$. Let $x, y \in V(G\backslash v)$ be adjacent vertices in $G\backslash v$. If both x and y are neighbors of v, then $\beta'(x) = \beta(x) \neq \beta(y) = \beta'(y)$. So we may assume that y is not a neighbor of v.

- If $x, y \in V(G_0)$, then $\alpha'(x) \neq \alpha'(y)$ because $\alpha'(x) \in \{0, h(x)\}$ and $\alpha'(y) = h(y) \neq 0$.
- If $x, y \in V(G_i)$ for some $i \in \{1, 2, \ldots, m\}$, then $(\alpha'(x), \beta'(x)) \neq (\alpha'(y), \beta'(y))$ because $\alpha'_i \times \beta'_i$ is a proper coloring of $G_i\backslash u_i$.
- If $x \in V(G_0)$ and $y \in V(G_i)$ for some $i \in \{1, 2, \ldots, m\}$, then x is adjacent to v_i in G_0 and y is adjacent to u_i in G_i. Since y is not adjacent to v, v_i is not adjacent to v in G_0 and so $\alpha'(y) = \alpha'_i(y) = h(v_i)$. As $\alpha'(x) \in \{0, h(x)\}$, we deduce that $\alpha'(x) \neq \alpha'(y)$.
- If $x \in V(G_i)$ and $y \in V(G_j)$ for distinct $i, j \in \{1, 2, \ldots, m\}$, then x is adjacent to u_i in G_i, v_i is adjacent to v_j in G_0, and u_j is adjacent to y in G_j. Since y is not adjacent to v, v_j is not adjacent to v in G_0 and so $\alpha'(y) = \alpha'_j(y) = h(v_j)$. Note that $\alpha'(x) \in \{0, h(v_i)\}$ and therefore $\alpha(x) \neq \alpha(y)$ because h is a proper coloring of G_0.

Therefore, c is a proper coloring of $G\backslash v$. This completes the proof. $\square$

Proof of Theorem 1.2. We may assume that $\mathcal{G}$ is hereditary, by replacing $\mathcal{G}$ with the closure of $\mathcal{G}$ under isomorphism and taking induced subgraphs, if necessary.

Let f be a χ-bounding function for $\mathcal{G}$ that is a polynomial. We may assume that $1 \leqslant f(0) \leqslant f(1) \leqslant f(2) \leqslant \cdots$, by replacing $f(x) = \sum_i a_i x^i$ with $\sum_i |a_i| x^i$ if needed. By Theorem 3.1, $\mathcal{G}^*$ is χ-bounded by a polynomial f^*. We may assume that $1 \leqslant f^*(0) \leqslant f^*(1) \leqslant f^*(2) \leqslant \cdots$.

We claim that

$$\chi(G) \leqslant (f(\omega(G)) + 1) f^*(\omega(G) - 1)$$

for all $G \in \mathcal{G}^{\&}$. This claim implies the theorem because $\mathcal{G}^{\&}$ is hereditary by Lemma 2.1.

Let $k = \omega(G)$. We may assume that $k > 1$. We may assume that G is connected because $\mathcal{G}^{\&}$ is hereditary and both f and f^* are non-decreasing. We may assume that G has at least three vertices. By Lemma 3.3, G has a composition tree (T, ϕ, ψ) with $\phi(x) \in \mathcal{G}$ for every node x of T. Note that $\omega(\phi(x)) \leqslant k$ because $\phi(x)$ is isomorphic to an induced subgraph of G and therefore $\chi(\phi(x)) \leqslant f(\omega(\phi(x))) \leqslant f(k)$. For each vertex $w \in V(G)$, $\omega(G[N_G(w)]) \leqslant k-1$ and $G[N_G(w)]$ belongs to $\mathcal{G}^*$ by Lemma 3.2, and so $G[N_G(w)]$ is $f^*(k-1)$-colorable. Let v be a vertex of G. By Lemma 3.4, there exist a proper $(f(k)+1)f^*(k-1)$-coloring $c = \alpha' \times \beta'$ of $G\backslash v$ such that $\alpha'(w) = 0$ for every neighbor w of v. Then we can easily extend this to a proper $(f(k)+1)f^*(k-1)$-coloring of G by taking $\alpha'(v) \neq 0$. $\square$

4 Graphs with no P_n vertex-minors

For a vertex v in a graph G, the *local complementation* at v results in the graph obtained from G by replacing the subgraph of G induced on $N_G(v)$ by its complement. We write $G * v$ to denote the graph obtained from G by applying local complementation at v. In other words, $G * v$ is a graph on $V(G)$ such that two distinct vertices x, y are adjacent in $G * v$ if and only if exactly one of the following holds.

(i) Both x and y are neighbors of v in G.

(ii) x is adjacent to y in G.

A graph H is *locally equivalent* to G if H can be obtained from G by a sequence of local complementations. See Figure 2 for an illustration. We say that a graph H is a *vertex-minor* of a graph G if H is an induced subgraph of a graph locally equivalent to G.

Let $K_n \boxslash \overline{K_n}$ be the graph on $2n$ vertices $\{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n\}$ such that $\{a_1, a_2, \ldots, a_n\}$ is a clique, $\{b_1, b_2, \ldots, b_n\}$ is a stable set, and for all $1 \leqslant i, j \leqslant n$, a_i is adjacent to b_j if and only if $i \geqslant j$. See Figure 1 for an illustration of

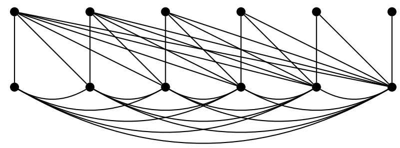

Figure 1: $K_6 \boxslash \overline{K_6}$.

$K_6 \boxslash \overline{K_6}$. The proof will use this graph $K_n \boxslash \overline{K_n}$ because it is locally equivalent to P_{2n}, shown by Kwon and Oum [12, Lemma 2.8].

Now we are ready to prove Proposition 1.5, which states that the class of graphs with no vertex-minor isomorphic to P_n is polynomially χ-bounded. Essentially we show that the class of graphs with no induced subgraph isomorphic to P_n or $K_n \boxslash \overline{K_n}$ is polynomially χ-bounded.

Lemma 4.1. *Let k, d be positive integers. Let G be a graph. If $\omega(G) \leqslant k$ and $\chi(G) > kd$, then there exist two vertices v and w of G and a component C of $G \backslash N_G(v)$ such that v and w are adjacent, w has a neighbor in C, and $\chi(C) > d$.*

Proof. We may assume that G is connected. Let K be a maximum clique of G. By assumption, $|K| \leqslant k$. For each vertex x of K, let $H_x = G \backslash N_G(x)$. Since K is a maximum clique, for every vertex y, there is $x \in K$ such that $y \in V(H_x)$ and therefore $\chi(G) \leqslant \sum_{x \in K} \chi(H_x)$. So there exists $v \in K$ such that $\chi(H_v) > d$. Let C be a component of H_v such that $\chi(C) = \chi(H_v)$. Since $\chi(C) > 1$ and v is an isolated vertex in H_v, $v \notin V(C)$. Since G is connected, G has a vertex w adjacent to both v and some vertex of C. □

For an induced path P from v to w in G, we write $\Omega(G, P)$ to denote $G \backslash (V(P) \cup N_G(V(P \backslash w)))$. A component of $\Omega(G, P)$ is *attached to* P if it contains a neighbor of w. A component C of $\Omega(G, P)$ is *d-good* if the neighbors of w in C induces a graph of chromatic number larger than d. We say C is *d-bad* if it is not d-good. We say P is *d-good* in G if $\Omega(G, P)$ has a d-good component.

Lemma 4.2. *If a graph G has an induced path P of length at least 1 and $\Omega(G, P)$ has a d-bad component C attached to P with $\chi(C) > d$, then there exist an induced path P' extending P by exactly 1 edge and a component C' of $\Omega(G, P')$ attached to P' such that*

$$\chi(C') \geqslant \chi(C) - d.$$

Proof. Let w be the last vertex of P. Let C_w be the subgraph of C induced by the neighbors of w. Since C is d-bad, $\chi(C_w) \leqslant d$ and therefore $\chi(C \backslash N_G(w)) \geqslant \chi(C) - \chi(C_w) \geqslant \chi(C) - d > 0$. So $C \backslash N_G(w)$ has a component C' with $\chi(C') \geqslant \chi(C) - d$. Since C is connected, there is a vertex $w' \in V(C_w)$ adjacent to some vertex in C'. We obtain P' by adding w' as a last vertex to P. Then C' is a component of $\Omega(G, P')$ attached to P'. □

Lemma 4.3. *Let $n \geqslant 4$. Let G be a graph having no induced subgraph isomorphic to P_n. Let P be a path of length 1. If $\Omega(G,P)$ has a component C attached to P with $\chi(C) > d(n-3)$, then G has a d-good induced path P' extending P.*

Proof. Suppose that G has no d-good induced path extending P. By applying Lemma 4.2 $(n-3)$ times, we can find an induced path P' of length $n-2$ extending P and a component C' of $\Omega(G,P')$ attached to P' such that $\chi(C') \geqslant \chi(C) - d(n-3) > 0$. We obtain an induced path of length $n-1$ by taking P' and one vertex in C' adjacent to the last vertex of P'. This contradicts the assumption that G has no induced path on n vertices. $\square$

Proposition 4.4. *Let $n \geqslant 4$ and k be integers. Let G be a graph. If*

$$\omega(G) \leqslant k \text{ and } \chi(G) > (n-3)^{\lceil n/2 \rceil - 1} k^{\lceil n/2 \rceil - 1},$$

then G has an induced subgraph isomorphic to P_n or $K_{\lceil n/2 \rceil} \boxslash \overline{K_{\lceil n/2 \rceil}}$.

Proof. Suppose that G has no induced subgraph isomorphic to P_n. We may assume that G is connected. Let $G_0 = G$. Let $d_i = (n-3)^{\lceil n/2 \rceil - i - 1} k^{\lceil n/2 \rceil - i - 1}$. Note that $\chi(G_0) > d_0$.

Inductively we will find, in G_{i-1} of $\chi(G_{i-1}) > d_{i-1}$, an induced path Q_i and connected induced subgraphs C_i, G_i of $\chi(G_i) > d_i$ as follows. For $i = 1, \ldots, \lceil n/2 \rceil - 1$, by Lemmas 4.1 and 4.3, G_{i-1} has a d_i-good induced path Q_i of length at least 1, because $d_{i-1} = d_i k(n-3)$. Let C_i be a d_i-good component of $\Omega(G_{i-1}, Q_i)$ attached to Q_i. Among all components of the subgraph of C_i induced by the neighbors of the last vertex of Q_i, we choose a component G_i of the maximum chromatic number. By definition of a d_i-good component, $\chi(G_i) > d_i$. This constructs $G_1, G_2, \ldots, G_{\lceil n/2 \rceil - 1}$.

As $\chi(G_{\lceil n/2 \rceil - 1}) > d_{\lceil n/2 \rceil - 1} = 1$, $G_{\lceil n/2 \rceil - 1}$ contains at least one edge xy. By collecting the last two vertices of $Q_1, Q_2, \ldots, Q_{\lceil n/2 \rceil - 1}$ and x, y, we obtain an induced subgraph isomorphic to $K_{\lceil n/2 \rceil} \boxslash \overline{K_{\lceil n/2 \rceil}}$. $\square$

Lemma 4.5 (Kwon and Oum [12, Lemma 2.8])**.** *The graph $K_n \boxslash \overline{K_n}$ is locally equivalent to P_{2n}.*

By Lemma 4.5, we deduce the following corollary, proving Proposition 1.5.

Corollary 4.6. *Let $n \geqslant 4$. If a graph G has no vertex-minor isomorphic to P_n, then*

$$\chi(G) \leqslant (n-3)^{\lceil n/2 \rceil - 1} \omega(G)^{\lceil n/2 \rceil - 1}.$$

5 Graphs with no P_5 vertex-minors

Corollary 4.6 provides some upper bound of the chromatic number for a graph G with no vertex-minor isomorphic to P_n in terms of $\omega(G)$. That bound is tight if $n = 4$, because a graph is perfect if it has no induced subgraphs isomorphic to P_4. We will present the best possible bound for $n = 5$.

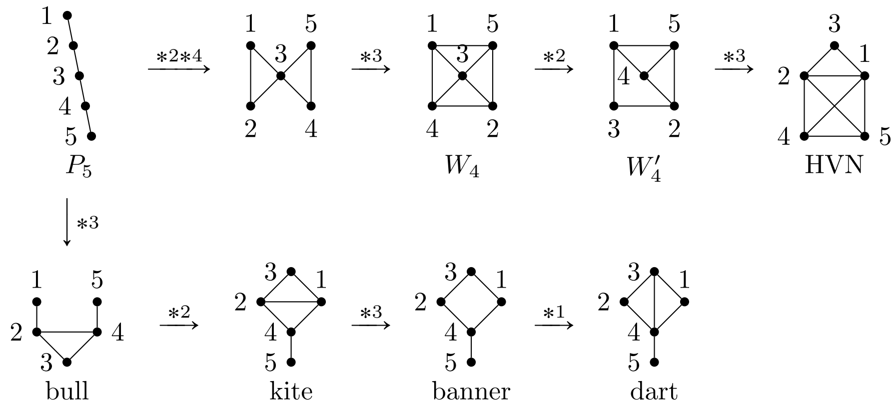

Figure 2: Some graphs locally equivalent to P_5.

Theorem 5.1. *If a graph G has no vertex-minor isomorphic to P_5, then*

$$\chi(G) \leqslant \omega(G) + 1.$$

The following proposition trivially implies Theorem 5.1. We denote by W_n the wheel graph on $n + 1$ vertices.

Proposition 5.2. *Every graph with no vertex-minor isomorphic to P_5 is perfect, unless it has a component isomorphic to C_5 or W_5.*

In order to prove Proposition 5.2, we need to define the following graph classes. See Figure 2 for an illustration.

- W_4': the graph obtained from W_4 by deleting a spoke.
- Banner: the graph obtained from C_4 by adding a pendant edge.
- Bull: the graph obtained from C_3 by adding two pendant edges to distinct vertices of C_3.
- Dart: the graph obtained from $K_4 \backslash e$ for some edge e of K_4 by adding a pendant edge to a vertex of degree 3.
- HVN: the graph obtained from K_4 by adding a vertex of degree 2.
- Kite: the graph obtained from $K_4 \backslash e$ for some edge e of K_4 by adding a pendant edge to a divalent vertex.

We say that G is *H-free* if G has no induced subgraph isomorphic to H. We write $\overline{G}$ to denote the *complement* of a graph G.

Proof of Proposition 5.2. From Figure 2, it is easy to check that W_4, W_4', a banner, a bull, a dart, an HVN and a kite are locally equivalent to P_5. Therefore, G has no induced subgraph isomorphic to any of those graphs.

We may assume that G is connected. If G is C_5-free, then G does not contain an odd hole because G is P_5-free. Since $\overline{C_5}$ is isomorphic to C_5, G is $\overline{C_5}$-free. In addition, since $\overline{W_4'}$ is the disjoint union of P_2 and P_3, G is $\overline{C_k}$-free for every odd $k \geqslant 7$. Therefore G is perfect by the strong perfect graph theorem [4].

Now we may assume that G contains C_5 as an induced subgraph. Let L_i be the set of vertices of G having the distance i to C_5. We may assume that G is not C_5, that is, L_1 is not empty.

We claim that L_1 is complete to L_0. Suppose $v \in L_1$ is not complete to L_0. Then v has exactly 1, 2, 3 or 4 neighbors in L_0. In each case it is easy to check that we can find an induced subgraph isomorphic to P_5, a bull, a banner or a kite, a contradiction.

Now we claim that $L_2 = \varnothing$. Suppose $v \in L_2$. Let $u \in L_1$ such that uv is an edge. Now we see that G contains a dart, a contradiction.

If two vertices u, v in L_1 are adjacent, then G contains a HVN as an induced subgraph, a contradiction. Thus, L_1 is stable.

If L_1 contains more than one vertex, then G contains W_4, a contradiction. So $|L_1| = 1$, and so $G = W_5$. □

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