A remark on the paper "Properties of intersecting families of ordered sets" by O. Einstein

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Abstract

O. Einstein (2008) proved Bollobás-type theorems on intersecting families of ordered sets of finite sets and subspaces. Unfortunately, we report that the proof of a theorem on ordered sets of subspaces had a mistake. We prove two weaker variants.

1 Introduction

The following theorem generalizing the theorem of Bollobás [2] is well known and proved by using the wedge product method (see [1]).

Theorem 1 (Lovász [6]; skew version)**.** *Let a, b be positive integers. Let $U_1, U_2, \ldots, U_m$, $V_1, V_2, \ldots, V_m$ be subspaces satisfying the following:*

(i) $\dim U_i \leq a$ *and* $\dim V_i \leq b$ *for all* $i = 1, 2, \ldots, m$.

(ii) $U_i \cap V_i = \{0\}$ *for all* $i = 1, 2, \ldots, m$.

(iii) $U_i \cap V_j \neq \{0\}$ *for all* $1 \leq i < j \leq m$.

Then $m \leq \binom{a+b}{a}$.

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Ori Einstein [3] published a paper on a generalization of the above theorem and its consequence on finite sets by Frankl [4]. We will show that his proof of Theorem 2.7 in [3] is incorrect and so we state it as a conjecture.

Conjecture 1 (Theorem 2.7 of [3])**.** *Let $\ell_1, \ell_2, \ldots, \ell_k$ be positive integers. Let U be a linear space over a field $\mathbb{F}$. Consider the following matrix of subspaces:*

$$\begin{matrix} U_{11} & U_{12} & \cdots & U_{1k} \\ U_{21} & U_{22} & \cdots & U_{2k} \\ \ldots & \ldots & \vdots & \ldots \\ U_{m1} & U_{m2} & \cdots & U_{mk} \end{matrix}$$

If these subspaces satisfy:

(i) for every $1 \le j \le k$, $1 \le i \le m$, $\dim U_{ij} \le \ell_j$;

(ii) for every fixed i, all subspaces U_{ij} are pairwise disjoint;

(iii) for each $i < i'$, there exist some $j < j'$ such that $U_{ij} \cap U_{i'j'} \neq \{0\}$;

then

$$m \le \frac{(\sum_{r=1}^{k} \ell_r)!}{\prod_{r=1}^{k} \ell_r!}.$$

Here is the overview of this note. In the next section, we will sketch the reason why the proof of Theorem 2.7 in [3] is incorrect and present a weaker theorem (Theorem 3) obtained by tightening condition (ii). In Section 3, we prove another weaker theorem (Theorem 4), by providing a weaker upper bound for m instead of modifying any assumptions. Section 4 will discuss the threshold versions.

2 The mistake and its first remedy

Let us first point out the mistake in the proof of Conjecture 1 in [3]. As it is typical in the wedge product method, we take $v_i = \bigwedge_{j=1}^{k-1} \wedge T_j(U_{ij})$ and $w_i = \bigwedge_{j=1}^{k-1} \bigwedge_{r=j+1}^{k} \wedge T_j(U_{ir})$ for some linear transformations $T_1, T_2, \ldots, T_{k-1}$. Then the following claim is made:

Claim (Page 41 in [3])**.** *For every $i \le i'$, $v_i \wedge w_{i'} \neq 0$ if and only if $i = i'$.*

This claim is false in general. For instance, if $U_{11} \cap (U_{12} + U_{13} + \cdots + U_{1,k-1}) \neq \{0\}$, then $\wedge T_1(U_{11}) \wedge \bigwedge_{r=2}^{k} \wedge T_1(U_{1r}) = 0$ and therefore $v_1 \wedge w_1' = 0$.

The crucial mistake is that condition (ii) in Conjecture 1 does not imply that $\dim(U_{i1}+U_{i2}+\cdots+U_{ik}) = \sum_{j=1}^{k} \dim U_{ij}$. (For instance the spans of $\binom{1}{0}$, $\binom{0}{1}$, and $\binom{1}{1}$ are pairwise disjoint and yet their sum has dimension 2 only.)

If $\dim(U_{i1}+U_{i2}+\cdots+U_{ik}) = \sum_{j=1}^{k} \dim U_{ij}$, then the claim is true and so we can recover the following weaker theorem by the proof in [3].

Theorem 2. *Let $\ell_1, \ell_2, \ldots, \ell_k$ be positive integers. Let U be a linear space over a field $\mathbb{F}$. Consider the following matrix of subspaces:*

$$\begin{matrix} U_{11} & U_{12} & \cdots & U_{1k} \\ U_{21} & U_{22} & \cdots & U_{2k} \\ \cdots & \cdots & \vdots & \cdots \\ U_{m1} & U_{m2} & \cdots & U_{mk} \end{matrix}$$

If these subspaces satisfy:

(i) for every $1 \le j \le k$, $1 \le i \le m$, $\dim U_{ij} \le \ell_j$;

(ii) for every fixed i, $\dim(\sum_{j=1}^{k} U_{ij}) = \sum_{j=1}^{k} \dim U_{ij}$;

(iii) for each $i < i'$, there exist some $j < j'$ such that $U_{ij} \cap U_{i'j'} \neq \{0\}$;

then

$$m \le \frac{(\sum_{r=1}^{k} \ell_r)!}{\prod_{r=1}^{k} \ell_r!}.$$

Though Theorem 2 is weaker than Conjecture 1, it allows us to recover Theorem 2.8 of [3].

Theorem 3 (Theorem 2.8 of [3])**.** *Let $\ell_1, \ell_2, \ldots, \ell_k$ be positive integers. Consider the following matrix of sets:*

$$\begin{matrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \cdots & \cdots & \vdots & \cdots \\ A_{m1} & A_{m2} & \cdots & A_{mk} \end{matrix}$$

If these sets satisfy:

(i) for every $1 \le j \le k$, $1 \le i \le m$, $|A_{ij}| \le \ell_j$;

(ii) for every fixed i, all sets A_{ij} are pairwise disjoint;

(iii) for each $i < i'$, there exist some $j < j'$ such that $A_{ij} \cap A_{i'j'} \neq \emptyset$;

then

$$m \leq \frac{(\sum_{r=1}^{k} \ell_r)!}{\prod_{r=1}^{k} \ell_r!}.$$

Note that Theorem 3 implies that Conjecture 1 is true when $\ell_1 = \ell_2 = \cdots = \ell_k = 1$.

3 Second remedy

Naturally we ask whether Conjecture 1 can be proven with some upper bound on m. Here we show that this is possible, while generalizing Theorem 1.

Theorem 4. *Under the same assumptions of Conjecture 1, we have*

$$m \leq \frac{\prod_{1 \leq a < b \leq k} (\ell_a + \ell_b)!}{(\prod_{r=1}^{k} \ell_r!)^{k-1}}.$$

Proof. We may assume that $\dim U_{ij} = \ell_j$ for all i, j and $\mathbb{F}$ is infinite. Let $V = V_{1,2} \oplus V_{1,3} \oplus \cdots \oplus V_{1,k} \oplus \cdots \oplus V_{2,3} \oplus \cdots \oplus V_{k-1,k} = \bigoplus_{a=1}^{k-1} \bigoplus_{b=a+1}^{k} V_{a,b}$ be a $\sum_{a=1}^{k-1} \sum_{b=a+1}^{k} (\ell_a + \ell_b)$-dimensional vector space over $\mathbb{F}$, decomposed into the direct sum of subspaces $V_{a,b}$, each of dimension $\ell_a + \ell_b$. By Corollary 3.14 of [1], for all $i < j$, there exists a linear transformation $T_{ab} : U \to V_{a,b}$ such that for all $1 \leq i \leq m$, $\dim T_{ab}(U_{ia}) = \ell_a$, $\dim T_{ab}(U_{ib}) = \ell_b$, and $\dim T_{ab}(U_{ia}) \cap T_{ab}(U_{jb}) = \dim U_{ia} \cap U_{jb}$ for all $1 \leq i, j \leq m$. Finally, for each $1 \leq i \leq m$, let $v_i = \bigwedge_{a=1}^{k-1} \bigwedge_{b=a+1}^{k} \wedge T_{ab}(U_{ia})$ and $w_i = \bigwedge_{a=1}^{k-1} \bigwedge_{b=a+1}^{k} \wedge T_{ab}(U_{ib})$.

We claim that for $i \leq i'$, $v_i \wedge w_{i'} \neq 0$ if and only if $i = i'$. If $i < i'$, then there exist $1 \leq j < j' \leq k$ such that $U_{ij} \cap U_{i'j'} \neq \{0\}$. By the choice of $T_{jj'}$, $T_{jj'}(U_{ij}) \cap T_{jj'}(U_{i'j'}) \neq \{0\}$ and so $(\wedge T_{jj'}(U_{ij})) \wedge (\wedge T_{jj'}(U_{i'j'})) = 0$, which implies that $v_i \wedge w_{i'} = 0$. If $i = i'$, then $v_i \wedge w_{i'}$ is the wedge product of disjoint subspaces and so $v_i \wedge w_{i'} \neq 0$.

Therefore $v_1, v_2, \ldots, v_m$ are linearly independent in the space $\bigwedge_{a=1}^{k-1} \bigwedge_{b=a+1}^{k} \bigwedge^{\ell_a} V_{a,b}$, whose dimension is $\prod_{a=1}^{k-1} \prod_{b=a+1}^{k} \binom{\ell_a+\ell_b}{\ell_a} = \frac{\prod_{1 \leq a < b \leq k} (\ell_a+\ell_b)!}{(\prod_{i=1}^{k} \ell_i!)^{k-1}}$. This proves that $m \leq \frac{\prod_{1 \leq a < b \leq k} (\ell_a+\ell_b)!}{(\prod_{i=1}^{k} \ell_i!)^{k-1}}$. □

4 Threshold versions

The paper [3] uses Conjecture 1 to deduce the threshold versions (Lemma 2.9 and Theorem 2.10) to generalize a result of Füredi [5]. We do not know how to prove Lemma 2.9 and Theorem 2.10 of [3] and so we leave them as conjectures. It is not clear how one can relax conditions in Lemma 2.9 and Theorem 2.10 of [3], while avoiding ugly conditions from (ii) of Theorem 3. (A necessary condition $\ell_i \geq t$ was missing in [3].)

Conjecture 2 (Lemma 2.9 of [3])**.** *Let $\ell_1, \ell_2, \ldots, \ell_k$ be positive integers such that $\ell_i \geq t$ for all i. Let U be a linear space over a field $\mathbb{F}$. Consider the following matrix of subspaces:*

$$\begin{array}{cccc} U_{11} & U_{12} & \cdots & U_{1k} \\ U_{21} & U_{22} & \cdots & U_{2k} \\ \cdots & \cdots & \vdots & \cdots \\ U_{m1} & U_{m2} & \cdots & U_{mk} \end{array}$$

If these subspaces satisfy:

(i) for every $1 \leq j \leq k$, $1 \leq i \leq m$, $\dim U_{ij} \leq \ell_j$;

(ii) for every fixed i, $\dim(U_{ij} \cap U_{ij'}) \leq t$;

(iii) for each $i < i'$, there exists some $j < j'$ such that $\dim(U_{ij} \cap U_{i'j'}) > t$;

then

$$m \leq \frac{[(\sum_{r=1}^{k} \ell_r) - kt]!}{\prod_{r=1}^{k}(\ell_r - t)!}.$$

Conjecture 3 (Theorem 2.10 of [3])**.** *Let $\ell_1, \ell_2, \ldots, \ell_k$ be positive integers such that $\ell_i \geq t$ for all i. Consider the following matrix of sets:*

$$\begin{array}{cccc} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \cdots & \cdots & \vdots & \cdots \\ A_{m1} & A_{m2} & \cdots & A_{mk} \end{array}$$

If these sets satisfy:

(i) for every $1 \leq j \leq k$, $1 \leq i \leq m$, $|A_{ij}| \leq \ell_j$;

(ii) for every i, j and j', $|A_{ij} \cap A_{ij'}| \leq t$;

(iii) for each $i < i'$, there exists some $j < j'$ such that $|A_{ij} \cap A_{i'j'}| > t$;

then

$$m \leq \frac{[(\sum_{r=1}^{k} \ell_r) - kt]!}{\prod_{r=1}^{k}(\ell_r - t)!}.$$

By using Theorem 4, we can prove the following weaker variants of Conjectures 2 and 3 by the same reduction in [3].

Theorem 5. *Under the same assumptions of Conjecture 2, we have*

$$m \leq \frac{\prod_{1\leq a<b\leq k}(\ell_a+\ell_b-2t)!}{(\prod_{r=1}^{k}(\ell_r-t)!)^{k-1}}.$$

Theorem 6. *Under the same assumptions of Conjecture 3, we have*

$$m \leq \frac{\prod_{1\leq a<b\leq k}(\ell_a+\ell_b-2t)!}{(\prod_{r=1}^{k}(\ell_r-t)!)^{k-1}}.$$

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