

# IMPROPER COLORING OF GRAPHS WITH NO ODD CLIQUE MINOR

DONG YEAP KANG AND SANG-IL OUM

ABSTRACT. As a strengthening of Hadwiger’s conjecture, Gerards and Seymour conjectured that every graph with no odd  $K_t$  minor is  $(t - 1)$ -colorable. We prove two weaker variants of this conjecture. Firstly, we show that for each  $t \geq 2$ , every graph with no odd  $K_t$  minor has a partition of its vertex set into  $6t - 9$  sets  $V_1, \dots, V_{6t-9}$  such that each  $V_i$  induces a subgraph of bounded maximum degree. Secondly, we prove that for each  $t \geq 2$ , every graph with no odd  $K_t$  minor has a partition of its vertex set into  $10t - 13$  sets  $V_1, \dots, V_{10t-13}$  such that each  $V_i$  induces a subgraph with no large component. The second theorem improves a result of Kawarabayashi (2008), which states that the vertex set can be partitioned into  $496t$  such sets.

## 1. INTRODUCTION

Every graph in this paper is finite and simple. For a nonnegative integer  $k$ , a graph  $G$  is  $k$ -colorable if there are  $k$  sets  $V_1, \dots, V_k$  with  $V(G) = \bigcup_{i=1}^k V_i$  such that every  $V_i$  induces a subgraph of maximum degree 0 for  $1 \leq i \leq k$ .

**1.1. Proper coloring of graphs with no clique minors.** In 1943, Hadwiger [8] proposed the following question, which is called “Hadwiger’s conjecture”, one of the most deepest conjectures in graph theory. For more on this conjecture and its variants, the readers are referred to the recent survey of Seymour [19].

**Conjecture 1.1** (Hadwiger [8]). *For an integer  $t \geq 1$ , every graph with no  $K_t$  minor is  $(t - 1)$ -colorable.*

Robertson, Seymour, and Thomas [18] proved that the conjecture is true for  $t \leq 6$ , but the conjecture remains open for  $t \geq 7$ . Kostochka [14, 15] and Thomason [20, 21] proved that graphs with no  $K_t$  minor is  $O(t\sqrt{\log t})$ -colorable, by showing that these graphs contain a vertex of degree  $O(t\sqrt{\log t})$ . It is still open whether every graph with no  $K_t$  minor is  $ct$ -colorable for some  $c > 0$  independent of  $t$ .

Gerards and Seymour (see [11, Section 6.5]) proposed the following conjecture that strengthens Hadwiger’s conjecture, which is called “the odd Hadwiger’s conjecture”.

**Conjecture 1.2** (Gerards and Seymour (see [11, Section 6.5])). *For an integer  $t \geq 1$ , every graph with no odd  $K_t$  minor is  $(t - 1)$ -colorable.*

Catlin [1] proved that this conjecture is true for  $t = 4$ , and Guenin (see [19]) announced that this is true for  $t = 5$ , but the proof has not been published. Geelen, Gerards, Reed, Seymour, and Vetta [6] proved that every graph with no odd  $K_t$  minor is  $O(t\sqrt{\log t})$ -colorable.

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**1.2. Improper coloring of graphs with no clique minors.** Recall a graph  $G$  is  $k$ -colorable if there are  $k$  sets  $V_1, \dots, V_k$  with  $V(G) = \bigcup_{i=1}^k V_i$  such that every  $V_i$  induces a subgraph of maximum degree 0 for  $1 \leq i \leq k$ . A *defective coloring* (see [2]) is a coloring that relaxes the degree condition. A graph  $G$  is  $(k, d)$ -colorable if there are  $k$  sets  $V_1, \dots, V_k$  with  $V(G) = \bigcup_{i=1}^k V_i$  such that every  $V_i$  induces a subgraph of maximum degree at most  $d$ . Note that  $G$  is  $k$ -colorable if and only if  $G$  is  $(k, 0)$ -colorable.

One may also investigate a variant of proper coloring that relaxes the condition that monochromatic subgraphs have the bounded component size. We investigate coloring of graphs with no odd clique minor in both directions.

Before mentioning the results of the paper, we survey some related results. Edwards, Kang, Kim, Oum, and Seymour [5] investigated defective coloring of graphs with no  $K_t$  minor, and proved that the number of colors  $t - 1$  is best possible.

**Theorem 1.3** (Edwards et al. [5]). *For an integer  $t \geq 1$ , there exists an integer  $s = s(t)$  such that for every graph  $G$  with no  $K_t$  minor, there are  $t - 1$  sets  $V_1, \dots, V_{t-1}$  with  $V(G) = \bigcup_{i=1}^{t-1} V_i$  such that  $V_i$  induces a subgraph of the maximum degree at most  $s$  for  $1 \leq i \leq t - 1$ . Moreover, this is sharp in the sense that we cannot reduce the number  $t - 1$  of sets to  $t - 2$ .*

Recently, Dvořák and Norin [4] announced that for  $t \geq 1$ , there is an integer  $s = s(t)$  such that every graph  $G$  with no  $K_t$  minor, there are  $t - 1$  sets  $V_1, \dots, V_{t-1}$  with  $V(G) = \bigcup_{i=1}^{t-1} V_i$  such that every component of the subgraph induced by  $V_i$  has at most  $s$  vertices for  $1 \leq i \leq t - 1$ . Note that this extends Theorem 1.3, as the maximum degree of a graph with  $s$  vertices is at most  $s - 1$ .

Kawarabayashi [13] proved the following result that every graph with no odd  $K_t$  minor can be colored with  $496t$  colors so that the size of monochromatic components is bounded by a function of  $t$ .

**Theorem 1.4** (Kawarabayashi [13]). *For an integer  $t \geq 2$ , there is an integer  $s = s(t)$  such that for every graph  $G$  with no odd  $K_t$  minor, there are  $496t$  sets  $V_1, \dots, V_{496t}$  with  $V(G) = \bigcup_{i=1}^{496t} V_i$  such that every  $V_i$  induces a subgraph with every component having at most  $s$  vertices for  $1 \leq i \leq 496t$ .*

Let  $I_t$  be a graph on  $t$  vertices with no edges, and for graphs  $G$  and  $H$ , let  $G + H$  be a graph obtained from disjoint union of  $G$  and  $H$  and joining edges from every vertex of  $G$  to all vertices of  $H$ . For positive integers  $s$  and  $t$ , let  $K_{s,t}^*$  be a graph obtained from  $K_s + I_t$  by subdividing every edge joining vertices of the subgraph  $K_s$  once.

Recently, Ossona de Mendez, Oum, and Wood [17] investigated defective coloring for various graph classes. One of their results implies the following, which extends Theorem 1.3 to a larger class of graphs.

**Theorem 1.5** (Ossona de Mendez et al. [17]). *For integers  $s, t \geq 1$  and real numbers  $\delta_1, \delta_2 > 0$ , there exists  $M = M(s, t, \delta_1, \delta_2)$  such that every graph  $G$  satisfying the following three conditions admits  $s$  sets  $V_1, \dots, V_s$  with  $V(G) = \bigcup_{i=1}^s V_i$  such that  $V_i$  induces a subgraph of the maximum degree at most  $M$  for all  $1 \leq i \leq s$ .*

- (1)  $G$  contains no  $K_{s,t}^*$  as a subgraph,
- (2) every subgraph of  $G$  has average degree at most  $\delta_1$ , and
- (3) for every graph  $H$  whose 1-subdivision is a subgraph of  $G$ , the average degree of  $H$  is at most  $\delta_2$ .

Moreover, this is sharp in the sense that we cannot reduce the number  $s$  to  $s - 1$ .

Since  $K_{s,t}^*$  contains a bipartite  $K_s + I_t$  subdivision, Theorem 1.5 implies the following.

**Corollary 1.6.** *For positive integers  $s$  and  $t$ , there is an integer  $N = N(s, t)$  such that for every graph  $G$  with no bipartite  $K_s + I_t$  subdivision, there are  $s$  sets  $V_1, \dots, V_s$  with  $V(G) = \bigcup_{i=1}^s V_i$  such that  $V_i$  induces a subgraph of the maximum degree at most  $N$  for  $1 \leq i \leq s$ .*

*Proof.* By [3, Lemma 7.2.1], there is  $c > 0$  such that for an integer  $p \geq 1$ , every  $n$ -vertex graph with at least  $cp^2n$  edges contains  $K_p$  as a topological minor. First of all, we prove the following claim.

**Claim.** *For an integer  $p \geq 1$ , every  $n$ -vertex with at least  $2cp^2n$  edges contains a bipartite  $K_p$  subdivision.*

*Proof of Claim.* Let  $H$  be an  $n$ -vertex graph with at least  $2cp^2n$  edges. Let  $H_0$  be a bipartite spanning subgraph of  $H$  with at least  $|E(H)|/2 \geq cp^2n$  edges. Since  $H_0$  contains  $K_p$  as a topological minor, it follows that  $H$  contains a bipartite  $K_p$  subdivision, as desired. ■

Since  $G$  contains no bipartite  $K_{s+t}$  subdivision, the graph  $G$  contains no  $K_{s,t}^*$  as a subgraph since  $K_{s,t}^*$  is a bipartite  $K_s + I_t$  subdivision, and every subgraph of  $G$  has average degree at most  $2c(s+t)^2$ . If there is a graph  $H$  whose 1-subdivision is a subgraph of  $G$ , then the average degree of  $H$  is at most  $c(s+t)^2$ , because otherwise  $H$  contains a bipartite  $K_{s+t}$  subdivision, and so does an 1-subdivision of  $H$  which is a subgraph of  $G$ .

By Theorem 1.5, there are  $s$  sets  $V_1, \dots, V_s$  with  $V(G) = \bigcup_{i=1}^s V_i$  such that  $V_i$  induces a subgraph of the maximum degree at most  $M(s, t, 2c(s+t)^2, c(s+t)^2)$  for  $1 \leq i \leq s$ . □

Since  $K_t + I_1$  is isomorphic to  $K_{t+1}$ , Theorem 1.3 can be extended to graphs with no bipartite clique subdivision.

**Corollary 1.7.** *For an integer  $t \geq 1$ , there is an integer  $N = N(t)$  such that for every graph  $G$  with no bipartite  $K_{t+1}$  subdivision, there are  $t$  sets  $V_1, \dots, V_t$  with  $V(G) = \bigcup_{i=1}^t V_i$  such that  $V_i$  induces a subgraph of the maximum degree at most  $N$  for  $1 \leq i \leq t$ .*

**1.3. Our theorems on improper coloring of graphs with no odd clique minors.** We aim to color graphs with no odd clique minor in a similar theme to those results. For a graph  $H$ , let  $\Delta(H)$  be the maximum degree of vertices in  $H$ . For  $S \subseteq V(H)$ , let  $H[S]$  be the subgraph of  $H$  induced by  $S$ . We prove the following.

**Theorem 1.8.** *For an integer  $t \geq 2$ , there exists an integer  $s = s(t)$  such that for every graph  $G$  with no odd  $K_t$  minor, there are  $6t - 9$  sets  $V_1, \dots, V_{6t-9}$  with  $V(G) = \bigcup_{i=1}^{6t-9} V_i$  such that  $\Delta(G[V_i]) \leq s$  for  $1 \leq i \leq 6t - 9$ .*

In other words, for  $t \geq 2$  there is an integer  $s = s(t)$  such that every graph  $G$  with no odd  $K_t$  minor is  $(6t - 9, s)$ -colorable. We also improve 496t of Theorem 1.4 to  $10t - 13$  for  $t \geq 2$ . This reduces a gap between the lower bound and the upper bound of the number of colors.

**Theorem 1.9.** *For an integer  $t \geq 2$ , there exists an integer  $s = s(t)$  such that for every graph  $G$  with no odd  $K_t$  minor, there are  $10t - 13$  sets  $V_1, \dots, V_{10t-13}$  with  $V(G) = \bigcup_{i=1}^{10t-13} V_i$  such that every component of  $G[V_i]$  has at most  $s$  vertices for  $1 \leq i \leq 10t - 13$ .*

We remark that both Theorems 1.8 and 1.9 cannot be extended in terms of list-coloring variant, contrasting Theorems 1.3 and 1.5, which we will discuss in Section 5.

**Organization.** The paper is organized as follows. We briefly introduce some basic notions in Section 2, discuss the structure of graphs with no odd  $K_t$  minor in Section 3, and prove Theorems 1.8 and 1.9 in Section 4. In Section 5, we make some further remarks, including extension of our main results to a slightly larger class of graphs. An appendix reviews elementary concepts of signed graphs and minors.

## 2. PRELIMINARIES

We follow the definitions in [3] unless stated otherwise. In this section,  $G$  and  $H$  always denote graphs.

A subset  $S \subseteq V(G)$  is *stable* if no two vertices in  $S$  are adjacent. Let  $G \cup H$  and  $G \cap H$  be graphs  $(V(G) \cup V(H), E(G) \cup E(H))$  and  $(V(G) \cap V(H), E(G) \cap E(H))$ , respectively. A pair  $(A, B)$  of subgraphs of  $G$  is a *separation* of  $G$  if  $G = A \cup B$ . The *order* of a separation  $(A, B)$  of  $G$  is  $|V(A \cap B)|$ . A *bipartition*  $\{X, Y\}$  of a bipartite graph  $H$  is a set of two disjoint stable subsets  $X \cup Y = V(G)$ .

**Coloring.** For a subset  $S$  of vertices of  $G$  and a set  $T$  of *colors*, a function  $\alpha : S \rightarrow T$  is a *coloring* on  $S$ . A *color class* of a coloring  $\alpha : S \rightarrow T$  is  $\alpha^{-1}(\{i\})$  for some  $i \in T$ . A subgraph  $H$  of  $G$  is *monochromatic* if every vertex of  $H$  has the same  $\alpha$  value.

A coloring  $\alpha$  on  $S$  is *proper* if  $\alpha(u) \neq \alpha(v)$  for every  $uv \in E(G[S])$ ; equivalently, every color class of  $\alpha$  is stable. For a nonnegative integer  $k$ , a graph  $G$  is  *$k$ -colorable* if there is a proper coloring  $\alpha : V(G) \rightarrow \{1, \dots, k\}$ . For integers  $k, d \geq 0$ , a graph  $G$  is  *$(k, d)$ -colorable* if there is a coloring such that every color class induces a graph with maximum degree at most  $d$ .

**Paths.** Let  $P$  be a path. For  $v \in V(P)$ ,  $v$  is an *end* if it is one of the endvertices of  $P$ , otherwise  $v$  is *internal*. The path  $P$  *joins*  $u, v \in V(G)$  if  $u$  and  $v$  are ends of  $P$ . For vertices  $v, w \in V(P)$ ,  $P(v, w)$  denotes the subpath of  $P$  with the ends  $v$  and  $w$ .

A path  $P$  *joins* two sets  $V_1, V_2 \subseteq V(G)$  if it joins a vertex in  $V_1$  and a vertex in  $V_2$ . The *length* of a path is its number of edges. The *parity* of a path  $P$  is the parity of its length.

Two paths  $P$  and  $Q$  in  $G$  are *vertex-disjoint* if  $V(P) \cap V(Q) = \emptyset$ . They are *internally disjoint* if no internal vertex of one is a vertex of the other.

For  $S \subseteq V(G)$ , an  *$S$ -path* is a path in  $G$  that joins two *distinct* vertices in  $S$ . For a coloring  $\alpha : S \rightarrow \{1, 2\}$ , an  $S$ -path  $P$  in  $G$  is *parity-breaking with respect to  $\alpha$*  if

$$|E(P)| \equiv \alpha(u) - \alpha(v) \pmod{2}.$$

For a connected bipartite subgraph  $H$  of  $G$  with a proper coloring  $\beta : V(H) \rightarrow \{1, 2\}$ , a path  $P$  in  $G$  is *parity-breaking with respect to  $H$*  if  $P$  is parity-breaking with respect to  $\beta$ . This is well defined since a proper coloring of  $H$  is unique up to permuting colors. We will use the following observation in Section 3.

**Observation 2.1.** (1) For  $S \subseteq V(G)$  and a coloring  $\alpha : S \rightarrow \{1, 2\}$ , let  $P, Q$  be internally disjoint  $S$ -paths sharing precisely one end. Then the  $S$ -path  $P \cup Q$  is parity-breaking with respect to  $\alpha$  if and only if exactly one of  $P$  and  $Q$  is parity-breaking with respect to  $\alpha$ .

(2) For a connected bipartite subgraph  $H$  of  $G$ , no path in  $H$  is parity-breaking with respect to  $H$ .

**Minors.** A graph  $H$  is a *minor* of  $G$  if a graph isomorphic to  $H$  can be obtained from  $G$  by deleting vertices or edges and contracting edges. If there are edges  $wu$  and  $wv$  for some vertex  $w \notin \{u, v\}$  and we contract an edge  $wu$ , then one of these two edges is removed after contraction to avoid parallel edges. A graph  $G$  *contains an  $H$ -minor* (or  $H$  as a *minor*) if  $H$  is a minor of  $G$ .

**Topological minors.** An  *$H$ -subdivision* is a graph obtained from  $H$  by subdividing edges, where edges may be subdivided more than once. A graph  $G$  *contains an  $H$ -subdivision* (or  $H$  as a *topological minor*), if  $G$  contains a subgraph isomorphic to an  $H$ -subdivision.

Since every  $H$ -subdivision  $H'$  is built from  $H$  by replacing all edges of  $H$  with internally disjoint paths called the *linking paths*, there are vertices of  $H'$  that correspond to vertices of  $H$ , which we call *branch vertices*.

**Odd minors.** For  $S \subseteq V(G)$  and a coloring  $\alpha : S \rightarrow \{1, 2\}$ , an edge  $uv \in E(G[S])$  is *bichromatic* if  $\alpha(u) \neq \alpha(v)$ , *monochromatic* otherwise.

For graphs  $G$  and  $H$ ,  $G$  contains  $H$  as an *odd minor* if there exist vertex-disjoint subgraphs  $\{T_u\}_{u \in V(H)}$  in  $G$  which are trees, and a coloring  $\alpha : \bigcup_{u \in V(H)} V(T_u) \rightarrow \{1, 2\}$  such that for every  $u \in V(H)$ , every edge in  $T_u$  is bichromatic, and for every edge  $vw \in E(H)$ , there is a monochromatic edge  $e \in E(G)$  that joins  $V(T_v)$  and  $V(T_w)$ .

We will use the following alternative definition in Section 3.

**Observation 2.2.** *For graphs  $G$  and  $H$ ,  $G$  contains  $H$  as an odd minor if and only if there exist vertex-disjoint subgraphs  $\{T_u\}_{u \in V(H)}$  in  $G$  which are trees, and a coloring  $\alpha : \bigcup_{u \in V(H)} V(T_u) \rightarrow \{1, 2\}$  such that*

- (1) *for every  $u \in V(H)$ , every edge in  $T_u$  is bichromatic,*
- (2) *there are internally disjoint paths  $\{P_e\}_{e \in E(H)}$  in  $G$ ,*
- (3) *for every  $e = vw \in E(H)$ ,  $P_e$  joins  $V(T_v)$  and  $V(T_w)$ , has no internal vertex in  $\bigcup_{u \in V(H)} V(T_u)$ , and is parity-breaking with respect to  $\alpha$ .*

We remark that a graph  $G$  contains  $H$  as an odd minor if and only if a signed graph  $(G, E(G))$  contains a signed graph  $(H, E(H))$  as a minor, which we will discuss in Appendix A.

### 3. THE STRUCTURE OF GRAPHS WITH NO ODD CLIQUE MINOR

The proof of Theorem 1.3 is based on the fact that every graph with no  $K_t$  minor has a vertex of degree at most  $c_t$  for some  $c_t$ . In contrast to graphs with no  $K_t$  minor, graphs with no odd  $K_t$  minor may have arbitrarily large minimum degree; for example, complete bipartite graphs have no odd  $K_3$  minor.

To prove Theorems 1.8 and 1.9, we use the following strategy similar to the one by Geelen et al. [6]. If a graph  $G$  has no bipartite subdivision of some graph, then we apply Corollary 1.6. Otherwise, we will show in Theorem 3.5 that  $G$  contains a bipartite block after removing few vertices, which allows us to use precoloring arguments in the following section.

First of all, we describe how to find an odd  $K_t$  minor in a graph  $G$  if the graph contains a bipartite  $K_{2t-2} + I_t$  subdivision with many vertex-disjoint parity-breaking paths between branch vertices of the subdivision.

**Lemma 3.1.** *For  $t \geq 2$ , let  $G$  be a graph that contains a bipartite  $K_{2t-2} + I_t$  subdivision  $H$ , and  $C$  be the set of all branch vertices of  $K_{2t-2} + I_t$  in  $H$ . If there are  $t - 1$  vertex-disjoint parity-breaking  $C$ -paths with respect to  $H$ , then  $G$  contains an odd  $K_t$  minor.*

*Proof.* For convenience, we identify each vertex in  $C$  with its corresponding vertex of  $K_{2t-2} + I_t$ .

For a collection  $\mathcal{Q}$  of paths, let  $\ell(\mathcal{Q}) = \sum_{P \in \mathcal{Q}} |E(P)|$ . Let  $\mathcal{P}$  be a collection of  $t - 1$  vertex-disjoint parity-breaking  $C$ -paths with respect to  $H$ , satisfying the following.

- (a)  $\sum_{P \in \mathcal{P}} |E(P) \setminus E(H)|$  is minimum, and
- (b) subject to (a),  $\ell(\mathcal{P})$  is minimum.

Note that every vertex in  $C$  is not an internal vertex of a path in  $\mathcal{P}$ . To see this, if a vertex in  $C$  is an internal vertex of a path  $Q \in \mathcal{P}$ , then  $Q$  contains a proper subpath  $Q'$  that is a parity-breaking  $C$ -path with respect to  $H$ . For  $\mathcal{Q} := (\mathcal{P} \setminus \{Q\}) \cup \{Q'\}$  it follows that

$\sum_{P \in \mathcal{Q}} |E(P) \setminus E(H)| \leq \sum_{P \in \mathcal{P}} |E(P) \setminus E(H)|$  and  $\ell(\mathcal{Q}) < \ell(\mathcal{P})$ , contradicting our choice of  $\mathcal{P}$ .

For distinct  $u, v \in C$ , if  $uv \in E(K_{2t-2} + I_t)$ , then let  $Q_{u,v}$  be the linking path from  $u$  to  $v$  in  $H$ . If  $uv \notin E(K_{2t-2} + I_t)$ , then let  $Q_{u,v}$  be a graph with two vertices  $u$  and  $v$  and no edges. Note that there are  $2t - 2$  branch vertices that appear in paths in  $\mathcal{P}$ , and  $t$  unused branch vertices. Let  $C_0 \subseteq C$  be the set of those unused  $t$  branch vertices.

**Claim 1.** *Let  $u, v$  be distinct vertices in  $C$ .*

- (1) *If  $u, v \in C_0$ , then no path  $P \in \mathcal{P}$  intersects  $Q_{u,v}$ .<sup>1</sup>*
- (2) *If  $u \in C_0$ , then  $Q_{u,v}$  intersects at most one path  $P$  in  $\mathcal{P}$ , and if so, then the intersection is a subpath of both  $Q_{u,v}$  and  $P$  and contains  $v$ .*

*Proof of Claim 1.* We may assume  $uv \in E(K_{2t-2} + I_t)$ , otherwise  $V(Q_{u,v}) \subseteq C$  and the claim is trivial. Let  $u \in C_0$ , and suppose  $Q_{u,v}$  intersects a path in  $\mathcal{P}$ . Since no path in  $\mathcal{P}$  intersects  $u$ , starting from  $u$  and following  $Q_{u,v}$  we arrive at the first vertex  $w \in V(Q_{u,v})$  on some path  $P \in \mathcal{P}$ .

Write  $P = A \cup B$ , where  $A$  and  $B$  are two subpaths of  $P$  with the only common vertex  $w$ . Since  $w \in V(H)$ , either  $A$  or  $B$  is parity-breaking with respect to  $H$ , and we may assume  $A$  is parity-breaking with respect to  $H$ . By Observation 2.1, a path  $R = A \cup Q_{u,v}(w, u)$  is parity-breaking with respect to  $H$ , and it intersects no path in  $\mathcal{P}$  other than  $P$ . Therefore, we conclude that  $(\mathcal{P} \setminus \{P\}) \cup \{R\}$  is a set of  $t - 1$  vertex-disjoint parity-breaking  $C$ -paths with respect to  $H$ . By our assumption on  $\mathcal{P}$ ,  $P$  does not have more edges not in  $H$  than  $R$ . This implies  $E(B) \subseteq E(H)$ , and thus  $B = Q_{u,v}(w, v)$  since one of the ends of  $B$  is in  $C$  and no path in  $\mathcal{P}$  intersects  $u$ . Note that  $A$  intersects  $Q_{u,v}$  only at  $w$ , since  $B = Q_{u,v}(w, v)$ .

Since  $P$  and  $Q_{u,v}$  share a common subpath from  $w$  to  $v$  and  $w$  is the only vertex that belongs to both  $Q_{u,v}(w, u)$  and some path in  $\mathcal{P}$ ,  $P$  is the only path that intersects  $Q_{u,v}$ . In particular,  $B = P \cap Q_{u,v}$  is a path that contains  $v$ , and  $v \notin C_0$ .  $\blacksquare$

Let  $\mathcal{P} = \{P_1, \dots, P_{t-1}\}$ , and for  $1 \leq i \leq t - 1$ , let  $x_i$  and  $y_i$  be the ends of  $P_i$ . Let  $C_1 = \{x_1, \dots, x_{t-1}\}$  and  $C_2 = \{y_1, \dots, y_{t-1}\}$ .

For  $v \in C$ , if  $v$  corresponds to a vertex in the subgraph  $K_{2t-2}$  of  $K_{2t-2} + I_t$ , then we call  $v$  *Type-A*. Otherwise we call  $v$  *Type-B*. Let  $q$  be the number of  $i$ 's ( $1 \leq i \leq t - 1$ ) such that both  $x_i$  and  $y_i$  are Type-A, and  $r$  be the number of  $i$ 's such that exactly one of  $x_i$  and  $y_i$  is Type-A, and  $s = t - 1 - q - r$ . Then there are  $(2t - 2) - (2q + r)$  vertices of Type-A in  $C_0$ . Since  $q + r + s = t - 1$ , it follows that  $(2t - 2) - (2q + r) \geq r + s$ . Therefore, the number of vertices in  $C_0$  of Type-A is at least  $r + s$ . Thus, we choose an ordering  $z_1, \dots, z_t$  of the vertices in  $C_0$  such that for  $1 \leq i \leq t - 1$  if  $x_i$  or  $y_i$  is Type-B, then  $z_i$  is a vertex of Type-A.

In summary,  $z_i x_i, z_i y_i \in E(H)$  for  $1 \leq i \leq t - 1$ . Equivalently, if  $z_i$  is Type-B, then both  $x_i$  and  $y_i$  are Type-A.

Let  $\beta : V(H) \rightarrow \{1, 2\}$  be a proper coloring of  $H$  unique up to permuting colors. In order to find an odd  $K_t$  minor, we now aim to construct vertex-disjoint subgraphs  $M_1, \dots, M_t$  in  $G$  which are trees and a coloring  $\alpha : \bigcup_{i=1}^t V(M_i) \rightarrow \{1, 2\}$  as in Observation 2.2.

For  $1 \leq i \leq t - 1$ , let  $M_i = P_i \cup Q_{z_i, y_i}$  if  $z_i$  is Type-A, and  $M_i = P_i \cup Q_{z_i, x_i}$  if  $z_i$  is Type-B. Let  $M_t$  be a graph with the only vertex  $z_t$ . If  $i < t$ , then by Claim 1,  $P_i \cap Q_{z_i, y_i}$  or  $P_i \cap Q_{z_i, x_i}$  is a subpath of  $P_i$  and so  $M_i$  is a tree with maximum degree at most 3 and at most one vertex of degree 3. We choose a coloring  $\alpha : \bigcup_{i=1}^t V(M_i) \rightarrow \{1, 2\}$  such that

- (1)  $\alpha$  on  $V(M_i)$  is a proper coloring of  $M_i$  and  $\alpha(x_i) = \beta(x_i)$  for all  $1 \leq i \leq t - 1$ , and
- (2)  $\alpha(z_t) = \beta(z_t)$  if and only if  $z_t$  is Type-B.

<sup>1</sup>Two subgraphs *intersect* if they share at least one common vertex.

Observation 2.1 implies that, for  $1 \leq i \leq t-1$ ,  $\alpha(y_i) \neq \beta(y_i)$  since  $P_i$  is parity-breaking with respect to  $H$  and  $\alpha(z_i) = \beta(z_i)$  if and only if  $z_i$  is Type-B.

For  $1 \leq i < j \leq t$ , we are now ready to construct a path  $P_{i,j}$  joining  $V(M_i)$  and  $V(M_j)$  that is parity-breaking with respect to  $\alpha$  and satisfies Observation 2.2. This will show that  $G$  contains an odd  $K_t$  minor. The structure of  $P_{i,j}$  depends on the types of  $z_i$  and  $z_j$ .

**Case 1.** Both  $z_i$  and  $z_j$  are Type-A.

Following  $Q_{z_j, x_i}$  from  $z_j$  to  $x_i$ , we arrive at the first vertex  $a_{i,j}$  in  $V(P_i) \cap V(Q_{z_j, x_i})$ . By Claim 1,  $Q_{z_j, x_i}$  and  $P_i$  share the subpath from  $a_{i,j}$  to  $x_i$ . Since  $P_i(x_i, a_{i,j})$  is in  $H$  and  $\alpha(x_i) = \beta(x_i)$ , it follows that  $\alpha(a_{i,j}) = \beta(a_{i,j})$  by Observation 2.1. Let us define  $P_{i,j} = Q_{z_j, x_i}(z_j, a_{i,j})$ . Since  $\alpha(z_j) \neq \beta(z_j)$ ,  $\alpha(a_{i,j}) = \beta(a_{i,j})$  and  $P_{i,j}$  is a subpath of  $Q_{z_j, x_i}$ , we conclude that  $P_{i,j}$  is parity-breaking with respect to  $\alpha$  by Observation 2.1.

**Case 2.**  $z_i$  and  $z_j$  are of different types.

Let us define  $P_{i,j} := Q_{z_i, z_j}$ . By Claim 1,  $P_{i,j}$  intersects no path in  $\mathcal{P}$ . Since  $\alpha(z_i) = \beta(z_i)$ ,  $\alpha(z_j) \neq \beta(z_j)$ , and  $Q_{z_i, z_j}$  is in  $H$ ,  $P_{i,j}$  is parity-breaking with respect to  $\alpha$  by Observation 2.1.

**Case 3.** Both  $z_i$  and  $z_j$  are Type-B.

Since  $z_i$  is Type-B,  $y_i$  is Type-A. Following  $Q_{z_j, y_i}$  from  $z_j$  to  $y_i$ , we arrive at the first vertex  $a_{i,j}$  in  $V(P_i) \cap V(Q_{z_j, y_i})$ . Claim 1 implies that  $Q_{z_j, y_i}$  and  $P_i$  share the subpath from  $a_{i,j}$  to  $y_i$ . Since  $P_i(y_i, a_{i,j})$  is in  $H$  and  $\alpha(y_i) \neq \beta(y_i)$ , it follows that  $\alpha(a_{i,j}) \neq \beta(a_{i,j})$ . Let us define  $P_{i,j} := Q_{z_j, y_i}(z_j, a_{i,j})$ . Since  $\alpha(z_j) = \beta(z_j)$ ,  $\alpha(a_{i,j}) \neq \beta(a_{i,j})$  and  $P_{i,j}$  is in  $H$ ,  $P_{i,j}$  is parity-breaking with respect to  $\alpha$  by Observation 2.1.  $\square$

We use the following lemma, which asserts that the family of  $S$ -paths of odd length satisfies Erdős-Pósa property.

**Lemma 3.2** (Geelen et al. [6, Lemma 11]). *Let  $G$  be a graph and  $S \subseteq V(G)$ . For every integer  $\ell \geq 1$ , either  $G$  contains  $\ell$  vertex-disjoint  $S$ -paths of odd length, or there is  $X \subseteq V(G)$  with  $|X| \leq 2\ell - 2$  such that  $G \setminus X$  contains no  $S$ -path of odd length.*

**Observation 3.3.** *Let  $G$  and  $H$  be graphs and  $X \subseteq V(G)$ . If  $G$  contains an  $H$ -subdivision  $K$ , then  $G \setminus X$  contains an  $H'$ -subdivision  $K'$  such that  $H' = H \setminus Y$  for some  $Y \subseteq V(H)$  with  $|Y| \leq |X|$  and  $K'$  is a subgraph of  $K$ .*

*Proof.* It is easy to see that if  $G$  has an  $H$ -subdivision  $K$  and  $v$  is a vertex of  $K$ , then there is a vertex  $w$  of  $H$  such that  $G \setminus v$  has a  $(H \setminus w)$ -subdivision.  $\square$

The following lemma is a variation of the lemma by Geelen et al. [6, Lemma 15].

**Lemma 3.4.** *Let  $\ell$  be a positive integer and  $G$  be a graph. Let  $H$  be a bipartite  $K_s + I_t$  subdivision in  $G$  for integers  $s \geq 2\ell$  and  $t \geq 1$ , and  $C$  be the set of all branch vertices in  $H$ . At least one of the following holds.*

- *There exists  $X \subseteq V(G)$  with  $|X| \leq 2\ell - 2$  such that  $G - X$  has a bipartite block  $U$  that contains at least  $s + t - |X|$  vertices in  $C \setminus X$  and all linking paths in  $H$  between them.*
- *$G$  has  $\ell$  vertex-disjoint parity-breaking  $C$ -paths with respect to  $H$ .*

*Proof.* (1) We claim that either there are  $\ell$  vertex-disjoint parity-breaking  $C$ -paths in  $G$  with respect to  $H$ , or there is  $X \subseteq V(G)$  with  $|X| \leq 2\ell - 2$  such that  $G \setminus X$  contains no parity-breaking  $C$ -path with respect to  $H$ .

Let  $\{L, R\}$  be the unique bipartition of  $H$ . Without loss of generality, we may assume that every linking path corresponding to an edge in  $K_s + I_t$  has the even length, because otherwise, for every branch vertex  $v \in C \cap L$ , we subdivide each edge  $e \in E(G)$  incident with

$v$  once. This gives an  $H$ -subdivision  $H'$  and a  $G$ -subdivision  $G'$  such that  $H'$  is a bipartite subgraph of  $G'$ . We may assume  $V(H) \subseteq V(H')$  and  $V(G) \subseteq V(G')$ , and then every vertex in  $V(G') \setminus V(G)$  has degree 2. Note that all vertices in  $C$  are in the same part of the bipartition of  $H'$ , and thus every path in  $H'$  between vertices in  $C$  has the even length. It is easy to check the following.

- A  $C$ -path of odd length in  $G'$  corresponds to a parity-breaking  $C$ -path in  $G$  with respect to  $H$ .
- If there is  $X' \subseteq V(G')$  with  $|X'| \leq 2\ell - 2$  such that  $G' \setminus X'$  contains no  $C$ -path of odd length, then we may assume  $X' \subseteq V(G)$  since every vertex in  $V(G') \setminus V(G)$  has degree 2.

Lemma 3.2 claims that either  $G'$  contains  $\ell$  vertex-disjoint  $C$ -paths of odd length, or there is  $X' \subseteq V(G')$  with  $|X'| \leq 2\ell - 2$  such that  $G' \setminus X'$  contains no  $C$ -path of odd length. This proves (1).

Suppose  $G$  contains no  $\ell$  vertex-disjoint parity-breaking  $C$ -paths with respect to  $H$ . By (1), there is  $X \subseteq V(G)$  with  $|X| \leq 2\ell - 2$  such that  $G \setminus X$  contains no parity-breaking  $C$ -path with respect to  $H$ . For convenience, we identify each vertex in  $C$  with its corresponding vertex of  $K_s + I_t$ . For distinct  $u, v \in C$ , if  $uv \in E(K_s + I_t)$  then let  $Q_{u,v}$  be the linking path from  $u$  to  $v$  in  $H$ . If  $uv \notin E(K_s + I_t)$ , then let  $Q_{u,v}$  be a graph with two vertices  $u$  and  $v$  and no edges.

(2) *We claim that there is a block  $U$  in  $G \setminus X$  containing at least  $s + t - |X|$  vertices in  $C \setminus X$  and all linking paths in  $H$  between them.*

By Observation 3.3,  $G \setminus X$  contains a  $(K_a + I_b)$ -subdivision  $K$  such that  $K_a + I_b = (K_s + I_t) \setminus Y$  for some  $Y \subseteq V(K_s + I_t)$  with  $|Y| \leq |X|$  and  $K$  is a subgraph of  $H$ . Let  $T$  be the set of all branch vertices in  $K$ , where  $|T| \geq a + b = s + t - |Y| \geq s + t - |X|$ .

Since  $a \geq s - |Y| \geq 2$  and  $a + b = s + t - |Y| \geq 3$ ,  $K_a + I_b$  is 2-connected. Therefore,  $K$  is 2-connected and all vertices in  $T$  are in the same block of  $G \setminus X$ .

(3) *We claim that  $U$  is bipartite.*

Suppose  $U$  contains an odd-length cycle  $D$ . For two distinct vertices  $u, v \in C \cap V(U)$ , there are two vertex-disjoint paths in  $U$  joining  $\{u, v\}$  and  $V(D)$  by Menger's theorem. Using these paths, we obtain both an odd-length path and an even-length path from  $u$  to  $v$  in  $U$ . One of those paths is a parity-breaking  $C$ -path with respect to  $H$ , contradicting that  $G \setminus X$  has no parity-breaking  $C$ -path with respect to  $H$ .  $\square$

Now we are ready to prove the main theorem of this section.

**Theorem 3.5.** *Let  $t \geq 2$  be an integer, and  $G$  be a graph. If  $G$  contains no odd  $K_t$  minor and contains a bipartite  $K_{2t-2} + I_t$  subdivision, then there is  $X \subseteq V(G)$  with  $|X| \leq 2t - 4$  such that  $G \setminus X$  contains a bipartite block  $U$  having at least  $t + 3$  vertices.*

*Proof.* Let  $H$  be a bipartite  $K_{2t-2} + I_t$  subdivision of  $G$ , and  $C = \{v_1, \dots, v_{3t-2}\}$  be the set of all branch vertices in  $H$ . For convenience, we identify each vertex in  $C$  with its corresponding vertex in  $V(K_{2t-2} + I_t)$ . Let  $C_1 \subseteq C$  be the set of branch vertices corresponding to vertices in  $K_{2t-2}$ , and  $C_2 = C \setminus C_1$  be the set of branch vertices corresponding to vertices in  $I_t$ .

By Lemmas 3.1 and 3.4, there exists  $X \subseteq V(G)$  with  $|X| \leq 2t - 4$  such that  $G \setminus X$  has a bipartite block  $U$  containing  $(3t - 2) - |X|$  vertices in  $C \setminus X$  and all linking paths between them. Let  $C' \subseteq C$  be those  $(3t - 2) - |X| \geq t + 2$  branch vertices in  $U$ , and  $H'$  be the union of all linking paths between vertices in  $C'$ , which is a subgraph of  $U$ .

Recall that we identified  $C \subseteq V(H)$  with  $V(K_{2t-2} + I_t)$ . Since vertices in  $C_1$  form a clique of  $K_{2t-2} + I_t$  and  $|C' \cap C_1| \geq |C'| - |C_2| \geq 2$ , the subgraph of  $K_{2t-2} + I_t$  induced by  $C'$  is not

bipartite. To obtain  $H'$  from the induced subgraph of  $K_{2t-2} + I_t$ , we should subdivide edges at least once, because  $H'$  is bipartite. Thus  $H'$  contains a vertex other than vertices in  $C'$ , implying  $|V(U)| \geq |V(H')| \geq |C'| + 1 \geq t + 3$ .  $\square$

#### 4. PROOFS OF THEOREMS 1.8 AND 1.9

For an integer  $N \geq 0$ , let  $[N]$  be the set  $\{1, \dots, N\}$ . If  $N = 0$ , then  $[N]$  is an empty set. For a class  $\mathcal{F}$  of graphs and an integer  $d \geq 0$ , a graph  $G$  has a  $(d, \mathcal{F})$ -coloring if there is  $f : V(G) \rightarrow [d]$  such that  $G[f^{-1}(\{i\})]$  is in  $\mathcal{F}$  for all  $i \in [d]$ . A class  $\mathcal{F}$  of graphs is *closed under isomorphisms* if for all  $G \in \mathcal{F}$ , every graph isomorphic to  $G$  is in  $\mathcal{F}$ . A class  $\mathcal{F}$  of graphs is *closed under taking disjoint unions* if for all  $G, H \in \mathcal{F}$ , the disjoint union of  $G$  and  $H$  is in  $\mathcal{F}$ .

**Lemma 4.1.** *Let  $t \geq 2$  and  $d \geq 3$  be integers and  $\mathcal{F}$  be a class of graphs closed under isomorphisms and taking disjoint unions, which satisfies the following.*

- (i)  $\mathcal{F}$  contains every graph with at most  $4t - 7$  vertices.
- (ii) If a graph  $H$  contains no odd  $K_t$  minor and no bipartite  $K_{2t-2} + I_t$  subdivision, then  $H$  has a  $(d, \mathcal{F})$ -coloring.

Then every graph with no odd  $K_t$  minor has a  $(d + 4t - 7, \mathcal{F})$ -coloring.

*Proof.* We prove the following stronger claim.

**Claim.** *Let  $G$  be a graph with no odd  $K_t$  minor,  $Z \subseteq V(G)$  with  $|Z| \leq 4t - 7$ , and  $f : Z \rightarrow [d + 4t - 7]$  be a coloring. Then  $G$  has a  $(d + 4t - 7, \mathcal{F})$ -coloring  $g$  that satisfies the following.*

- (a) For every  $z \in Z$ ,  $f(z) = g(z)$ .
- (b) For every  $v \in Z$  and its neighbor  $w \notin Z$ ,  $g(v) \neq g(w)$ .

Let  $G$  be a counterexample with the minimum  $|V(G)| + |E(G)|$ . As the claim is true for graphs with at most  $4t - 7$  vertices by giving distinct colors to each vertex not in  $Z$ ,  $|V(G)| \geq 4t - 6$ .

- (1)  $Z$  is stable.

Suppose there are adjacent  $z_1, z_2 \in Z$ . Applying the claim on  $G' = G \setminus z_1 z_2$  with the same  $Z$  and  $f$ ,  $G'$  has a  $(d + 4t - 7, \mathcal{F})$ -coloring  $g$  that satisfies the claim. We claim that every component of  $G[g^{-1}(\{i\})]$  for some  $i \in [d + 4t - 7]$  is in  $\mathcal{F}$ . Let  $C$  be a component of  $G[g^{-1}(\{i\})]$  for some  $i \in [d + 4t - 7]$ . If  $V(C) \cap Z \neq \emptyset$  then  $V(C) \subseteq Z$  by (b), implying  $C \in \mathcal{F}$  as  $|V(C)| \leq |Z| \leq 4t - 7$ . If  $V(C) \cap Z = \emptyset$  then  $C$  is a component of  $G'[g^{-1}(\{i\})]$ , which implies  $C \in \mathcal{F}$ . Therefore,  $g$  is a  $(d + 4t - 7, \mathcal{F})$ -coloring of  $G$  satisfying (a) and (b), contradicting our assumption.

- (2) For every separation  $(A, B)$  of order at most  $2t - 3$ , either  $V(A) \setminus V(B) \subseteq Z$  or  $V(B) \setminus V(A) \subseteq Z$ .

Suppose  $G$  has a separation  $(A, B)$  of order at most  $2t - 3$  such that both  $V(A) \setminus V(B) \setminus Z$  and  $V(B) \setminus V(A) \setminus Z$  are nonempty. Since  $|Z| = |V(A) \cap Z| + |(V(B) \setminus V(A)) \cap Z|$ , we may assume  $|(V(B) \setminus V(A)) \cap Z| \leq \lfloor \frac{|Z|}{2} \rfloor \leq 2t - 4$ . Note that  $V(B) \setminus V(A) \setminus Z \neq \emptyset$  implies that  $|V(A) \cup Z| < |V(G)|$  and we can apply the claim on  $A \cup G[Z]$  with  $Z$  and  $f$ . Let  $g_1$  be a  $(d + 4t - 7, \mathcal{F})$ -coloring of  $A \cup G[Z]$  satisfying (a) and (b). Let  $Z' = V(A \cap B) \cup (V(B) \cap Z)$ . Since  $|Z'| = |V(A \cap B)| + |(V(B) \setminus V(A)) \cap Z| \leq 4t - 7$ , we can apply the claim on  $B$  with  $Z'$  and  $g_1|_{Z'}$ . Let  $g_2$  be a  $(d + 4t - 7, \mathcal{F})$ -coloring of  $B$  satisfying (a) and (b).

Let  $g$  be a coloring on  $V(G)$  such that for each vertex  $v$  of  $G$ ,

$$g(v) = \begin{cases} g_1(v) & \text{for } v \in V(A), \text{ and} \\ g_2(v) & \text{for } v \in V(B). \end{cases}$$

This is well defined since  $g_1$  is identical to  $g_2$  on  $Z'$ . We claim that  $g$  is a  $(d+4t-7, \mathcal{F})$ -coloring of  $G$  satisfying (a) and (b), which contradicts our assumption.

By the definition of  $g_1$ , it follows that  $g(z) = g_1(z) = f(z)$  for every  $z \in Z$ . For every  $vw \in E(G)$  with  $v \in Z$  and  $w \notin Z$ ,  $g(v) \neq g(w)$  since  $g_1(v) = g(v) \neq g(w) = g_2(w)$  if  $w \in V(A)$  and  $g_2(v) = g(v) \neq g(w) = g_2(w)$  if  $w \in V(B)$ . This verifies (a) and (b).

Let  $C$  be a component of  $G[g^{-1}(\{i\})]$  for some  $i \in [d+4t-7]$ . If  $V(C) \cap Z \neq \emptyset$  then  $V(C) \cap (V(A) \setminus Z) = \emptyset$  by the definition of  $g_1$  and (b), and  $V(C) \cap (V(B) \setminus V(A) \setminus Z) = \emptyset$  by the definition of  $g_2$  and (b). This implies  $V(C) \subseteq Z$  and thus  $C \in \mathcal{F}$  as  $|V(C)| \leq |Z| \leq 4t-7$ . If  $V(C) \cap Z = \emptyset$  and  $V(A) \cap V(B) \cap V(C) \neq \emptyset$  then  $V(C) \subseteq Z' \setminus Z \subseteq V(A) \cap V(B)$  by the definition of  $g_2$  and (b). Thus  $C$  is a component of  $G[g_1^{-1}(\{i\})]$ , implying  $C \in \mathcal{F}$ . Finally, if  $V(C) \cap Z = \emptyset$  and  $V(A) \cap V(B) \cap V(C) = \emptyset$ , then either  $V(C) \subseteq V(A) \setminus V(B) \setminus Z$  or  $V(C) \subseteq V(B) \setminus V(A) \setminus Z$ , which implies that  $C \in \mathcal{F}$  as  $C$  is a component of either  $G[g_1^{-1}(\{i\})]$  or  $G[g_2^{-1}(\{i\})]$ .

(3)  $G \setminus Z$  contains a bipartite  $K_{2t-2} + I_t$  subdivision.

Since  $|Z| \leq 4t-7$ , we may assume  $f(Z) \subseteq \{d+1, \dots, d+4t-7\}$  by permuting colors. Suppose  $G \setminus Z$  does not contain a bipartite  $K_{2t-2} + I_t$  subdivision. Let  $g_0$  be a  $(d, \mathcal{F})$ -coloring of  $G \setminus Z$ . Let  $g : V(G) \rightarrow [d+4t-7]$  be a coloring such that for each vertex  $v$  of  $G$ ,

$$g(v) = \begin{cases} g_1(v) & \text{for every } v \in V(G) \setminus Z \text{ and,} \\ f(z) & \text{for every } z \in Z. \end{cases}$$

We claim that  $g$  is a  $(d+4t-7, \mathcal{F})$ -coloring of  $G$  satisfying (a) and (b), which contradicts our assumption. Let  $C$  be a component of  $G[g^{-1}(\{i\})]$  for some  $i \in [d+4t-7]$ . Since  $g$  is identical to  $g_1$  on  $V(G) \setminus Z$  and  $g(u) \neq g(v)$  for every  $u \in V(G) \setminus Z$  and  $v \in Z$ ,  $C$  is a component of either  $G[g_1^{-1}(\{i\})]$  or  $G[f^{-1}(\{i\})]$ . This implies  $C \in \mathcal{F}$ . This proves (3).

Since  $G$  contains a bipartite  $K_{2t-2} + I_t$  subdivision, Theorem 3.5 implies that there exists  $X \subseteq V(G)$  with  $|X| \leq 2t-4$  such that  $G \setminus X$  admits a block decomposition with a bipartite block  $U$  having at least  $t+3$  vertices.

(4) Every component of  $G \setminus X \setminus V(U)$  is a subgraph of  $G[Z]$ .

Let  $C$  be a component of  $G \setminus X \setminus V(U)$ . Let  $V_C$  be the set of vertices in  $U$  adjacent to a vertex in  $C$ . As  $U$  is a block and  $C$  is a component of  $G \setminus X \setminus V(U)$ , it follows that  $|V_C| \leq 1$ . If  $|V_C| = 1$  then let  $v_C$  be the unique vertex in  $V_C$ . Let  $A_C = G[V(C) \cup X \cup V_C]$  and  $B_C = G \setminus V(C)$ . Note that  $V(A_C) \cap V(B_C) = X \cup V_C$  and  $(A_C, B_C)$  is a separation of  $G$  of order at most  $2t-3$ , since  $|X| + |V_C| \leq 2t-3$ . By (2), either  $V(A_C) \setminus V(B_C)$  or  $V(B_C) \setminus V(A_C)$  is in  $Z$ . Since  $Z$  is stable,  $V(U) \setminus V_C \subseteq V(B_C) \setminus V(A_C)$  and  $U$  is 2-connected as  $|V(U)| \geq t+3$ ,  $V(B_C) \setminus V(A_C)$  is not a subset  $Z$ . Therefore,  $V(A_C) \setminus V(B_C) = V(C)$  is a subset of  $Z$ . This proves (4).

Since  $U \setminus Z$  is a bipartite subgraph of  $G$ , let  $\{X_1, X_2\}$  be its bipartition. By (4), it follows that  $V(G) = Z \cup (X \setminus Z) \cup X_1 \cup X_2$ . Let us choose three colors  $\{c_1, c_2, c_3\} \subseteq [4t-4] \setminus f(Z)$ .

Let  $g : V(G) \rightarrow [4t - 4] \subseteq [d + 4t - 7]$  be a coloring defined as follows:

$$g(x) = \begin{cases} f(x) & \text{for } x \in Z, \\ c_1 & \text{for } x \in X \setminus Z, \\ c_2 & \text{if } x \in X_1, \\ c_3 & \text{if } x \in X_2. \end{cases}$$

We claim that  $g$  is a  $(d + 4t - 7, \mathcal{F})$ -coloring of  $G$  satisfying (a) and (b), which contradicts our assumption.

Let  $C$  be a component of  $G[g^{-1}(\{i\})]$  for some  $i \in [d + 4t - 7]$ . One of the following cases hold: either  $V(C) \subseteq Z$  or  $V(C) \subseteq X \setminus Z$  or  $V(C) \subseteq X_1$  or  $V(C) \subseteq X_2$ . Since  $|Z|$  and  $|X|$  are at most  $4t - 7$  and both  $X_1$  and  $X_2$  are stable in  $G$ ,  $C$  is in  $\mathcal{F}$ .  $\square$

Now we prove Theorem 1.8.

*Proof of Theorem 1.8.* Let  $\mathcal{F}$  be the set of graphs of maximum degree at most  $\max(N(2t - 2), 4t - 8)$  where  $N$  is in Corollary 1.6. Corollary 1.6 implies that every graph with no bipartite  $K_{2t-2} + I_t$  subdivision has a  $(2t - 2, \mathcal{F})$ -coloring. By Lemma 4.1,  $G$  has a  $(6t - 9, \mathcal{F})$ -coloring.  $\square$

Liu and Oum [16] proved that if a graph  $G$  contains no odd  $K_t$  minor, then we can color  $G$  with only three colors so that every monochromatic component has the size bounded by a function of  $t$  and the maximum degree of  $G$ , as follows.

**Theorem 4.2** (Liu and Oum [16]). *Let  $t \geq 1$  and  $\Delta \geq 0$  be integers. There is  $C = C(t, \Delta)$  such that for every graph  $G$  with the maximum degree at most  $\Delta$  and no odd  $K_t$  minor, there are subsets  $V_1, V_2, V_3$  of  $V(G)$  such that  $V_1 \cup V_2 \cup V_3 = V(G)$  and every component of  $G[V_i]$  has at most  $C$  vertices for  $i = 1, 2, 3$ .*

Now we prove Theorem 1.9.

*Proof of Theorem 1.9.* Let  $u(t) := C(t, N(2t - 2, t))$  where  $C$  is in Theorem 4.2 and  $N$  is in Corollary 1.6. Let  $\mathcal{F}$  be the set of graphs that every component has at most  $\max(u(t), 4t - 7)$  vertices. By Corollary 1.6 and Theorem 4.2, every graph with no odd  $K_t$  minor and no bipartite  $K_{2t-2} + I_t$  subdivision has a  $(3(2t - 2), \mathcal{F})$ -coloring. By Lemma 4.1,  $G$  has a  $(10t - 13, \mathcal{F})$ -coloring.  $\square$

## 5. CONCLUDING REMARKS

**5.1. List-coloring variant.** We may consider a list-coloring variant of defective coloring. For integers  $s, N \geq 0$ , a graph  $G$  is  $(s, N)$ -choosable if for every set of lists  $\{L_v\}_{v \in V(G)}$  with  $|L_v| \geq s$  for every  $v \in V(G)$ , there is a map  $f : V(G) \rightarrow \bigcup_{v \in V(G)} L_v$  with  $f(v) \in L_v$  for each  $v \in V(G)$  such that  $\Delta(G[f^{-1}(\{i\})]) \leq N$  for every  $i \in \bigcup_{v \in V(G)} L_v$ .

As we remarked in Section 1, Theorems 1.3 and 1.5 can be extended in terms of list-coloring variant. For instance, the theorem of Ossona de Mendez et al. [17] combined with the argument in Section 1 implies that for integers  $s, t \geq 1$  and every graph  $G$  with no bipartite  $K_s + I_t$  subdivision, there is  $N = N(s, t)$  such that  $G$  is  $(s, N)$ -choosable. It follows that for  $t \geq 1$ , graphs with no  $K_{t+1}$  minor are  $(t, s)$ -choosable for some  $s = s(t)$ .

Note that every  $n$ -vertex graph with no  $K_t$  minor contains  $O(t\sqrt{\log t n})$  edges [14, 15, 20, 21]. In contrast to graphs with no clique minor, an  $n$ -vertex graph with no odd  $K_3$  minor may contain  $O(n^2)$  edges. For example, complete bipartite graphs have no odd  $K_3$  minor.

**Theorem 5.1** (Kang [12]). *For each fixed integer  $N \geq 0$ , there is a function*

$$s(d) = (1/2 + o(1)) \log_2 d$$

*as  $d \rightarrow \infty$  such that every graph of minimum degree at least  $d$  is not  $(s(d), N)$ -choosable.*

By Theorem 5.1, it follows that for integers  $t \geq 3$  and  $s, N \geq 0$ , there are graphs with no odd  $K_t$  minor not  $(s, N)$ -choosable; for instance, large complete bipartite graphs.

**5.2. Extending Theorems 1.8 and 1.9.** We extend our main results to a slightly larger class of graphs. As we mentioned in Section 2,  $G$  contains  $H$  as an odd minor if and only if a signed graph  $(G, E(G))$  contains a signed graph  $(H, E(H))$  as a minor. We review the concepts of signed graphs and their minors in Appendix A.

Given  $\Sigma \subseteq E(H)$ , we provide an alternative characterization for signed graphs  $(G, E(G))$  containing  $(H, \Sigma)$  as a minor. A signed graph  $(G, E(G))$  contains a signed graph  $(H, \Sigma)$  as a *minor* if and only if

- (1) there exist vertex-disjoint subgraphs  $\{T_u\}_{u \in V(H)}$  in  $G$  which are trees, and
- (2) a coloring  $\alpha : \bigcup_{u \in V(H)} V(T_u) \rightarrow \{1, 2\}$  such that for every  $u \in V(H)$ , every edge in  $T_u$  is bichromatic, and for every edge  $vw \in E(H)$ , there is an edge  $e \in E(G)$  that joins  $V(T_v)$  and  $V(T_w)$  where  $e$  is monochromatic if and only if  $vw \in \Sigma$ .

Note that for every  $\Sigma \subseteq E(K_t)$ ,  $(K_{2t}, E(K_{2t}))$  contains  $(K_t, \Sigma)$  as a minor. Replacing  $t$  by  $2t$ , Theorem 1.8 implies that for every  $t \geq 2$  and every  $\Sigma \subseteq E(K_t)$ , if  $(G, E(G))$  contains no  $(K_t, \Sigma)$  as a minor, then  $V(G)$  admits a partition  $V_1, \dots, V_{12t-9}$  such that  $\Delta(G[V_i]) \leq s(2t)$  for  $1 \leq i \leq 12t - 9$ . Theorem 1.9 also implies that for every  $t \geq 2$  and every  $\Sigma \subseteq E(K_t)$ , if  $(G, E(G))$  contains no  $(K_t, \Sigma)$  as a minor, then  $V(G)$  admits a partition  $V_1, \dots, V_{20t-13}$  such that every component of  $G[V_i]$  has at most  $s(2t)$  vertices for  $1 \leq i \leq 20t - 13$ .

By modifying the proofs in Section 3, we can improve these bounds further. In the proof of Lemma 3.1, we join  $V(M_i)$  and  $V(M_j)$  with a parity-breaking path with respect to  $\alpha$  for  $1 \leq i < j \leq t$ . Because  $\alpha(x_i) = \beta(x_i)$  and  $\alpha(y_i) \neq \beta(y_i)$ , we can also join  $V(M_i)$  and  $V(M_j)$  with a path that is not parity-breaking with respect to  $\alpha$ . In particular, Lemma 3.1 forces not only an odd  $K_t$  minor, but also a signed  $(K_t, \Sigma)$  minor for *every*  $\Sigma \subseteq E(K_t)$ . This extends Theorems 1.8 and 1.9 as follows.

**Corollary 5.2.** *For an integer  $t \geq 2$ , there exists an integer  $s = s(t)$  such that for every  $\Sigma \subseteq E(K_t)$  and  $(G, E(G))$  with no  $(K_t, \Sigma)$  minor, there are  $6t - 9$  sets  $V_1, \dots, V_{6t-9}$  with  $V(G) = \bigcup_{i=1}^{6t-9} V_i$  such that  $\Delta(G[V_i]) \leq s$  for  $1 \leq i \leq 6t - 9$ .*

**Corollary 5.3.** *For an integer  $t \geq 2$ , there exists an integer  $s = s(t)$  such that for every  $\Sigma \subseteq E(K_t)$  and  $(G, E(G))$  with no  $(K_t, \Sigma)$  minor, there are  $10t - 13$  sets  $V_1, \dots, V_{10t-13}$  with  $V(G) = \bigcup_{i=1}^{10t-13} V_i$  such that every component of  $G[V_i]$  has at most  $s$  vertices for  $1 \leq i \leq 10t - 13$ .*

**5.3. Upper bound of the maximum degree.** In Theorem 1.5, the function  $M(s, t, \delta_1, \delta_2)$  is defined as follows.

$$M(s, t, \delta_1, \delta_2) = \begin{cases} t - 1 & \text{if } s = 1 \\ \frac{\delta_2 t (\delta_1 - 2)}{2} + \delta_1 & \text{if } s = 2 \\ (\delta_1 - s) \left( \binom{\delta_2}{s-1} (t-1) + \frac{\delta_2}{2} \right) + \delta_1 & \text{if } s > 2 \end{cases}$$

Therefore, it follows that  $N(2t - 2, t) = M(2t - 2, t, O(t^2), O(t^2)) = \exp(O(t \log t))$  in Corollary 1.6. Hence we have the upper bound of  $s(t) = \exp(O(t \log t))$  in Theorem 1.8.

If one replaces a bipartite  $K_{2t-2} + I_t$  subdivision of Lemma 3.1 with a bipartite  $K_{3t-2}$  subdivision, one may set  $s(t) = O(t^4)$  since  $N(3t-3, 1) = O(t^4)$ . However, this will increase the number of colors from  $6t-9$  to  $7t-10$  of Theorem 1.8, as graphs with no bipartite  $K_{3t-2}$  subdivision are defectively colored with  $3t-3$  colors, which is more than  $2t-2$  colors in defective coloring of graphs with no bipartite  $K_{2t-2} + I_t$  subdivision.

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## APPENDIX A. SIGNED GRAPHS

We review elementary concepts of signed graphs, following definitions in [9] and [10] unless stated otherwise.

A *signed graph*  $(G, \Sigma)$  is a graph  $G = (V, E)$  equipped with a *signature*  $\Sigma \subseteq E$ . To avoid confusion, graphs always denote unsigned graphs. Every signed graph is assumed to be simple; parallel edges and loops are not allowed. If an edge  $e \in E(G)$  is in  $\Sigma$ ,  $e$  is *negative*. Otherwise,  $e$  is *positive*.

For two sets  $A$  and  $B$ ,  $A\Delta B$  denotes the set  $(A\setminus B)\cup(B\setminus A)$ . For a graph  $G$  and  $X\subseteq V(G)$ , let  $\delta_G(X)$  be the set of edges joining  $X$  and  $V(G)\setminus X$ . For a signature  $\Sigma$ , a *re-signing* on  $X$  is an operation that replaces  $\Sigma$  with another signature  $\Sigma\Delta\delta_G(X)$  for some  $X\subseteq V(G)$ . Note that for  $X, Y\subseteq V(G)$ , applying re-signing on  $X$  and  $Y$  is identical to applying re-signing on  $X\Delta Y$ . In particular, *re-signing at  $v$*  is the operation that replaces  $\Sigma$  with  $\Sigma\Delta\delta_G(\{v\})$ . Note that applying a re-signing operation on  $X$  is identical to applying re-signing operations at all vertices in  $X$ .

Two signatures  $\Sigma$  and  $\Sigma'$  are *equivalent* if  $\Sigma'$  can be obtained from  $\Sigma$  by re-signing operations;  $\Sigma$  and  $\Sigma'$  are equivalent if and only if there is  $X\subseteq V(G)$  such that  $\Sigma' = \Sigma\Delta\delta_G(X)$ . Two signed graphs  $(G, \Sigma)$  and  $(G, \Sigma')$  are *equivalent* if  $\Sigma$  is equivalent to  $\Sigma'$ .

A cycle  $C$  is called *balanced* if it contains an even number of negative edges. Two signed graphs  $(G, \Sigma)$  and  $(G, \Sigma')$  have the same set of balanced cycles if and only if  $\Sigma$  and  $\Sigma'$  are equivalent (see [9]).

A map  $f : V(G) \rightarrow V(H)$  is an *isomorphism* from  $(G, \Sigma)$  and  $(H, \Sigma')$  if  $f$  is an isomorphism from  $G$  to  $H$ , and  $uv \in \Sigma$  if and only if  $f(u)f(v) \in \Sigma'$ . If there is an isomorphism from  $(G, \Sigma)$  to  $(H, \Sigma')$ ,  $(G, \Sigma)$  is *isomorphic* to  $(H, \Sigma')$ .

For two signed graphs  $(G, \Sigma)$  and  $(H, \Sigma')$ ,  $(H, \Sigma')$  is a *minor* of  $(G, \Sigma)$  if a signed graph isomorphic to  $(H, \Sigma')$  can be obtained from  $(G, \Sigma)$  by deleting vertices, deleting edges, applying re-signing operations, and contracting positive edges. To avoid parallel edges, if  $G$  has edges  $wu, vw \in E(G)$  of different signs, then when we contract a positive edge  $uv$ , we should remove either  $wu$  or  $vw$  before contracting  $uv$ .

When applying a series of operations to find minors, we may assume that deleting vertices and edges always precede re-signing operations; contracting a positive edge  $uv$  into a new vertex  $t$  and re-signing at  $t$  is identical to re-signing on  $\{u, v\}$  and contracting a positive edge  $uv$  into a vertex  $t$ . This implies the following, which can be found in [7].

**Lemma A.1.** *For graphs  $G, H$  and a signature  $\Sigma \subseteq E(H)$ , a signed graph  $(G, E(G))$  contains a signed graph  $(H, \Sigma)$  as a minor if and only if*

- (1) *there are vertex-disjoint subgraphs  $\{T_u\}_{u \in V(H)}$  of  $G$  assigned to vertices in  $V(H)$ ,*
- (2) *for every  $u \in V(H)$ ,  $T_u$  is a tree and has a proper 2-coloring  $c_u : V(T_u) \rightarrow \{1, 2\}$ , and*
- (3) *for every edge  $uv \in E(H)$ , there is an edge  $e = ab \in E(G)$  that joins  $T_u$  and  $T_v$  such that  $c_u(a) = c_v(b)$  if and only if  $uv \in \Sigma$ .*

DEPARTMENT OF MATHEMATICAL SCIENCES, KAIST, 291 DAEHAK-RO YUSEONG-GU DAEJEON, 34141 SOUTH KOREA

*E-mail address:* dyk90@kaist.ac.kr

*E-mail address:* sangil@kaist.edu