Coloring graphs without fan vertex-minors and graphs without cycle pivot-minors

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Abstract

A fan F_k is a graph that consists of an induced path on k vertices and an additional vertex that is adjacent to all vertices of the path. We prove that for all positive integers q and k, every graph with sufficiently large chromatic number contains either a clique of size q or a vertex-minor isomorphic to F_k. We also prove that for all positive integers q and $k \geq 3$, every graph with sufficiently large chromatic number contains either a clique of size q or a pivot-minor isomorphic to a cycle of length k.

1 Introduction

All graphs in this paper are simple, which means no loops and no parallel edges. Given a graph, a *clique* is a set of pairwise adjacent vertices and an *independent set* is a set of pairwise non-adjacent vertices. For a graph G, let $\chi(G)$ denote the *chromatic number* of G and let $\omega(G)$ denote the maximum size of a clique of G. Since two vertices in a clique cannot receive the same color in a proper coloring, the clique number is a trivial lower bound for the chromatic number. If $\chi(H) = \omega(H)$ for every induced subgraph H of a graph G, then we say G is *perfect*. Gyárfás [19] introduced the notion of a χ-bounded class as a generalization of perfect graphs. A class $\mathcal{C}$ of graphs is χ*-bounded* if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for all graphs $G \in \mathcal{C}$, and all induced subgraphs H of G, $\chi(H) \leq f(\omega(H))$. Therefore the class of perfect graphs is χ-bounded with the identity function.

Chudnovsky, Robertson, Seymour, and Thomas [9] proved the strong perfect graph theorem, which states that a graph G is perfect if and only if neither G nor its complement contains an

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induced odd cycle of length at least 5. This shows that there is a deep connection between the chromatic number and the structure of the graph. Gyárfás [19] proved that for each integer k, the class of graphs with no induced path of length k is χ-bounded. Gyárfás also made the following three conjectures for χ-boundedness in terms of forbidden induced subgraphs. Note that (iii) implies both (i) and (ii).

Conjecture 1.1 (Gyárfás [19]). *The following classes are χ-bounded:*

(i) *The class of graphs with no induced odd cycle of length at least* 5.

(ii) *The class of graphs with no induced cycle of length at least k for a fixed k.*

(iii) *The class of graphs with no induced odd cycle of length at least k for a fixed k.*

There are recent works by Chudnovsky, Scott, and Seymour [10, 11, 12] and Scott and Seymour [34, 35] regarding χ-boundedness and induced subgraphs; in this series of papers they prove (i) and (ii) of Conjecture 1.1, and also solve the case when $k = 5$ for (iii). The full conjecture of (iii) is still open. One result in this paper (Theorem 4.1) gives further evidence on (iii) of Conjecture 1.1, as the half of Theorem 4.1 is implied by (iii) of Conjecture 1.1.

Scott and Seymour [35] proved that the class of triangle-free graphs having no long induced even (or odd) cycles have bounded chromatic number, thus extending the result of Lagoutte [25] who claimed a proof for triangle-free graphs having no induced even cycles of length at least 6. It has also been shown that the class of graphs having no induced even cycle [1] is χ-bounded.

The following graph classes are also known to be χ-bounded:

- Bipartite graphs, distance-hereditary graphs, and parity graphs are perfect graphs and therefore χ-bounded [2, 8].
- Circle graphs are χ-bounded, shown by Kostochka and Kratochvíl [23].
- For each integer k, the class of graphs of rank-width at most k is χ-bounded, shown by Dvořák and Král' [15].

Vertex-minors and *pivot-minors* are graph containment relations introduced by Bouchet [3, 4, 5, 6] while conducting research of circle graphs (intersection graphs of chords in a cycle) and 4-regular Eulerian digraphs. Furthermore, these graph operations have been used for developing theory on rank-width [20, 26, 27, 28, 29]. We review these concepts in Section 2. Interestingly, the aforementioned graph classes can be characterized in terms of forbidden vertex-minors or pivot-minors.

- Bipartite graphs are graphs having no pivot-minor isomorphic to C_3.
- Parity graphs are graphs having no pivot-minor isomorphic to C_5 [1].
- Distance-hereditary graphs are graphs having no vertex-minor isomorphic to C_5, shown by Bouchet [3, 5].

[1]Parity graphs are known as graphs admitting a split decomposition whose bags are bipartite graphs or complete graphs [13], and it implies that parity graphs are closed under taking pivot-minors. One can easily verify that parity graphs are C_5-pivot-minor-free graphs using the fact that parity graphs are the graphs in which every odd cycle has two crossing chords [8].

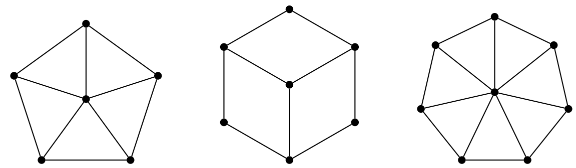

Figure 1: The three forbidden vertex-minors for circle graphs.

- Circle graphs are graphs having no vertex-minor isomorphic to the three graphs in Figure 1, shown by Bouchet [7]. Circle graphs are graphs having no pivot-minor isomorphic to the fifteen graphs, shown by Geelen and Oum [17].
- Graphs of rank-width at most k can be characterized by a finite list of forbidden pivot-minors, shown by Oum [26, 27].

In 2009, Geelen (see [15]) conjectured the following, which includes all aforementioned results regarding classes of graphs excluding certain vertex-minors.

Conjecture 1.2 (Geelen). *For every graph H, the class of graphs having no vertex-minor isomorphic to H is χ-bounded.*

Dvořák and Král' [15] showed that Conjecture 1.2 is true when $H = W_5$, where W_5 is the wheel graph on 6 vertices, depicted in Figure 1. Chudnovsky, Scott, and Seymour [11] showed that (ii) of Conjecture 1.1 holds and this implies that Conjecture 1.2 is true when H is a cycle.

In 1997, Scott [33] made a stronger conjecture claiming that for every graph H, the class of graphs having no subdivision of H as an induced subgraph is χ-bounded and proved the conjecture when H is a tree as follows. However, the conjecture of Scott turned out to be false, shown by Pawlik et al. [31].

Theorem 1.3 (Scott [33]). *For every tree H, the class of graphs having no induced subdivision of H is χ-bounded.*

Theorem 1.3 implies that Conjecture 1.2 is true when H is a vertex-minor of a tree. Kwon and Oum [24] showed that a graph is a vertex-minor of some tree if and only if it is a distance-hereditary graph, or equivalently, a graph of rank-width 1. Thus, Theorem 1.3 implies that Conjecture 1.2 is true if H is a distance-hereditary graph.

Our main theorem (Theorem 3.1) adds another infinite class of graphs for which Conjecture 1.2 is true. A *fan* F_k is a graph that consists of an induced path on k vertices and an additional vertex not on the path that is adjacent to all vertices of the path. We prove the following.

Theorem 3.1. *For each integer k, the class of graphs having no vertex-minor isomorphic to F_k is χ-bounded.*

We further ask whether the stronger statement for pivot-minors is also true. Conjecture 1.2 would be true if Conjecture 1.4 were to be true, because every pivot-minor of a graph is a vertex-minor.

Conjecture 1.4. *For every graph H, the class of graphs having no pivot-minor isomorphic to H is χ-bounded.*

Theorem 1.3 implies that if H is a subdivision of $K_{1,n}$, then Conjecture 1.4 is true. Thus, Conjecture 1.4 is true when H is a pivot-minor of a subdivision of $K_{1,n}$.

Scott and Seymour [34] proved that the class of graphs with no odd hole is χ-bounded, proving (i) of Conjecture 1.1. Thus, Conjecture 1.4 holds when $H = C_5$. Our second theorem provides another evidence to Conjecture 1.4 as follows.

Theorem 4.1. *For each integer $k \geq 3$, the class of graphs having no pivot-minor isomorphic to a cycle of length k is χ-bounded.*

Theorem 4.1 does not follow from the result of Chudnovsky, Scott, and Seymour [11] on long holes. The reason is that for every pair of integers k and ℓ with $k > \ell$ and $k - \ell \equiv 1 \pmod 2$, C_k has no pivot-minor isomorphic to C_ℓ [2] (but has a pivot-minor isomorphic to every shorter induced cycle with the same parity). We would like to mention that if (iii) of Conjecture 1.1 were to be true, then this would imply Conjecture 1.4 is true when H is an odd cycle.

The paper is organized as follows. In Section 2, we provide necessary definitions including vertex-minors, pivot-minors, and a leveling of a graph. Section 3 proves Theorem 3.1. We show that for a leveling of a graph, if a level contains a sufficiently long induced path, then the graph contains a large fan as a vertex-minor. We devote in Subsections 3.1 and 3.2 to show how to find a simple structure containing a fan vertex-minor from a leveling with a long induced path in a level. With the help of a result by Gyárfás [19] (Theorem 2.1) we show Theorem 3.1 in Subsection 3.3. Section 4 presents a proof of Theorem 4.1 by using a similar strategy. However, there is an issue of finding a pivot-minor isomorphic to a long induced cycle from a graph consisting of a long induced path with a vertex having many neighbors on it. In fact, this is not always true; for instance, a graph obtained from a fan by subdividing each edge on the path once is bipartite, and thus, it contains no odd cycles. We need a relevant result regarding the parity of a cycle, and we show in Subsection 4.1 that for every fixed k, there exists ℓ with $\ell \equiv k \pmod 2$ such that every graph consisting of an induced path P of length ℓ and a vertex v not on P where v is adjacent to the end vertices of P and may be adjacent to some other vertices contains a pivot-minor isomorphic to C_k. Based on this result, we show Theorem 4.1 in Subsection 4.2. We conclude the paper by further discussions in Section 5.

2 Preliminaries

For a graph G, let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. For $S \subseteq V(G)$, let $G[S]$ denote the subgraph of G induced on the vertex set S. For $v \in V(G)$ and $S \subseteq V(G)$, let $G \setminus v$ be the graph obtained from G by removing v, and let $G \setminus S$ be the graph obtained by removing all vertices in S. For $F \subseteq E(G)$, let $G \setminus F$ denote the graph obtained from G by removing all edges in F. For $v \in V(G)$, the set of *neighbors* of v in G is denoted by $N_G(v)$.

The *length* of a path is the number of edges on the path.

For two positive integers k and ℓ, let $R(k, \ell)$ be the *Ramsey number*, which is the minimum integer satisfying that every graph with at least $R(k, \ell)$ vertices contains either a clique of size k or an independent set of size ℓ. By Ramsey's Theorem [32], $R(k, \ell)$ exists for every pair of positive integers k and ℓ.

[2] This can be checked using the result of Bouchet [5] that if H is a pivot-minor of G and $v \in V(G) \setminus V(H)$, then H is a pivot-minor of one of $G \setminus v$ and $G \wedge vw \setminus v$ for a neighbor w of v. It implies that if C_ℓ is isomorphic to a pivot-minor of C_k and $k > \ell$, then C_ℓ is isomorphic to a pivot-minor of C_{k-2}.

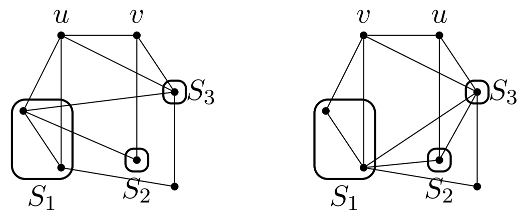

Figure 2: Pivoting an edge uv.

Vertex-minors and pivot-minors

Given a graph G and a vertex $v \in V(G)$, let $G * v$ denote the graph obtained from G by applying local complementation at v; the *local complementation* at v is an operation to replace the subgraph induced on $N_G(v)$ with its complement. A graph H is a *vertex-minor* of G if H can be obtained from G by applying a sequence of local complementations and vertex deletions.

The graph obtained from G by *pivoting* an edge $uv \in E(G)$ is defined by $G \wedge uv := G*u*v*u$. A graph H is a *pivot-minor* of G if H can be obtained from G by pivoting edges and deleting vertices. By the definition of pivoting edges, every pivot-minor of a graph G is also its vertex-minor.

For an edge uv of a graph G, let $S_1 := N_G(u) \setminus (N_G(v) \cup \{v\})$, $S_2 := N_G(v) \setminus (N_G(u) \cup \{u\})$, and $S_3 := N_G(v) \cap N_G(u)$. See Figure 2 for an example. It is easy to verify that $G \wedge uv$ is identical to the graph obtained from G by complementing the adjacency relations of vertices between distinct sets S_i and S_j, and swapping the labels of the vertices u and v. See [26, Proposition 2.1] for a formalized proof.

For a vertex v of G with exactly two neighbors v_1 and v_2, if v_1 and v_2 are non-adjacent, then the operation of replacing G with $G * v \setminus v$ is called *smoothing* a vertex v. Smoothing a vertex v is equivalent to removing v and adding the edge between the two neighbors of v.

Leveling in a graph

A sequence $L_0, L_1, \ldots, L_m$ of disjoint subsets of the vertex set of a graph G is called a *leveling* in G if

1. $|L_0| = 1$, and
2. for each $i \in \{1, \ldots, m\}$, every vertex in L_i has a neighbor in L_{i-1}, and has no neighbors in L_j for all $j \in \{0, \ldots, i-2\}$.

Each L_i is called a *level*. For $i \in \{1, \ldots, m\}$, a vertex $v \in L_{i-1}$ is called a *parent* of a vertex $w \in L_i$ if v and w are adjacent in G. For $u \in L_i$ and $v \in L_j$ where $0 \leq i \leq j \leq m$, u is called an *ancestor* of v if there is a path between u and v of length $j - i$ with one vertex in each of $L_i, L_{i+1}, \ldots, L_j$.

One natural way to obtain a leveling that covers all vertices in a graph is to fix a vertex v, and define L_i as the set of all vertices at distance i from v.

Our basic strategy to color a graph is to color each level of this leveling. If each level can be colored with N colors, then all levels can be colored with $2N$ colors, by using two disjoint sets of N colors for even levels and odd levels. So, we may assume that some level has sufficiently large chromatic number. The following theorem of Gyárfás [19] implies that we may assume that a level contains a sufficiently long induced path, and this gives a starting point of proving Theorems 3.1 and 4.1.

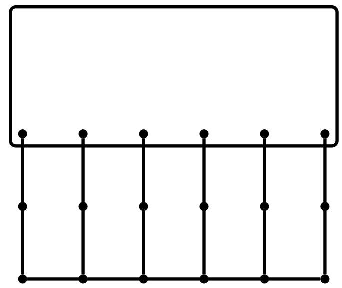

Figure 3: A graph obtained from E_6 and a connected graph by identifying 6 vertices.

Theorem 2.1 (Gyárfás [19]). *If $k \geq 2$ and a graph G has no induced path on k vertices, then $\chi(G) \leq (k-1)^{\omega(G)-1}$.*

3 Coloring graphs without F_k vertex-minors

We prove that every class of graphs excluding a fixed fan as a vertex-minor is χ-bounded.

Theorem 3.1. *For each integer k, the class of graphs having no vertex-minor isomorphic to F_k is χ-bounded.*

3.1 A structure containing a fan vertex-minor

To show Theorem 3.1, we essentially prove that for a fixed k and a graph G with a leveling, if a level contains a sufficiently long induced path, then G contains a vertex-minor isomorphic to F_k. In this subsection, we introduce an intermediate structure having a vertex-minor isomorphic to F_k.

We will use the following two theorems.

Theorem 3.2 (Erdős and Szekeres [16]). *Every sequence of n^2+1 integers contains an increasing or decreasing subsequence of length $n+1$.*

Theorem 3.3 (folklore; see Diestel [14]). *For $k \geq 1$ and $\ell \geq 3$, every connected graph on at least $k^{\ell-2}+1$ vertices contains a vertex of degree at least k or an induced path on ℓ vertices.*

For $k \geq 2$, let E_k be a graph on $3k$ vertices constructed in the following way: start with the disjoint union of k 2-edge paths $P_1, \ldots, P_k$ having $v_1, \ldots, v_k$ as an end vertex, respectively and then add $k-1$ edges that make the graph induced on $\{v_1, \ldots, v_k\}$ a path (of length $k-1$). Note that E_k is a tree with k vertices of degree 1, $k+2$ vertices of degree 2, and $k-2$ vertices of degree 3.

Proposition 3.4. *Let k be a positive integer and let $\ell \geq R(k,k)^{2(k-1)^2-1}+1$. Let H be a connected graph with at least ℓ vertices. Then the graph obtained from the disjoint union of H and E_ℓ by identifying ℓ distinct vertices of H with the leaves of E_ℓ contains a vertex-minor isomorphic to F_k.*

See Figure 3 for an illustration of a graph described in Proposition 3.4.

We first observe that for every connected graph H and a vertex v in H, either $H \setminus v$ or $H * v \setminus v$ is connected. This allows us to reduce H into a graph on exactly ℓ vertices.

Lemma 3.5. *Let H be a connected graph with at least 2 vertices. For each vertex v of H, either $H \setminus v$ or $H * v \setminus v$ is connected.*

Proof. If $H[N_H(v)]$ is connected, then $H \setminus v$ is connected trivially. Otherwise, $(H * v)[N_H(v)]$ is connected and therefore $H * v \setminus v$ is connected. □

This implies that in Proposition 3.4, if H contains a vertex v that will not be identified with a leaf of E_ℓ, then we can reduce H into one of $H \setminus v$ or $H * v \setminus v$, which is connected. In the end, we may assume that H is a connected graph on the vertex set $\{v_1, \dots, v_\ell\}$. We now aim to obtain a fan vertex-minor in either case, by using Theorem 3.3, which says that every sufficiently large connected graph contains a vertex of large degree or a long induced path,

The following lemma proves the case when H contains a long induced path. For a positive integer t, the *ladder* of order t is a graph G that consists of two vertex-disjoint paths $P = p_1p_2 \cdots p_t$, $Q = q_1q_2 \cdots q_t$ such that

- $V(G) = V(P) \cup V(Q)$, and
- for each $i, j \in \{1, \dots, t\}$, $p_iq_j \in E(G)$ if and only if $i = j$.

The 1-*subdivision* of a graph G is the graph obtained from G by replacing each edge by a 2-edge path.

Lemma 3.6. *The* 1*-subdivision of the ladder of order k contains a vertex-minor isomorphic to F_k.*

Proof. Let H be the ladder of order k with two vertex-disjoint paths $P = p_1p_2 \cdots p_k$ and $Q = q_1q_2 \cdots q_k$ such that for each $i, j \in \{1, \dots, k\}$, $p_iq_j \in E(G)$ if and only if $i = j$. Let G be the 1-subdivision of H, and let v_{xy} be the degree-2 vertex adjacent to x and y in G for each edge xy of H. We claim that for each $1 \le j \le k-1$, the vertex p_{j+1} is adjacent to $v_{p_iq_i}$ for all $1 \le i \le j+1$ in the graph

$$G \wedge p_1v_{p_1p_2} \wedge \cdots \wedge p_jv_{p_jp_{j+1}}.$$

It is easy to observe that this is true when $j = 1$. Suppose $j \ge 2$. By the induction hypothesis, p_j is adjacent to $v_{p_iq_i}$ for all $1 \le i \le j$ in the graph $G \wedge p_1v_{p_1p_2} \wedge \cdots \wedge p_{j-1}v_{p_{j-1}p_j}$. Note that $v_{p_jp_{j+1}}$ still has two neighbors p_j and p_{j+1} in the graph $G \wedge p_1v_{p_1p_2} \wedge \cdots \wedge p_{j-1}v_{p_{j-1}p_j}$ because it is adjacent to no vertex of $\{p_1, v_{p_1p_2}, \dots, p_{j-1}, v_{p_{j-1}p_j}\}$ in G and thus, it was not affected by the previous pivotings. By the definition of pivoting, p_{j+1} becomes adjacent to $v_{p_iq_i}$ for all $1 \le i \le j+1$ in

$$(G \wedge p_1v_{p_1p_2} \wedge \cdots \wedge p_{j-1}v_{p_{j-1}p_j}) \wedge p_jv_{p_jp_{j+1}}.$$

By the above claim, p_k is adjacent to $v_{p_iq_i}$ for all $1 \le i \le k$ in $G \wedge p_1v_{p_1p_2} \wedge \cdots \wedge p_{k-1}v_{p_{k-1}p_k}$. Note that there are no edges between the vertices of $\{v_{p_iq_i} : 1 \le i \le k\}$ as this graph is bipartite. Therefore, by removing all vertices in $\{p_1, v_{p_1p_2}, \dots, p_{k-1}, v_{p_{k-1}p_k}\}$ and smoothing all degree-2 vertices in the remaining graph, we obtain a vertex-minor isomorphic to F_k. □

Proof of Proposition 3.4. Let $w_1, \dots, w_\ell$ be the leaves of E_ℓ in the order following the main path. For all $i \in \{1, \dots, \ell\}$, let x_i be the neighbor of w_i in E_ℓ and let y_i be the neighbor of x_i other than w_i. Let $v_1, \dots, v_\ell$ be the vertices of H to be identified with $w_1, \dots, w_\ell$, respectively. Let G be the graph obtained from the disjoint union of H and E_ℓ by identifying v_i and w_i for each i.

Suppose there is a vertex v in H other than $v_1, \dots, v_\ell$. By Lemma 3.5, either $H \setminus v$ or $H * v \setminus v$ is connected. Since applying local complementation at v in G does not change adjacency with a vertex in $V(E_\ell) \setminus \{w_1, \dots, w_\ell\}$, we can reduce G to one of $G \setminus v$ or $G * v \setminus v$. By this observation, we may assume that H is a connected graph on the vertex set $\{v_1, \dots, v_\ell\}$.

Since $\ell \geq R(k,k)^{2(k-1)^2-1}+1$, by Theorem 3.3, H contains a vertex of degree at least $R(k,k)$, or an induced path on $2(k-1)^2+1$ vertices.

Case 1: H has an induced path $v_{i_1}v_{i_2}\dots v_{i_{2(k-1)^2+1}}$.

By Theorem 3.2, $i_1, i_2, \dots, i_{2(k-1)^2+1}$ contains an increasing or decreasing subsequence $j_1, j_2, \dots, j_k$, where all of $j_1, \dots, j_k$ have the same parity. We may assume $j_1 < j_2 < \dots < j_k$ by relabeling the indices if necessary and let $j_1 = i_p$ and $j_k = i_q$. Now, the graph induced on

$$\{w_z : z \in \{i_p, i_{p+1}, \dots, i_q\}\} \cup \{x_z : z \in \{j_1, j_2, \dots, j_k\}\} \cup \{y_z : z \in \{j_1, j_1+1, \dots, j_k\}\}$$

is a subdivision of a ladder of order k, where each edge of the ladder is subdivided at least once. We apply local complementations to degree-2 vertices to transform this graph into the 1-subdivision of the ladder of order k. By Lemma 3.6, it contains a vertex-minor isomorphic to F_k.

Case 2: H has a vertex v_s of degree at least $R(k,k)$.

Using Ramsey's Theorem on $N_H(v_s)$, we get either a clique of size k or an independent set of size k. If there is an independent set $\{v_{i_1}, \dots, v_{i_k}\}$ in $N_H(v_s)$ where $i_1 < i_2 < \dots < i_k$, then the graph induced on $\{v_s\} \cup \{v_{i_z}, x_{i_z} : z \in \{1, \dots, k\}\} \cup \{y_z : z \in \{i_1, i_1+1, \dots, i_k\}\}$ is a subdivision of F_k. Thus, it contains a vertex-minor isomorphic to F_k. If there is a clique $\{v_{i_1}, \dots, v_{i_k}\}$ in $N_H(v_s)$ where $i_1 < i_2 < \dots < i_k$, then first apply local complementation at v_s to change $\{v_{i_1}, \dots, v_{i_k}\}$ into an independent set. Similar to above, the graph induced on $\{v_s\} \cup \{v_{i_z}, x_{i_z} : z \in \{1, \dots, k\}\} \cup \{y_z : z \in \{i_1, i_1+1, \dots, i_k\}\}$ is a subdivision of F_k, which contains a vertex-minor isomorphic to F_k. □

Now, it is sufficient to find a vertex-minor isomorphic to a graph described in Proposition 3.4. In Subsection 3.2, we show how to extract an induced matching between two levels in a leveling where one contains a long induced path.

3.2 ℓ-patched paths

The following proposition will be used to extract an induced matching between two levels in a leveling where one level contains a long induced path.

Proposition 3.7. *Let $k \geq 3$ and $\ell \geq 1$ be integers. Let G be a graph on the disjoint union of vertex sets S and T such that $G[T]$ is an induced path and each vertex of T has a neighbor in S. If $|T| \geq (k-1)^{(k-1)^{2\ell+1}+1}$, then either S has a vertex having at least k neighbors in T, or there exist $S' \subseteq S$, $T' \subseteq T$ with $S' = \{s'_j : 1 \leq j \leq \ell\}$, $T' = \{q'_j : 1 \leq j \leq \ell\}$ and a graph G' on the vertex set $S' \cup T'$ such that*

- *$G'[S'] = G[S']$ and $G'[T']$ is an induced path $q'_1 q'_2 \cdots q'_\ell$,*
- *s'_i is adjacent to q'_j in G' if and only if $i = j$, and*
- *G' is obtained from G by applying a sequence of local complementations at vertices in T and removing vertices in $V(G) \setminus (S' \cup T')$.*

For $\ell \geq 1$, an *ℓ-patched path* is a graph G on two disjoint sets $S = \{s_1, s_2, \dots, s_\ell\}$ and $T = \{q_1, q_2, \dots, q_n\}$ satisfying the following.

- $G[T]$ is an induced path $q_1 q_2 \cdots q_n$, called its *underlying path*.

- There exists a sequence $b_1 < \ldots < b_2 < \cdots < b_\ell \le n$ such that for each $j \in \{1, 2, \ldots, \ell\}$, s_j is adjacent to q_{b_j} and non-adjacent to q_m for all $m > b_j$.

In particular, if s_j has no neighbors in $\{q_1, \ldots, q_{b_{j-1}}\}$ for all $j \in \{2, \ldots, k\}$, then we call it a *simple ℓ-patched path*.

We first find an ℓ-patched path with sufficiently large ℓ from the structure given in Proposition 3.7. In the next step, we will find a long simple patched path from a patched path.

Lemma 3.8. *Let $k \ge 3$ and $\ell \ge 1$ be integers. Let G be a graph on the disjoint union of vertex sets S and T such that $G[T]$ is an induced path and each vertex of T has a neighbor in S. If $|T| \ge 1 + (k-1) + (k-1)^2 + \cdots + (k-1)^\ell$, then either S has a vertex having at least k neighbors in T, or there exist $S' \subseteq S$ and $T' \subseteq T$ such that $G[S' \cup T']$ is an ℓ-patched path whose underlying path is $G[T']$.*

Proof. Suppose that every vertex of S has less than k neighbors in T. Let $q_1 q_2 \ldots q_{|T|}$ be the path induced by T. Assume that $|T| \ge 1 + (k-1) + (k-1)^2 + \cdots + (k-1)^\ell$.

Let $s_1 \in S$ be a neighbor of q_1. Since s_1 has at most $k-1$ neighbors on T, there exists b_1 such that q_{b_1} is adjacent to s_1 and q_{b_1+j} is non-adjacent to s_1 for all

$$1 \le j \le \left\lceil \frac{1 + (k-1) + (k-1)^2 + \cdots + (k-1)^\ell}{(k-1)} - 1 \right\rceil = 1 + (k-1) + (k-1)^2 + \cdots + (k-1)^{\ell-1}$$

and $b_1 \le (k-1)^\ell$.

Let i be the maximum i such that there exist distinct vertices $s_1, s_2, \ldots, s_i$ of S and a sequence $b_1 < b_2 < \cdots < b_i$ such that

- $b_1 \le (k-1)^\ell$, and $b_{m+1} - b_m \le (k-1)^{\ell-m}$ for all $1 \le m < i$,
- for all $1 \le m \le i$, s_m is adjacent to q_{b_m} but non-adjacent to q_{b_m+j} for all $1 \le j \le 1 + (k-1) + (k-1)^2 + \cdots + (k-1)^{\ell-m}$.

Such i exists, because $i = 1$ satisfies the conditions.

Suppose that $i < \ell$. Let $s_{i+1} \in S$ be a neighbor of q_{b_i+1}. For each $m \le i$, since $b_i + 1 - b_m \le (k-1)^{\ell-m} + (k-1)^{\ell-(m+1)} + \cdots + (k-1)^{\ell-(i-1)} + 1 \le 1 + (k-1) + (k-1)^2 + \cdots + (k-1)^{\ell-m}$, s_m is non-adjacent to q_{b_i+1} and therefore $s_m \ne s_{i+1}$.

Since s_{i+1} has at most $k-1$ neighbors in $\{q_{b_i+j} : 1 \le j \le 1 + (k-1) + (k-1)^2 + \cdots + (k-1)^{\ell-i}\}$, there exists b_{i+1} such that $b_i + 1 \le b_{i+1} \le b_i + (k-1)^{\ell-i}$ and s_{i+1} is adjacent to $q_{b_{i+1}}$ but non-adjacent to $b_{i+1} + j$ for all

$$1 \le j \le \left\lceil \frac{1 + (k-1) + \cdots + (k-1)^{\ell-i}}{k-1} - 1 \right\rceil = 1 + (k-1) + \cdots + (k-1)^{\ell-i-1}.$$

This contradicts our assumption that i was maximum.

Thus $i \ge \ell$. We take $S' = \{s_1, s_2, \ldots, s_\ell\}$ and $T' = \{q_1, q_2, \ldots, q_{b_\ell}\}$. For all $m < \ell$, since $b_\ell - b_m = (k-1)^{\ell-m} + (k-1)^{\ell-(m+1)} + \cdots + (k-1)^1 + 1$, s_m is non-adjacent to all q_i with $b_m < i \le b_\ell$. $\square$

Lemma 3.9. *Let $k \ge 3$ and $\ell \ge 1$ be integers. If G is a graph on the disjoint union of vertex sets S and T such that G is a $(1 + (k-1) + (k-1)^2 + \cdots + (k-1)^{\ell-1})$-patched path whose underlying path is $G[T]$, then either S has a vertex having at least k neighbors in T, or there exist $S' \subseteq S$, $T' \subseteq T$ such that $G[S' \cup T']$ is a simple ℓ-patched path whose underlying path is $G[T']$.*

Proof. Suppose that every vertex of S has at most $k-1$ neighbors in T. Suppose that $S = \{s_1, s_2, \ldots, s_{|S|}\}$ and $G[T]$ is an underlying induced path $q_1 q_2 \cdots q_m$. Furthermore let us assume that there exists a sequence $b_1 < b_2 < \cdots < b_{(k-1)^\ell} \leq m$ such that for all i, s_i is adjacent to q_{b_i} but non-adjacent to q_j for all $j > b_i$.

We prove a stronger claim that T' can be chosen so that $T' = \{q_i, q_{i+1}, q_{i+2}, \ldots, q_m\}$ for some i. We proceed by induction on ℓ. The statement is trivial if $\ell = 1$ and so we may assume $\ell > 1$.

We say that a vertex q_j of T is *paired* with s_i if $b_i = j$. There are $|S| = 1 + (k-1) + (k-1)^2 + \cdots + (k-1)^{\ell-1}$ paired vertices in T. We say that a paired vertex q_j is an *s-friend* of q_t for $s \in S$ if $j < t$ and $q_j, q_{j+1}, \ldots, q_{t-1}$ are non-neighbors of s and q_t is a neighbor of s.

Let $s' = s_{|S|}$. Since s' has at most $k-1$ neighbors in T, there exists b' such that s' is adjacent to $q_{b'}$ and the number of s'-friends of $q_{b'}$ is at least

$$\left\lceil \frac{(1 + (k-1) + (k-1)^2 + \cdots + (k-1)^{\ell-1}) - (k-1)}{k-1} \right\rceil = 1 + (k-1) + \cdots + (k-1)^{\ell-2}.$$

Let S_1 be a set of all $s_i \in S$ such that q_{b_i} is an s'-friend of $q_{b'}$ and $|S_1| = 1 + (k-1) + \cdots + (k-1)^{\ell-2}$. Let i be the minimum such that q_i is paired with some $s \in S_1$. Let $T_1 = \{q_i, q_{i+1}, \ldots, q_{b'-1}\}$. Then $G[S_1 \cup T_1]$ is a $(1 + (k-1) + \cdots + (k-1)^{\ell-2})$-patched path and therefore by the induction hypothesis, there exist $S_1' \subseteq S_1$, $T_1' \subseteq T_1$ such that $G[S_1' \cup T_1']$ is a simple $(\ell-1)$-patched path whose underlying path is $G[T_1']$ and furthermore $T_1' = \{q_p, q_{p+1}, \ldots, q_{b'-1}\}$ for some p.

By the definition of an s'-friend, no vertex in T_1 is adjacent to s'. Let $S' = S_1' \cup \{s'\}$ and $T' = T_1 \cup \{q_{b'}, q_{b'+1}, \ldots, q_m\}$. Then $G[S' \cup T']$ is a simple ℓ-patched path whose underlying path is $G[T']$. □

Lemma 3.10. *Let ℓ be a positive integer. If G is a graph on the disjoint union of vertex sets S and T such that G is a simple 2ℓ-patched path whose underlying path is $G[T]$, then there exist $S' \subseteq S$, $T' \subseteq T$ with $S' = \{s_j' : 1 \leq j \leq \ell\}$, $T' = \{q_j' : 1 \leq j \leq \ell\}$ and a graph G' on the vertex set $S' \cup T'$ such that*

- *$G'[S'] = G[S']$ and $G'[T']$ is an induced path $q_1' q_2' \cdots q_\ell'$,*
- *s_i' is adjacent to q_j' in G' if and only if $i = j$, and*
- *G' is obtained from G by applying a sequence of local complementations at vertices in T and removing vertices in $V(G) \setminus (S' \cup T')$.*

Proof. Suppose that $S = \{s_1, s_2, \ldots, s_{2\ell}\}$ and $G[T]$ is an underlying induced path $q_1 q_2 \cdots q_m$. Furthermore let us assume that there exists a sequence $0 = b_0 < b_1 < b_2 < \cdots < b_{2\ell} \leq m$ such that for all i, s_i is adjacent to q_{b_i} but non-adjacent to q_j for all $j > b_i$ and all $j \leq b_{i-1}$. We proceed by induction on $|V(T)|$. The statement is trivial if $|V(T)| = 2\ell$. We assume that $|V(T)| > 2\ell$.

If T contains a vertex of degree 2 in G, then we smooth it. Since the resulting graph is still a simple 2ℓ-patched path, we are done by induction hypothesis.

If s_i is adjacent to 4 consecutive neighbors $q_{x+1}, q_{x+2}, q_{x+3}, q_{x+4}$, then we apply local complementation at q_{x+2} and remove it. This operation removes the edges $s_i q_{x+1}$ and $s_i q_{x+3}$. Since s_i has at least one neighbor q_{x+4}, the resulting graph is a simple 2ℓ-patched path, and it contains the required structure by induction hypothesis.

By these two reductions, we may assume that each vertex in T has a neighbor in S, and each vertex in S has at most 3 neighbors in T.

Now, we take a subset $S' = \{s_2, s_4, \ldots, s_{2\ell}\}$ of S, and let $G' := G[T \cup S']$. For each $1 \le i \le \ell$, we shrink the path $q_{b_{2(i-1)}+1} q_{b_{2(i-1)}+2} \cdots q_{b_{2i}}$ into some vertex q'_i such that q'_i is adjacent to s_{2i}.

If $|N_G(s_{2i}) \cap T| = 1$, then let $q'_i := q_{b_{2i}}$. If $|N_G(s_{2i}) \cap T| = 2$, then we apply local complementation at $q_{b_{2i}-1}$ and remove it. Then $s_{2i}q_{b_{2i}}$ is removed and $s_{2i}q_{b_{2i}-2}$ is added. We assign $q'_i := q_{b_{2i}-2}$. In case when $|N_G(s_{2i}) \cap T| = 3$, we pivot $q_{b_{2i}-2}q_{b_{2i}-1}$ and remove both end vertices. Then $s_{2i}q_{b_{2i}}$ is removed and $s_{2i}q_{b_{2i}-3}$ is added. We assign $q'_i := q_{b_{2i}-3}$. We can observe that in each case, s_{2i} has exactly one neighbor on the remaining path from $q_{b_{2(i-1)}+1}$ to $q_{b_{2i}}$. Finally, we smooth all vertices of $q_{b_{2(i-1)}+1}, \ldots, q_{b_{2i}}$ except q'_i in the remaining path. Then we obtain an induced path $q'_1 q'_2 \cdots q'_\ell$ such that s_{2i} is adjacent to q'_j if and only if $i = j$. □

Proof of Proposition 3.7. Suppose that every vertex of S has at most $k-1$ neighbors in T. Since $|T| \ge (k-1)^{(k-1)^{2\ell+1}+1}$ and $k \ge 3$, by Lemma 3.8, there exist $S_1 \subseteq S$ and $T_1 \subseteq T$ such that $G[S_1 \cup T_1]$ is an $(k-1)^{2\ell+1}$-patched path whose underlying path is $G[T_1]$. Then, by Lemma 3.9, there exist $S_2 \subseteq S_1$, $T_2 \subseteq T_1$ such that $G[S_2 \cup T_2]$ is a simple 2ℓ-patched path whose underlying path is $G[T_2]$. Lastly, by Lemma 3.10, there exist $S_3 \subseteq S_2$, $T_3 \subseteq T_2$ with $S_3 = \{s'_j : 1 \le j \le \ell\}$, $T_3 = \{q'_j : 1 \le j \le \ell\}$ and a graph G' on the vertex set $S_3 \cup T_3$ such that

- $G'[S_3] = G[S_3]$ and $G'[T_3]$ is an induced path $q'_1 q'_2 \cdots q'_\ell$,
- s'_i is adjacent to q'_j in G' if and only if $i = j$, and
- G' is obtained from G by applying a sequence of local complementations at vertices in $T_2 \subseteq T$ and removing vertices in $V(G) \setminus (S_3 \cup T_3)$. □

3.3 Proof of Theorem 3.1

Proof of Theorem 3.1. Let q and k be positive integers. If $k = 1$, then it is trivial. Since F_2 is isomorphic to C_3, graphs having no vertex-minor isomorphic to F_2 are exactly forests, and we can color such graphs with 2 colors. Therefore, we may assume that $k \ge 3$. Let $\ell := R(k,k)^{2(k-1)^2-1}+1$ and $m := (k-1)^{(k-1)^{2R(k+1,k\ell)+1}+1}$. Let G be a graph with maximum clique size q such that it has no vertex-minor isomorphic to F_k. We claim that G can be colored with $2(m-1)^{q-1}$ colors.

We may assume that G is connected as we can color each connected component separately. Let v be a vertex of G and for $i \ge 0$, let L_i be the set of all vertices of G whose distance to v is i in G. If each L_j is $(m-1)^{q-1}$-colorable, then G is $2(m-1)^{q-1}$-colorable. By Theorem 2.1, we may assume that there exists a level L_n containing an induced path P on m vertices.

By Proposition 3.4, it is sufficient to find a vertex-minor that is isomorphic to a graph obtained from the disjoint union of E_ℓ with the leaves $w_1, \ldots, w_\ell$ and a connected graph H on at least ℓ vertices with pairwise distinct vertices $v_1, \ldots, v_\ell$, by identifying v_i and w_i for all $1 \le i \le \ell$. We construct this graph based on the path P and the leveling $L_0, \ldots, L_n$.

Since $L_0, \ldots, L_n$ is a leveling, each vertex in P has a neighbor in L_{n-1}. If $n = 1$, then we directly obtain a vertex-minor isomorphic to F_k. We may assume that $n \ge 2$. Since $m = (k-1)^{(k-1)^{2R(k+1,k\ell)+1}+1}$, by Proposition 3.7, there exist $S = \{s_j : 1 \le j \le R(k+1,k\ell)\} \subseteq L_{n-1}$, $T = \{q_j : 1 \le j \le R(k+1,k\ell)\} \subseteq V(P)$, and a graph G' on the vertex set $L_0 \cup \cdots \cup L_{n-2} \cup S \cup T$ such that

- G and G' are identical on the vertex set $L_0 \cup \cdots \cup L_{n-2} \cup S$,
- $G'[T]$ is an induced path $q_1 q_2 \cdots q_{R(k+1,k\ell)}$,

- s_i is adjacent to q_j in G' if and only if $i = j$, and
- G' is obtained from G by applying a sequence of local complementations at vertices in P and removing vertices in $V(G) \setminus V(G')$.

Since $|S| = R(k+1, k\ell)$, by Ramsey's Theorem, $G'[S]$ contains a clique of size $k+1$ or an independent set of size $k\ell$. If $G'[S]$ has a clique C of size $k+1$, then for a vertex $s_i \in C$ with minimum i, $G' * v$ contains an induced subgraph isomorphic to a subdivision of F_k and so G has a vertex-minor isomorphic to F_k. Thus we may assume that $G'[S]$ contains an independent set S' of size $k\ell$.

Now, if there is a vertex in L_{n-2} that has k neighbors on S' in G', then G' contains an induced subgraph isomorphic to a subdivision of F_k. Thus, we may assume that each vertex in L_{n-2} has at most $k-1$ neighbors on S' in G'. It implies that $n \geq 3$. Since each vertex of S' has a neighbor in L_{n-2} and $k\ell \geq (k-1)\ell + 1$, there exist $\{w_1, \ldots, w_\ell\} \subseteq L_{n-2}$ and $\{x_1, \ldots, x_\ell\} \subseteq S'$ where w_i is adjacent to x_j in G' if and only if $i = j$. For each $1 \leq i \leq \ell$, let y_i be the neighbor of x_i contained in T.

Let G'' be the graph obtained from

$$G'[L_0 \cup \cdots \cup L_{n-3} \cup \{w_z, x_z : z \in \{1, \ldots, \ell\}\} \cup T]$$

by repeatedly removing degree-1 vertices and smoothing degree-2 vertices in T other than $y_1, \ldots, y_\ell$. In the resulting graph, the vertices $y_1, \ldots, y_\ell$ remain among vertices of T. Note that $G'[L_0 \cup \cdots \cup L_{n-3} \cup \{w_z : z \in \{1, \ldots, \ell\}\}]$ is connected because there is a path from each vertex to the vertex in L_0. Also, the graph obtained from $G''[\{w_z, x_z, y_z : z \in \{1, \ldots, \ell\}\}]$ by removing edges in $G''[\{w_z : z \in \{1, \ldots, \ell\}]$ is isomorphic to E_ℓ. Therefore, by Proposition 3.4, it contains a vertex-minor isomorphic to F_k. □

4 Coloring graphs without C_k pivot-minors

In this section, we prove the second main result.

Theorem 4.1. *For each integer $k \geq 3$, the class of graphs having no pivot-minor isomorphic to a cycle of length k is χ-bounded.*

4.1 Obtaining C_k pivot-minor from a large incomplete fan

We show that for every fixed k, there exists ℓ with the same parity as k such that every graph consisting of an induced path P of length ℓ and a vertex v not on P where v is adjacent to the end vertices of P contains a pivot-minor isomorphic to C_k. This will support Theorem 4.1.

Proposition 4.2. *Let $k \geq 3$ be an integer and $n \geq 6k^3 - 26k^2 + 25k - 2$ such that $k \equiv n \pmod 2$. If G is a graph with a vertex v such that $G \setminus v$ is an induced path P of length n and v is adjacent to the end vertices of P, then G contains a pivot-minor isomorphic to C_k.*

We remark that the parity condition in Proposition 4.2 cannot be removed as C_n has no pivot-minor isomorphic to C_k if $n \not\equiv k \pmod 2$.

To prove Proposition 4.2, we prove some useful lemmas.

Lemma 4.3. *Every induced cycle of length $k+2$ contains an induced cycle of length k as a pivot-minor.*

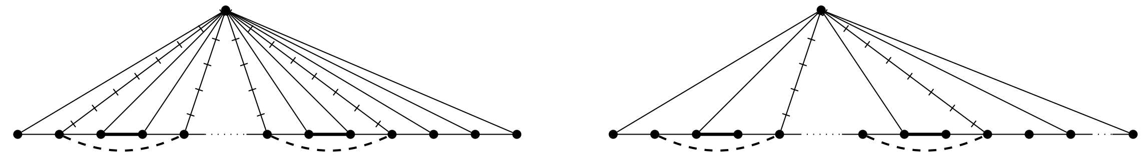

Figure 4: Configurations in (2) and (3) of Lemma 4.4.

Proof. By pivoting an edge xy on an induced cycle and deleting x, y from the resulting graph, we obtain an induced cycle that is of length 2 shorter than the initial one. $\square$

Lemma 4.4. *Let G be a graph with a vertex v such that $G \setminus v$ is an induced path $P := p_0p_1 \cdots p_n$. Let $i_1 = 0 < i_2 < i_3 < \cdots < i_t = n$ be a sequence of integers such that $p_{i_1}, \ldots, p_{i_t}$ are all neighbors of v on P. Then the following hold.*

(1) If $k := i_2 - i_1 > 1$ and $i_2 \equiv i_3 \equiv \cdots \equiv i_{t-1} \not\equiv i_t \pmod 2$, then G contains a pivot-minor isomorphic to C_{k+1}.

(2) For a positive integer k, if $t \geq 4k$ and $i_j = j - 1$ for all $j \in \{1, 2, \ldots, t\}$, then G contains a pivot-minor isomorphic to C_{2k+1} and a pivot-minor isomorphic to C_{2k+2}.

(3) For a positive integer k, if $t \geq 2k + 1$ and $i_j = 2(j - 1)$ for all $j \in \{1, \ldots, t - 1\}$, then G contains a pivot-minor isomorphic to C_{2k+2}. Moreover, if $i_t - i_{t-1}$ is odd, then G contains a pivot-minor isomorphic to C_{2k+1}.

Proof. (1) We proceed by induction on $i_t - i_2$.

If $i_t - i_2 = 1$, then we can create the edge vp_{i_2-1} by pivoting the edge $p_{i_2}p_{i_t}$. Since p_{i_2}, p_{i_t} have no neighbors in $\{p_{i_1}, \ldots, p_{i_2-2}\}$, $vp_0p_1 \cdots p_{i_2-1}v$ is an induced cycle of length $k + 1$ in $G \wedge p_{i_2}p_{i_t}$.

If $i_t - i_2 \geq 3$, then we can create the edge vp_{i_t-2} by pivoting $p_{i_{t-1}}p_{i_t}$. Then the (new) neighborhood of v on the path from p_{i_1} to p_{i_t-2} satisfies the condition of our assumption as the new edge vp_{i_t-2} divides either an even interval into two odd intervals or an odd interval into an odd interval and an interval of length 2. Thus, by the induction hypothesis, $G \wedge p_{i_{t-1}}p_{i_t}$ contains a pivot-minor isomorphic to C_{k+1} and so does G.

(2) For $j \in \{1, \ldots, t - 3\}$, if we pivot p_jp_{j+1}, then the edges vp_{j-1}, vp_{j+2} are removed and $p_{j-1}p_{j+2}$ is added. If $k \geq 2$, then by pivoting $p_{4j-2}p_{4j-1}$ and removing the vertices p_{4j-2} and p_{4j-1} for all $j \in \{1, \ldots, k - 1\}$, we can obtain an induced cycle

$$vp_0p_1p_4p_5 \cdots p_{4k-4}p_{4k-3}v$$

of length $2k + 1$. If $k = 1$, then vp_0p_1 is an induced cycle of length $3 = 2k + 1$. Now, by pivoting $p_{4k-2}p_{4k-1}$, we can remove the edge vp_{4k-3} and thus, we obtain an induced cycle of length $2k + 2$, which is $vp_0p_1 \cdots p_{4k-4}p_{4k-3}p_{4k-2}v$.

(3) For $j \in \{1, \ldots, t - 3\}$, if we pivot $p_{2j}p_{2j+1}$, then the edge vp_{2j+2} is removed and $p_{2j-1}p_{2j+2}$ is added. Therefore, pivoting $p_2p_3, p_6p_7, p_{10}p_{11}, \ldots, p_{4k-6}p_{4k-5}$ and removing the vertices p_2, p_3, p_6, p_7, p_{10}, p_{11}, $\ldots$, p_{4k-6}, p_{4k-5} creates an induced cycle

$$vp_0p_1p_4p_5 \cdots p_{4k-6}p_{4k-5}p_{4k-4}v$$

of length $2k+2$. If $i_t - i_{t-1}$ is odd, then the last odd interval is still an odd interval after pivotings, and by (1), it also contains a pivot-minor isomorphic to a cycle of length $2k+1$. $\square$

For positive integers k, ℓ, a *(k,ℓ)-fan* is a graph F with a specified vertex p, called the *central vertex*, such that

- $F \setminus p$ is a path $p_0p_1\cdots p_n$, and let $i_1 = 0 < i_2 < i_3 < \cdots < i_t = n$ be a sequence of integers such that $p_{i_1}, \ldots, p_{i_t}$ are all neighbors of v on P,
- $i_{j+1} - i_j$ is odd for $j \in \{1, \ldots, k\}$,
- $|j \in \{1, \ldots, t-1\} : i_{j+1} - i_j \text{ is odd}\}| \geq \ell$.

Lemma 4.5. *Every (k,ℓ)-fan contains a pivot-minor isomorphic to $F_{k+\lfloor(\ell-k)/3\rfloor}$.*

Proof. Let $m = k + \lfloor(\ell-k)/3\rfloor$. Let G be the (k,ℓ)-fan with the central vertex v such that $G \setminus v$ is an induced path $P := p_0p_1\cdots p_n$ and let $i_1 = 0 < i_2 < i_3 < \cdots < i_t = n$ be a sequence of integers such that $p_{i_1}, \ldots, p_{i_t}$ are all neighbors of v on P.

We proceed by induction on $|V(G)| - k$. If there exists j such that both p_j and p_{j+1} are non-adjacent to v, then $G \wedge p_jp_{j+1} \setminus p_jp_{j+1}$ is a (k,ℓ)-fan, thus having a pivot-minor isomorphic to F_m by the induction hypothesis. Thus we may assume that $i_{j+1} - i_j \in \{1,2\}$ for all $j \in \{1, 2, \ldots, t-1\}$. If $\ell - k < 3$, then G contains an induced subgraph isomorphic to F_m. Thus we may assume that $\ell - k \geq 3$.

If $i_{k+2} - i_{k+1}$ is odd, then G is a $(k+1,\ell)$-fan and therefore by the induction hypothesis, F_m is isomorphic to a pivot-minor of G. Thus we may assume that $i_{k+2} - i_{k+1} = 2$ and therefore $i_j = j - 1$ for all $j \in \{1, 2, \ldots, k+1\}$ and $i_{k+2} = k+2$.

If p_{k+3} is non-adjacent to v, then p_{k+4} is adjacent to v and $G \wedge p_{k+2}p_{k+3} \setminus p_{k+2}p_{k+3}$ is a $(k+1,\ell)$-fan, proving this lemma by the induction hypothesis. Thus we may assume that p_{k+3} is adjacent to v and $i_{k+3} = k+3$.

If p_{k+4} is non-adjacent to v, then $G \wedge p_{k+2}p_{k+3} \setminus p_{k+2}p_{k+3}$ is a $(k+1,\ell)$-fan. Thus, we may assume that p_{k+4} is adjacent to v and $i_{k+4} = k+4$.

Now, $G \wedge p_{k+2}p_{k+3} \setminus p_{k+2} \setminus p_{k+3}$ is a $(k+1, \ell-3)$-fan, thus having a pivot-minor isomorphic to F_m by the induction hypothesis. $\square$

Now we are ready to prove Proposition 4.2.

Proof of Proposition 4.2. Let $P := p_0p_1\cdots p_n$ and let $i_1 = 0 < i_2 < i_3 < \cdots < i_t = n$ be a sequence of integers such that $p_{i_1}, \ldots, p_{i_t}$ are all neighbors of v on P.

If $i_{j+1} - i_j \geq k-2$ and $i_{j+1} - i_j \equiv k \pmod 2$ for some j, then G has a pivot-minor isomorphic to C_k by Lemma 4.3.

If $i_{j+1} - i_j \geq k-2$ and $i_{j+1} - i_j \not\equiv k \pmod 2$ for some j, then there exists m such that $i_{m+1} - i_m$ is odd, because $n \equiv k \pmod 2$. By symmetry, we may assume that $m > j$. We may assume that m is chosen to be minimum. Then, $i_{j+1} \equiv i_{j+2} \equiv \cdots \equiv i_m \not\equiv i_{m+1} \pmod 2$ and therefore G contains a pivot-minor isomorphic to C_k by (1) of Lemma 4.4. Thus we may assume that $i_{j+1} - i_j \leq k-3$ for all j and therefore $n \leq (k-3)(t-1)$.

If there exist at least $6k-2$ values of j such that $i_{j+1} - i_j$ is odd, then G has a $(1, 6k-2)$-fan as an induced subgraph and therefore by Lemma 4.5, G has a pivot-minor isomorphic to F_{2k}. By (2) of Lemma 4.4, if k is even, then F_{2k} contains a pivot-minor isomorphic to C_{k+2}. If k is odd, then

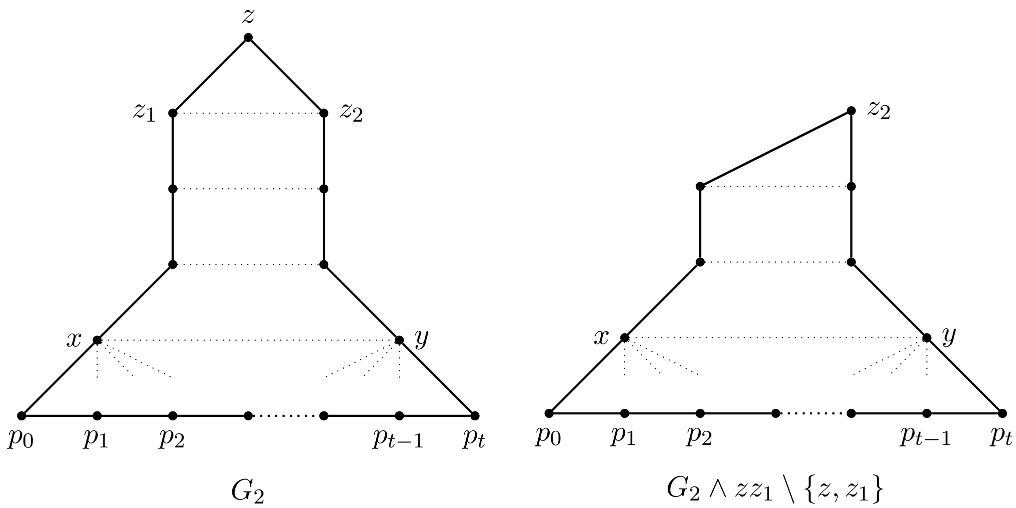

Figure 5: Reducing the length of the path $x - P_1 - z - P_2 - y$ in Theorem 4.1.

$F_{2(k-1)}$ contains a pivot-minor isomorphic to C_k by (2) of Lemma 4.4. Therefore we may assume that there are at most $6k-3$ values of j such that $i_{j+1} \not\equiv i_j \pmod 2$.

Suppose that $i_j \equiv i_{j+1} \equiv i_{j+2} \equiv \cdots \equiv i_{j+k-1} \pmod 2$ for some $j \leq t-k+1$. If k is even, then by (3) of Lemma 4.4, G has a pivot-minor isomorphic to C_k. If k is odd, then there exists m such that $i_{m+1} - i_m$ is odd. By (3) of Lemma 4.4, G has a pivot-minor isomorphic to C_k. Thus we may assume that at least one of $i_{j+1}-i_j, i_{j+2}-i_{j+1}, \ldots, i_{j+k-1}-i_j$ is odd for all $j \leq t-k+1$. We conclude that $t \leq (k-1)(6k-2)$ and therefore $n \leq (k-3)((k-1)(6k-2)-1) = 6k^3 - 26k^2 + 25k - 3$. □

4.2 Proof of Theorem 4.1

Proof of Theorem 4.1. Let q and k be positive integers with $k \geq 3$. If $k = 3$, then graphs having no pivot-minor isomorphic to C_3 are bipartite graphs, and we can color such graphs with 2 colors. We may assume that $k \geq 4$. Let $\ell := 6k^3 - 26k^2 + 25k - 2$. Let G be a graph such that it has no pivot-minor isomorphic to C_k. We claim that $\chi(G) \leq 2(\ell+1)^{q-1}$ if $\omega(G) \leq q$.

We may assume that G is connected as we can color each connected component separately. Let v be a vertex of G and for $i \geq 0$, let L_i be the set of all vertices of G that are at distance i away from v. If each L_j is $(\ell+1)^{q-1}$-colorable, then G is $2(\ell+1)^{q-1}$-colorable. By Theorem 2.1, we may assume that there exists a level L_n containing an induced path of length $t \in \{\ell, \ell+1\}$ where t and k have the same parity. Let $P := p_0p_1p_2\cdots p_t$. If $n = 1$, then by Proposition 4.2, $G[V(P) \cup \{v\}]$ contains a pivot-minor isomorphic to C_k. We may assume that $n \geq 2$.

Let x be a parent of p_0. If x is adjacent to p_t, then by Proposition 4.2, G contains a pivot-minor isomorphic to C_k. We may assume that x is not adjacent to p_t. Let y be a parent of p_t. By the same reason, we can assume that y is not adjacent to p_0. We choose a first common ancestor of x and y in the leveling $L_0, \ldots, L_{n-1}$, and call it z. Such a vertex z exists because v is a common ancestor of x and y. Let P_1 be the path from x to z in G_1 with exactly one vertex in each level, and similarly, let P_2 be the path from y to z in G_1 with exactly one vertex in each level. Since P_1 and P_2 have the same length, the path $x - P_1 - z - P_2 - y$ has even length. Note that the path $x - P_1 - z - P_2 - y$ is not necessary an induced path in G_1 as there may be an edge between two vertices on the same level. See Figure 5.

We claim that $G[V(P) \cup V(P_1) \cup V(P_2)]$ contains a pivot-minor isomorphic to C_k. Let $G_1 := G[V(P) \cup V(P_1) \cup V(P_2)]$. Note that by construction, all internal vertices of the path $x - P_1 - z - P_2 - y$ have no neighbors on the path P. If there are at least two internal vertices in $x - P_1 - z - P_2 - y$, then let z_1 and z_2 be the neighbors of z on P_1 and P_2, respectively. We pivot zz_1 and remove z and z_1 from G_1. Then z_2 becomes adjacent to the neighbor of z_1 on P_1 other than z. This operation reduces the length of the path $x - P_1 - z - P_2 - y$ by 2. Thus, we can do this until the remaining path has length exactly 2. From this operation, we may assume that the path $x - P_1 - z - P_2 - y$ has length exactly 2, which is xzy.

Now, we pivot xz in G_2. Note that

- p_0 is adjacent to x but not adjacent to z, and
- y is either a common neighbor of x and z, or adjacent to z but not to x.

From these two facts, p_0y becomes an edge after pivoting xz. Since all vertices on P are not adjacent to z, $V(P)$ still induces the same path after pivoting xz. So, y is adjacent to p_0 and p_t in $G_2 \wedge xz$, and by Proposition 4.2, $G_2 \wedge xz$ contains a pivot-minor isomorphic to C_k. □

5 Further discussions

Let us conclude our paper by summarizing known cases for Conjectures 1.2 and 1.4. As far as we know, the class of graphs having no H vertex-minor is χ-bounded if

- H is a distance-hereditary graph (due to Theorem 1.3),
- H is a vertex-minor of a fan graph (Theorem 3.1),
- $H = W_5$ (due to Dvořák and Král' [15]),

and the class of graphs having no H pivot-minor is χ-bounded if

- H is a pivot-minor of a cycle graph (Theorem 4.1),
- H is a pivot-minor of a 1-subdivision of a tree, which we can deduce easily from Theorem 1.3,
- H is a pivot-minor of a tree satisfying Gyárfás-Sumner conjecture, which we describe below.

Gyárfás [18] and Sumner [36] independently conjectured that for a fixed tree T, the class of graphs having no induced subgraph isomorphic to T is χ-bounded. So far this conjecture is known to be true for the following cases:

- T is a subdivision of a star (due to Scott [33]),
- T is a tree of radius 2 (due to Kierstead and Penrice [21]),
- T is a tree of radius 3 obtained from a tree of radius 2 by making exactly one subdivision in every edge adjacent to the root (due to Kierstead and Zhu [22]).

Note that a cycle is a vertex-minor of a large fan graph. Thus, Conjecture 1.2 holds when H is a cycle graph, by two reasons, one by Theroem 3.1 and another by the proof of (ii) of Conjecture 1.1 by Scott and Seymour [34].

One may wish to have a structure theorem describing graphs with no fixed vertex-minors or no fixed pivot-minors in order to extend these theorems to other forbidden graphs. Indeed, Oum [28] conjectured the following. A graph is a *circle graph* if it is an intersection graph of chords in a circle. Rank-width is a width parameter of graphs introduced by Oum and Seymour [30].

Conjecture 5.1. *Let H be a bipartite circle graph. Every graph with sufficiently large rank-width contains a pivot-minor isomorphic to H.*

This conjecture, if true, implies χ-boundedness by the following theorem of Dvořák and Král' [15].

Theorem 5.2 (Dvořák and Král' [15])**.** *For each integer k, the class of graphs of rank-width at most k is χ-bounded.*

Let F_n' be a graph obtained from F_n by subdividing each edge on the induced path precisely once. It can be easily seen that F_n' is a bipartite circle graph and F_n is a vertex-minor of F_n'. Thus if Conjecture 5.1 holds, then the class of graphs with no F_n vertex-minor has bounded rank-width and therefore by Theorem 5.2, it will be χ-bounded, implying Theorem 3.1. Similarly we can also see easily that Conjecture 5.1 implies Theorem 4.1. However, we do not know yet whether Conjecture 5.1 holds when $H = F_n'$ or H is an even cycle.

Furthermore it would be interesting to see whether Conjectures 1.2 and 1.4 hold when H is a wheel graph on at least 6 vertices, since such a graph H is not a circle graph and therefore Conjectures 1.2 and 1.4 are independent of Conjecture 5.1.

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