

Partitioning H -minor free graphs into three subgraphs with no large components

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July 27, 2016

Abstract

We prove that for every graph H , if a graph G has no (odd) H minor, then its vertex set $V(G)$ can be partitioned into three sets X_1, X_2, X_3 such that for each i , the subgraph induced on X_i has no component of size larger than a function of H and the maximum degree of G . This improves a previous result of Alon, Ding, Oporowski and Vertigan (2003) stating that $V(G)$ can be partitioned into four such sets if G has no H minor. Our theorem generalizes a result of Esperet and Joret (2014), who proved it for graphs embeddable on a fixed surface and asked whether it is true for graphs with no H minor.

As a corollary, we prove that for every positive integer t , if a graph G has no K_{t+1} minor, then its vertex set $V(G)$ can be partitioned into $3t$ sets X_1, \dots, X_{3t} such that for each i , the subgraph induced on X_i has no component of size larger than a function of t . This corollary improves a result of Wood (2010), which states that $V(G)$ can be partitioned into $\lceil 3.5t + 2 \rceil$ such sets.

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‡Supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT & Future Planning (2011-0011653).

1 Introduction

The famous Four Color Theorem states that every planar graph G admits a partition of its vertex set into four sets X_1, X_2, X_3, X_4 such that for $1 \leq i \leq 4$, every component of the subgraph $G[X_i]$ induced on X_i has at most one vertex. Certainly there are planar graphs whose vertex set cannot be partitioned into three such sets. However, Esperet and Joret [9] proved that the number of sets can be reduced to three, if we relax each X_i to induce a subgraph having no component of size larger than a function of the maximum degree of G .

Theorem 1.1 (Esperet and Joret [9]). *Let Σ be a surface of Euler genus g . If a graph G is embeddable on Σ and has maximum degree at most Δ , then $V(G)$ can be partitioned into three sets X_1, X_2, X_3 such that for $1 \leq i \leq 3$, every component of $G[X_i]$ has at most $(5\Delta)^{2^g-1}(15\Delta)^{(32\Delta+8)2^g}$ vertices.*

The number of sets in Theorem 1.1 is best possible, since a $k \times k$ triangular grid has maximum degree six but its vertex set cannot be partitioned into two sets such that each set induces a subgraph with no component of size less than k by the famous HEX lemma [10]. In contrast, Alon, Ding, Oporowski, and Vertigan [1] showed that for graphs of bounded tree-width and bounded maximum degree, it is possible to partition the vertex set into two sets inducing subgraphs having no large components.

Theorem 1.2 (Alon et al. [1, Theorem 2.2]¹). *Let $w \geq 3$ and Δ be positive integers. If a graph G has maximum degree at most Δ and tree-width at most w , then $V(G)$ can be partitioned into X_1, X_2 such that for $1 \leq i \leq 2$, every component of $G[X_i]$ has at most $24w\Delta$ vertices.*

It was pointed out by Esperet and Joret [private communication, 2015] that the condition of maximum degree mentioned in Theorem 1.2 cannot be removed. See Theorem 4.1 for details.

Though it is impossible to partition all planar graphs of bounded maximum degree into two induced subgraphs with components of bounded size, it is possible to partition them such that the tree-width of every component is small. More precisely, DeVos, Ding, Oporowski, Sanders, Reed, Seymour, and Vertigan [3] proved the following result, which was conjectured by

¹In [1], Theorem 1.2 is stated without requiring $w \geq 3$. However, [1] cites [5, (3.7)], which requires $w \geq 3$. However, Theorem 1.2 is true even if $w < 3$, because a stronger statement was proved by Wood [20].

Thomas [19]. A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges.

Theorem 1.3 (DeVos et al. [3]). *For every graph H , there exists an integer N such that if H is not a minor of G , then $V(G)$ can be partitioned into two sets X_1, X_2 such that the tree-width of $G[X_i]$ is at most N for $1 \leq i \leq 2$.*

Alon, Ding, Oporowski, and Vertigan [1] combined Theorems 1.2 and 1.3 to prove the following theorem.

Theorem 1.4 (Alon et al. [1, Theorem 6.7]). *For every graph H and every positive integer Δ , there exists an integer N such that if H is not a minor of a graph G of the maximum degree at most Δ , then $V(G)$ can be partitioned into four sets X_1, X_2, X_3, X_4 such that for $1 \leq i \leq 4$, every component of $G[X_i]$ has at most N vertices.*

In this paper, we prove the following strengthening of Theorems 1.1 and 1.4 and answer a question of Esperet and Joret [9, Question 5.1].

Theorem 1.5. *For every graph H and every positive integer Δ , there exists an integer N such that if H is not a minor of a graph G of the maximum degree at most Δ , then $V(G)$ can be partitioned into three sets X_1, X_2, X_3 such that for $1 \leq i \leq 3$, every component of $G[X_i]$ has at most N vertices.*

Strengthening to odd minors Indeed, we prove a stronger theorem in terms of odd minors as follows. A graph H is an *odd minor* of a graph G if there exists a set $\{(T_v)\}_{v \in V(H)}$ of vertex-disjoint subgraphs of G that are trees such that each tree T_v is properly colored by colors 1 and 2 and for each edge uw of H , there exists an edge joining T_u and T_w whose ends have the same color.

Theorem 1.6. *For every graph H and every positive integer Δ , there exists an integer N such that if H is not an odd minor of a graph G of the maximum degree at most Δ , then $V(G)$ can be partitioned into three sets X_1, X_2, X_3 such that for $1 \leq i \leq 3$, every component of $G[X_i]$ has at most N vertices.*

Since every odd minor of a graph G is also a minor of G , Theorem 1.6 trivially implies Theorem 1.5.

Interestingly Demaine, Hajiaghayi, and Kawarabayashi [2] proved a result analogous to Theorem 1.3 for odd minors, claiming that graphs with no odd

H -minor can be partitioned into two induced subgraphs of bounded tree-width. This with Theorem 1.2 would imply that graphs with no odd H -minor having bounded maximum degree can be partitioned into 4 induced subgraphs each having no large components. Theorem 1.6 reduces 4 to 3.

Applications to a weaker version of Hadwiger’s conjecture As an application of Theorem 1.5, we investigate the following relaxation of Hadwiger’s conjecture: what is the minimum k as a function of t such that for some N , every graph G with no K_{t+1} minor admits a partition of $V(G)$ into k sets X_1, X_2, \dots, X_k with the property that each $G[X_i]$ has no component on more than N vertices? Hadwiger’s conjecture [12], if true, would imply that $k = t$ (with $N = 1$). Kawarabayashi and Mohar [13] proved that $k \leq \lceil 15.5(t + 1) \rceil$, and Wood [21] proved that $k \leq \lceil 3.5t + 2 \rceil$. We improve these results by using a recent result of Edwards, Kang, Kim, Oum, and Seymour [7].

Theorem 1.7 ([7]). *For every positive integer t , there exists s such that if K_{t+1} is not a minor of a graph G , then $V(G)$ can be partitioned into t sets X_1, X_2, \dots, X_t such that for $1 \leq i \leq t$, $G[X_i]$ has maximum degree at most s .*

Theorem 1.8. *For every positive integer t , there exists N such that if K_{t+1} is not a minor of a graph G , then $V(G)$ can be partitioned into $3t$ sets X_1, X_2, \dots, X_{3t} such that for $1 \leq i \leq 3t$, every component of $G[X_i]$ has at most N vertices.*

Proof. By Theorem 1.7, there exists an integer s such that $V(G)$ can be partitioned into t sets V_1, V_2, \dots, V_t such that the maximum degree of $G[V_i]$ is at most s for $1 \leq i \leq t$. By Theorem 1.5, there exists an integer N depending only on t such that for $1 \leq i \leq t$, V_i can be partitioned into three sets V_{i1}, V_{i2}, V_{i3} and each of $G[V_{i1}]$, $G[V_{i2}]$, $G[V_{i3}]$ has no component having size larger than N vertices. \square

In this paper, *graphs* are simple. A k -*coloring* of a graph G is a function mapping the vertices of G into the set $\{1, 2, \dots, k\}$. A *monochromatic component* is a component of the subgraph induced by the vertices of the same color in a given k -coloring. The *size* of a component is the number of its vertices. For a graph G and a set X of vertices, let $N_G(X)$ be the set of vertices not in X but adjacent to some vertex in X and $N_G[X] = N_G(X) \cup X$. For a vertex v of a graph G , let $N_G(v) = N_G(\{v\})$.

The proof of Theorem 1.6 uses the machinery in the Graph Minors series of Robertson and Seymour and the structure theorem of graphs with no odd minors by Geelen, Gerards, Reed, Seymour, and Vetta [11]. A theorem by Robertson and Seymour [18] states that every graph that excludes a fixed graph as a minor can be “decomposed” into pieces satisfying certain structure properties. We will review some tools in the Graph Minors series and modify the mentioned pieces such that they are relatively easier to be 3-colored with small monochromatic components in Section 2. In Section 3, we complete the proof of Theorem 1.6 by first 3-coloring the mentioned pieces and then extending the coloring to the whole graph. Finally, we will make some remarks in Section 4.

2 Structure theorems

In this section, we review some notions in the Graph Minors series of Robertson and Seymour and derive a structure for graphs without a fixed graph as a minor.

A *tree-decomposition* of a graph G is a pair (T, \mathcal{X}) such that T is a tree and $\mathcal{X} = \{X_t : t \in V(T)\}$ is a collection of subsets of $V(G)$ with the following properties.

- $\bigcup_{t \in V(T)} X_t = V(G)$.
- For every $e \in E(G)$, there exists $t \in V(T)$ such that X_t contains both ends of e .
- For every $v \in V(G)$, the subgraph of T induced by $\{t \in V(T) : v \in X_t\}$ is connected.

For every $t \in V(T)$, X_t is called the *bag* of t . The *width* of (T, \mathcal{X}) is $\max\{|X_t| : t \in V(T)\} - 1$. The *adhesion* of (T, \mathcal{X}) is $\max\{|X_t \cap X_{t'}| : tt' \in E(T)\}$. A tree-decomposition (T, \mathcal{X}) is a *path-decomposition* if T is a path. The *tree-width* of G is the minimum width of a tree-decomposition of G .

A *separation* of a graph G is an ordered pair (A, B) of subgraphs with $A \cup B = G$ and $E(A \cap B) = \emptyset$, and the *order* of a separation (A, B) is $|V(A) \cap V(B)|$. A *tangle* \mathcal{T} in G of *order* θ is a set of separations of G , each of order less than θ such that

- (T1) for every separation (A, B) of G of order less than θ , either $(A, B) \in \mathcal{T}$ or $(B, A) \in \mathcal{T}$;

(T2) if $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{T}$, then $A_1 \cup A_2 \cup A_3 \neq G$;

(T3) if $(A, B) \in \mathcal{T}$, then $V(A) \neq V(G)$.

Tangles were first introduced by Robertson and Seymour [16]. We call (T1), (T2) and (T3) the first, second and third tangle axiom, respectively. For a subset Z of $V(G)$ with $|Z| < \theta$, we define $\mathcal{T} - Z$ to be the set of all separations (A', B') of $G - Z$ of order less than $\theta - |Z|$ such that there exists $(A, B) \in \mathcal{T}$ with $Z \subseteq V(A) \cap V(B)$, $A' = A - Z$ and $B' = B - Z$. We remark that $\mathcal{T} - Z$ is a tangle in $G - Z$ of order $\theta - |Z|$ by [16, Theorem 8.5].

Given a graph H , an H -minor of a graph G is a map α with domain $V(H) \cup E(H)$ such that the following hold.

- For every $h \in V(H)$, $\alpha(h)$ is a nonempty connected subgraph of G .
- If h_1 and h_2 are different vertices of H , then $\alpha(h_1)$ and $\alpha(h_2)$ are disjoint.
- For each edge e of H with ends h_1, h_2 , $\alpha(e)$ is an edge of G with one end in $\alpha(h_1)$ and one end in $\alpha(h_2)$; furthermore, if $h_1 = h_2$, then $\alpha(e) \in E(G) - E(\alpha(h_1))$.
- If e_1, e_2 are two different edges of H , then $\alpha(e_1) \neq \alpha(e_2)$.

We say that G contains an H -minor if such a function α exists. A tangle \mathcal{T} in G controls an H -minor α if \mathcal{T} has no (A, B) of order less than $|V(H)|$ such that $V(\alpha(h)) \subseteq V(A)$ for some $h \in V(H)$.

A *society* is a pair (S, Ω) , where S is a graph and Ω is a cyclic permutation of a subset $\bar{\Omega}$ of $V(S)$. For a nonnegative integer ρ , a society (S, Ω) is a ρ -vortex if for all distinct $u, v \in \bar{\Omega}$, there do not exist $\rho + 1$ mutually disjoint paths of S between $I \cup \{u\}$ and $J \cup \{v\}$, where I is the set of vertices in $\bar{\Omega}$ after u and before v in the order Ω , and J is the set of vertices in $\bar{\Omega}$ after v and before u . For a society (S, Ω) with $\bar{\Omega} = \{v_1, v_2, \dots, v_{|\bar{\Omega}|}\}$ in order, a *vortical decomposition* of (S, Ω) is a path-decomposition $(t_1 t_2 \cdots t_{|\bar{\Omega}|}, \mathcal{X})$ such that the i -th bag of \mathcal{X} contains the i -th vertex v_i for each i .

Theorem 2.1 (Robertson and Seymour [15, (8.1)]). *Every ρ -vortex has a vortical decomposition with adhesion at most ρ .*

A *segregation* of a graph G is a set \mathcal{S} of societies such that

- S is a subgraph of G for every $(S, \Omega) \in \mathcal{S}$, and $\bigcup\{S : (S, \Omega) \in \mathcal{S}\} = G$,
- for every distinct (S, Ω) and $(S', \Omega') \in \mathcal{S}$, $V(S \cap S') \subseteq \bar{\Omega} \cap \bar{\Omega}'$ and $E(S \cap S') = \emptyset$.

We write $V(\mathcal{S}) = \bigcup\{\bar{\Omega} : (S, \Omega) \in \mathcal{S}\}$. For a tangle \mathcal{T} in G , a segregation \mathcal{S} of G is \mathcal{T} -central if for every $(S, \Omega) \in \mathcal{S}$, there is no $(A, B) \in \mathcal{T}$ with $B \subseteq S$.

A *surface* is a nonnull compact connected 2-manifold without boundary. Let Σ be a surface. For every subset Δ of Σ , we denote the closure of Δ by $\bar{\Delta}$ and the boundary of Δ by $\partial\Delta$. An *arrangement* of a segregation $\mathcal{S} = \{(S_1, \Omega_1), \dots, (S_k, \Omega_k)\}$ in Σ is a function α with domain $\mathcal{S} \cup V(\mathcal{S})$, such that the following hold.

- For $1 \leq i \leq k$, $\alpha(S_i, \Omega_i)$ is a closed disk $\Delta_i \subseteq \Sigma$, and $\alpha(x) \in \partial\Delta_i$ for each $x \in \bar{\Omega}_i$.
- For $1 \leq i \leq k$, if $x \in \Delta_i \cap \Delta_j$, then $x = \alpha(v)$ for some $v \in \bar{\Omega}_i \cap \bar{\Omega}_j$.
- For all distinct $x, y \in V(\mathcal{S})$, $\alpha(x) \neq \alpha(y)$.
- For $1 \leq i \leq k$, Ω_i is mapped by α to the natural order of $\alpha(\bar{\Omega}_i)$ determined by $\partial\Delta_i$.

An arrangement is *proper* if $\Delta_i \cap \Delta_j = \emptyset$ whenever $|\bar{\Omega}_i|, |\bar{\Omega}_j| > 3$, for all $1 \leq i < j \leq k$.

An *O-arc* is a subset homeomorphic to a circle, and a *line* is a subset homeomorphic to $[0, 1]$. A *drawing* Γ in a surface Σ is a pair (U, V) , where $V \subseteq U \subseteq \Sigma$, U is closed, V is finite, $U - V$ has only finitely many arc-wise connected components, called *edges*, and for every edge e , either \bar{e} is a line with set of ends $\bar{e} \cap V$, or \bar{e} is an O-arc and $|\bar{e} \cap V| = 1$. The components of $\Sigma - U$ are called *regions*. The members of V are called *vertices*. If v is a vertex of a drawing Γ and e is an edge or a region of Γ , we say that e is *incident with* v if v is contained in the closure of e . Note that the incidence relation between vertices and edges of Γ defines a multigraph, and we say that Γ is a *drawing* of a multigraph G in Σ if G is defined by this incident relation. In this case, we say that G is *embeddable* in Σ , or G can be *drawn* in Σ . A drawing is *2-cell* if Σ is connected and every region is an open disk.

A drawing $\Gamma = (U, V)$ in Σ is the *skeleton* of a proper arrangement α of a segregation \mathcal{S} in Σ if $V = \bigcup_{v \in V(\mathcal{S})} \alpha(v)$ and U consists of the boundary of $\alpha(S, \Omega)$ for each $(S, \Omega) \in \mathcal{S}$ with $|\bar{\Omega}| = 3$, and a line in $\alpha(S', \Omega')$ with ends

$\bar{\Omega}'$ for each $(S', \Omega') \in \mathcal{S}$ with $|\bar{\Omega}'| = 2$. Note that we do not add any edges into the skeleton for (S, Ω) with $|\bar{\Omega}| \leq 1$ or $|\bar{\Omega}| > 3$.

A segregation \mathcal{S} of G is *maximal* if there exists no segregation \mathcal{S}' such that $\{(S, \Omega) \in \mathcal{S} : |\bar{\Omega}| > 3\} = \{(S', \Omega') \in \mathcal{S}' : |\bar{\Omega}'| > 3\}$ and for every $(S, \Omega) \in \mathcal{S}$ with $|\bar{\Omega}| \leq 3$, there exists $(S', \Omega') \in \mathcal{S}'$ with $|\bar{\Omega}'| \leq 3$ such that $S' \subseteq S$, and the containment is strict for at least one society. Note that if \mathcal{S} is maximal, then for every $(S, \Omega) \in \mathcal{S}$ with $|\bar{\Omega}| \leq 3$ and every $v \in \bar{\Omega}$, there exist $|\bar{\Omega}| - 1$ paths in S from v to $\bar{\Omega} - \{v\}$ intersecting only in v . In particular, the maximum degree of the skeleton of a proper arrangement of a maximal segregation of G is at most the maximum degree of G .

By taking advantage of a theorem by Robertson and Seymour [17, Theorem (9.2)], the following statement is an easy corollary of a theorem in Dvořák [6, Theorem 7] by choosing the function ϕ in [6, Theorem 7] to be the constant function $4d + 5$. (We omit the statements of [17, Theorem (9.2)] and [6, Theorem 7] as they require a couple of definitions to be formally stated but will not be further used in the rest of the paper.)

Corollary 2.2. *For every graph L , there exists an integer κ such that for every positive integer d , there exist integers θ, ξ, ρ with the following property. If a graph G has a tangle \mathcal{T} of order at least θ controlling no L -minor of G , then there exist $Z \subseteq V(G)$ with $|Z| \leq \xi$, a maximal $(\mathcal{T} - Z)$ -central segregation $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ of $G - Z$ with $|\mathcal{S}_2| \leq \kappa$ and a proper arrangement α of \mathcal{S} in some surface Σ in which L cannot be drawn, such that $|\bar{\Omega}| \leq 3$ for all $(S, \Omega) \in \mathcal{S}_1$, every member in \mathcal{S}_2 is a ρ -vortex, and the skeleton G' of α of \mathcal{S} in Σ satisfies the following.*

1. G' is 2-cell embedded in Σ .
2. For every $(S, \Omega) \in \mathcal{S}_2$, there exists a closed disk D_S in Σ containing $\alpha(S)$ and disjoint from $\bigcup_{(S', \Omega') \in \mathcal{S}_2 - \{(S, \Omega)\}} D_{S'}$ such that D_S contains every vertex of G' that can be connected by a path in G' of length at most d from a vertex in $\bar{\Omega}$.
3. For distinct $(S, \Omega), (S', \Omega') \in \mathcal{S}_2$, there exists no path of length at most $2d + 2$ in G' from $\bar{\Omega}$ to $\bar{\Omega}'$.

Let G_0 be a drawing in a surface Σ with k pairwise disjoint closed disks D_1, D_2, \dots, D_k such that each disk intersects G_0 only in vertices of G_0 and contains no vertex of G_0 in its interior. For $1 \leq i \leq k$, let $v_{i,1}, v_{i,2}, \dots, v_{i,n_i}$

be the vertices of $G_0 \cap \partial D_i$ appearing on ∂D_i in order. For a positive integer w , a graph G is an *outgrowth by k w -rings of a graph G_0 in Σ* [18] if

- there exist k societies $(S_1, \Omega_1), (S_2, \Omega_2), \dots, (S_k, \Omega_k)$ such that $G = G_0 \cup \bigcup_{i=1}^k S_i$ and $\Omega_i = \{v_{i,1}, v_{i,2}, \dots, v_{i,n_i}\}$ in order and $S_i \cap G_0 = \overline{\Omega_i}$ for $1 \leq i \leq k$,
- for $1 \leq i \leq k$, S_i has a path-decomposition $(t_{i,1}t_{i,2} \cdots t_{i,n_i}, \mathcal{X}_i)$ of width at most w such that $v_{i,j} \in X_{i,j}$ for $1 \leq j \leq n_i$, where $X_{i,j}$ is the bag at $t_{i,j}$.

In addition, for $d \geq 0$, we say that G is *d -local* if G_0 satisfies the following.

- For $1 \leq i \leq k$, there exists a closed disk D in Σ containing D_i and disjoint from $\bigcup_{j \neq i} D_j$ such that D contains every vertex of G_0 that can be connected by a path in G_0 of length at most d from a vertex in $V(G_0) \cap \partial D_i$.
- For $1 \leq i < j \leq k$, G_0 has no path of length at most $2d + 2$ from a vertex in $V(G_0) \cap \partial D_i$ to a vertex in $V(G_0) \cap \partial D_j$.

Let \mathcal{S} be a segregation of a graph G . Assume that for every $(S, \Omega) \in \mathcal{S}$ with $|\overline{\Omega}| > 3$, there exists a path-decomposition $(P_S = t_1 t_2 \cdots t_{|\overline{\Omega}|}, \mathcal{X}_S)$ such that the bag at t_i , denoted by $X_{S,i}$, contains the i -th vertex $v_{S,i}$ in $\overline{\Omega}$, for every $1 \leq i \leq |\overline{\Omega}|$. For every $(S, \Omega) \in \mathcal{S}$ with $|\overline{\Omega}| > 3$, let G_S be the graph obtained from the subgraph of S induced by $\overline{\Omega} \cup \bigcup_{i=1}^{|\overline{\Omega}|-1} (X_{S,i} \cap X_{S,i+1})$ by adding three new vertices $x_{S,i,1}, x_{S,i,2}, x_{S,i,3}$ for each $i \in \{1, 2, \dots, |\overline{\Omega}|\}$ with the same set of neighbors

$$N_G(X_{S,i} - (\{v_{S,i}\} \cup X_{S,i-1} \cup X_{S,i+1})) \cap (\{v_{S,i}\} \cup X_{S,i-1} \cup X_{S,i+1}),$$

where $X_{S,0} = X_{S,|\overline{\Omega}+1} = \emptyset$.

Let G_0 be the skeleton of a proper arrangement α of \mathcal{S} in a surface Σ . We define the *extended skeleton* of α of \mathcal{S} in Σ with respect to $\{(P_S, \mathcal{X}_S) : (S, \Omega) \in \mathcal{S}, |\overline{\Omega}| > 3\}$ to be the graph obtained from the disjoint union of G_0 and G_S for every $(S, \Omega) \in \mathcal{S}$ with $|\overline{\Omega}| > 3$ by identifying the copies of the i -th vertex of $\overline{\Omega}$ in G_0 and G_S for each $(S, \Omega) \in \mathcal{S}$ with $|\overline{\Omega}| > 3$ and for every $1 \leq i \leq |\overline{\Omega}|$. Note that if there are at most κ members (S, Ω) of \mathcal{S} with $|\overline{\Omega}| > 3$ and the adhesion of each (P_S, \mathcal{X}_S) is at most ρ , then the extended skeleton of α of \mathcal{S} is an outgrowth by κ $(2\rho + 3)$ -rings of G_0 in Σ . Furthermore, if \mathcal{S} is a maximal segregation, then the maximum degree of the extended skeleton of α of \mathcal{S} in Σ is at most $\max\{3\Delta, 2\rho + 1\}$, where Δ is the maximum degree of G .

Theorem 2.3. *For every graph L and positive integer d , there exist integers $\kappa, \theta, \xi, \rho$ with the following property.*

If a graph G has a tangle \mathcal{T} of order at least θ controlling no L -minor of G , then there exist $Z \subseteq V(G)$ with $|Z| \leq \xi$ and a maximal $(\mathcal{T} - Z)$ -central segregation \mathcal{S} of $G - Z$ with a proper arrangement α in a surface Σ in which L cannot be drawn, such that if $\mathcal{S}_1 = \{(S, \Omega) \in \mathcal{S} : |\bar{\Omega}| \leq 3\}$ and $\mathcal{S}_2 = \mathcal{S} - \mathcal{S}_1$, then

1. $|\mathcal{S}_2| \leq \kappa$ and every $(S, \Omega) \in \mathcal{S}_2$ is a ρ -vortex with a vortical decomposition (P_S, \mathcal{X}_S) of adhesion at most ρ ,
2. the extended skeleton of α of \mathcal{S} in Σ with respect to $\{(P_S, \mathcal{X}_S) : (S, \Omega) \in \mathcal{S}_2\}$ is a d -local outgrowth by $\kappa(2\rho + 3)$ -rings of the skeleton of α of \mathcal{S} in Σ , whose maximum degree is at most $\max\{3\Delta, 2\rho + 1\}$, where Δ is the maximum degree of G .

Proof. Let $\kappa, \theta, \xi, \rho$ be the numbers, $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ the segregation of G , Σ the surface, α the arrangement of \mathcal{S} in Σ obtained by applying Corollary 2.2. By Theorem 2.1, for every $(S, \Omega) \in \mathcal{S}_2$, S has a vortical decomposition (P_S, \mathcal{X}_S) of adhesion at most ρ . Therefore, the extended skeleton of α of \mathcal{S} in Σ with respect to $\{(P_S, \mathcal{X}_S) : (S, \Omega) \in \mathcal{S}_2\}$ is a d -local outgrowth by $\kappa(2\rho + 3)$ -rings of the skeleton of α of \mathcal{S} in Σ . Since \mathcal{S} is maximal, the maximum degree of the extended skeleton of α of \mathcal{S} is at most $\max\{3\Delta, 2\rho + 1\}$ \square

3 Monochromatic components

For an integer $q > 0$, a q -necklace with chain $v_1v_2 \dots v_n$ is a multigraph G with $V(G) = \{v_1, v_2, \dots, v_n\}$ such that

- $v_1v_2 \dots v_nv_1$ is a cycle C ,
- G contains pairwise edge-disjoint complete subgraphs M_1, M_2, \dots, M_k each having at most q vertices such that $E(G) - E(C) = \bigcup_{i=1}^k E(M_i)$, and
- there exist no integers i, j, a, b, c, d with $i \neq j$ and $a < b < c < d$ such that $\{v_a, v_c\} \subseteq V(M_i)$ and $\{v_b, v_d\} \subseteq V(M_j)$.

Note that every 2-connected outerplanar multigraph is a 2-necklace.

Lemma 3.1. *Every q -necklace has tree-width at most $\max\{q - 1, 2\}$.*

Proof. Let G be a q -necklace with n vertices v_1, v_2, \dots, v_n and k complete subgraphs M_1, M_2, \dots, M_k each having at most q vertices. Since outerplanar multigraphs have tree-width at most 2, we may assume that $q \geq 3$ and $k \geq 1$.

We claim that there is a tree-decomposition (T, \mathcal{X}) of width at most $q - 1$. We proceed by induction on k . If $k = 1$, then it is trivial to find such a tree-decomposition (T, \mathcal{X}) , as the graph G is isomorphic to a graph obtained from M_1 by adding many paths. In (T, \mathcal{X}) , one bag is M_1 and other bags have at most three vertices.

Now suppose that $k > 1$. We may assume that $V(M_1) = \{v_{i_1}, v_{i_2}, \dots, v_{i_q}\}$ with $1 \leq i_1 < i_2 < \dots < i_q \leq n$. We may assume that $i_1 = 1$ by rotating labels. Let $i_{q+1} = n + 1$ and $v_{n+1} = v_1$. For $j \in \{1, 2, \dots, q\}$, let $W_j = \{v_{i_j}, v_{i_{j+1}}, \dots, v_{i_{j+1}}\}$. It is easy to see that $G[W_j]$ is a q -necklace and so it has a tree-decomposition (T_j, \mathcal{X}_j) of width at most $q - 1$. Since v_{i_j} is adjacent to $v_{i_{j+1}}$, T_j has a node t_j whose bag contains v_{i_j} and $v_{i_{j+1}}$.

Let T be the tree obtained from the disjoint union of all T_j by adding a node t adjacent to all t_j . Let M_1 be the bag corresponding to t and we assign bags to all other nodes of T according to \mathcal{X}_j for some j . It is easy to see that this is a tree-decomposition of width at most $q - 1$. \square

We use the same idea of the proof of [4, Lemma 8.1] to prove the following generalization.

Lemma 3.2. *For an integer $q \geq 3$, let H be a q -necklace with chain $u_1 u_2 \dots u_n$. Let (S, Ω) be a society with a vortical decomposition $(t_1 t_2 \dots t_n, \mathcal{X})$ of width w . If G is the multigraph obtained from the disjoint union of S and H by identifying u_i with the i -th vertex of Ω for each $1 \leq i \leq n$, then G has tree-width at most $q(w + 1) - 1$.*

Proof. Let X_i be the i -th bag of \mathcal{X} . By Lemma 3.1, H has a tree-decomposition (T, \mathcal{X}') of width at most $q - 1$. We denote \mathcal{X}' by $\{X'_t : t \in V(T)\}$. For every $t \in V(T)$, define $X''_t = X'_t \cup \bigcup\{X_i : u_i \in X'_t, 1 \leq i \leq n\}$ and $\mathcal{X}'' = \{X''_t : t \in V(T)\}$. Since there exists a path in H passing through u_1, u_2, \dots, u_n in order, (T, \mathcal{X}'') is a tree-decomposition of G and

$$|X''_t| \leq (w + 1)|X'_t| \leq q(w + 1).$$

So the width of (T, \mathcal{X}'') is at most $q(w + 1) - 1$. \square

For a positive integer k and a graph G , we say that a k -coloring c of a subgraph H of G can be *extended* to a k -coloring of G or can be *extended* to G if G has a k -coloring c' such that $c'(v) = c(v)$ for every $v \in V(H)$.

Lemma 3.3. *Let Δ, k, w and g be positive integers. Let Σ be a surface of Euler genus g , and let $d = (5\Delta)^{2g-1}(15\Delta)^{(32\Delta+8)2^g}$. If G is a d -local outgrowth by k w -rings of a graph G_0 in Σ and G has maximum degree Δ , then G can be 3-colored in such a way that every monochromatic component has at most $48d^4w\Delta^5$ vertices.*

Proof. We may assume that $\Delta \geq 3$. Let D_1, D_2, \dots, D_k be the closed disks and $(S_1, \Omega_1), \dots, (S_k, \Omega_k)$ the societies mentioned in the definition of an outgrowth G by k w -rings of G_0 in Σ .

By Theorem 1.1, there exists a 3-coloring c of $G - \bigcup_{i=1}^k V(S_i)$ such that every monochromatic component has at most d vertices. Let L_i be the set of vertices of $G - \bigcup_{i=1}^k V(S_i)$ that has a monochromatic path to a vertex in $N_G(V(S_i))$ with respect to c . Since G is d -local, $L_i \cap L_j = \emptyset$ for $1 \leq i < j \leq k$. Let $G_i = G[L_i \cup V(S_i)]$. To prove this lemma, it suffices to show that for every $1 \leq i \leq k$, the 3-coloring on $G_i - V(S_i)$ can be extended to a 3-coloring of G_i such that every monochromatic component of G_i has at most $48d^4w\Delta^5$ vertices.

For $1 \leq i \leq k$, define H_i to be the multigraph obtained from $G[V(S_i)]$ by first adding a cycle passing through $\overline{\Omega_i}$ in order and adding a complete graph on $N_G(V(C)) \cap V(S_i)$ for each monochromatic component C of $G_i - V(S_i)$. Since each C contains at most d vertices, each of added complete subgraphs has at most $d\Delta$ vertices. Since G is d -local, $H_i[\overline{\Omega_i}]$ is a $d\Delta$ -necklace. Hence, the tree-width of H_i is at most $d\Delta(w+1) - 1$ by Lemma 3.2, and the maximum degree of H_i is at most $(d\Delta - 1)\Delta + 2 \leq d\Delta^2$, as $\Delta \geq 3$. By Theorem 1.2, there exists a 3-coloring of H_i (in fact, a 2-coloring) such that every monochromatic component of H_i contains at most $24 \cdot d\Delta(w+1) \cdot d\Delta^2$ vertices.

Now, we extend the 3-coloring c of $G_i - V(S_i)$ to the 3-coloring c' of G_i by taking the 3-coloring of H_i on $V(S_i)$. Let Q be a monochromatic component of G_i with respect to c' . We know that $|V(Q) \cap V(S_i)| \leq 24 \cdot d\Delta(w+1) \cdot d\Delta^2$. Since H_i has maximum degree at most $d\Delta^2$, each vertex of H_i may join at most $d\Delta^2$ distinct monochromatic components of $H_i - V(S_i)$, each having at most d vertices. Thus, Q contains at most

$$(24 \cdot d\Delta(w+1) \cdot d\Delta^2) \cdot d\Delta^2 \cdot d \leq 48d^4w\Delta^5$$

vertices. This completes the proof. \square

The following simple lemma is a stronger statement of [9, Observation 3.9]. This lemma is obvious, so we omit the proof.

Lemma 3.4. *Let G be a graph of maximum degree Δ and Z a subset of $V(G)$. Assume that G has a coloring such that every monochromatic component has size at most an integer k . If we recolor some vertices in Z , then the union of the monochromatic components intersecting in Z in the new coloring has at most $|Z|(\Delta k + 1)$ vertices, and every monochromatic component disjoint from Z in the new coloring has at most k vertices.*

We use the following theorem of Geelen et al. [11] on odd minors.

Theorem 3.5 (Geelen et al. [11, Theorem 13]). *There is a constant c such that if G contains a K_t -minor α where $t = \lceil c\ell\sqrt{\log 12\ell} \rceil$, then either G contains an odd K_ℓ -minor, or there exists a set X of vertices with $|X| < 8\ell$ such that the (unique) block U of $G - X$ that intersects all branch vertices of α disjoint from X is bipartite.*

Our main theorem, Theorem 1.6, is an immediate corollary of the following stronger theorem by taking $Y = \emptyset$.

Theorem 3.6. *For every graph W and positive integer Δ , there exists an integer η such that if W is not an odd minor of a graph G of maximum degree at most Δ , then for every subset Y of $V(G)$ with $|Y| \leq \eta$, every 3-coloring of Y can be extended to that of G satisfying the following.*

- (i) *The union of all monochromatic components of G meeting Y contains at most $|Y|^2\Delta$ vertices.*
- (ii) *Every monochromatic component of G contains at most $\eta^2\Delta$ vertices.*

Proof. We may assume that $\Delta > 1$. Since G does not have an odd W -minor, by Theorem 3.5, there exist sufficiently large integers c and $t = \lceil c|V(W)|\sqrt{\log 12|V(W)|} \rceil$ such that if G has a K_t -minor α , then it has a set X of vertices such that $|X| < 8|V(W)|$ and the (unique) block U of $G - X$ intersecting all branch vertices of α is bipartite. Let $L = K_t$.

Let $d = (5\Delta)^{2g-1}(15\Delta)^{(32\Delta+8)2^g}$, where g is the maximum genus of a surface in which L cannot be drawn. (If L is planar, then let $g = 0$.) Let κ, ξ, θ and ρ be given by Theorem 2.3 for L and d . We may assume that $\theta > 8|V(W)| + 1$. Let $M = 48d^4(2\rho + 3)(3\Delta + 2\rho)^5$ and $\eta = 2000\rho\theta^3M\Delta^6$.

We proceed by induction on $|V(G)|$. It is trivial if $|V(G)| \leq 1$, because $\eta \geq 1$ and $\Delta > 0$. Thus we may assume that $|V(G)| \geq 2$. Let Y be a subset of $V(G)$ with at most η vertices. We may assume that Y is nonempty, because otherwise we can add one vertex to Y . Let $c_Y : Y \rightarrow \{1, 2, 3\}$ be a given 3-coloring of Y . We say that a 3-coloring c of G is Y -good with respect to c_Y if it extends c_Y and satisfies the conditions 1 and 2 of the theorem. Suppose that G has no Y -good 3-coloring with respect to c_Y . Note that for the condition 2, it is unnecessary to consider monochromatic components meeting Y because it follows from the condition 1.

For a subset X of $V(G)$, we write 1_X to denote a 3-coloring of X coloring all vertices of X by 1. Similarly we define 2_X and 3_X .

Claim 1: $|Y| > \frac{\eta}{\Delta^2}$.

Proof of Claim 1: Let $Y_1 = N_G(Y)$ and $Y_2 = N_G(Y \cup Y_1)$. Note that $|Y_2| \leq |Y|\Delta(\Delta - 1) \leq \eta$. By permuting colors, we may assume that $3 \in c_Y(Y)$. We apply the induction hypothesis to $G - (Y \cup Y_1)$ with the 3-coloring 1_{Y_2} of Y_2 to obtain a Y_2 -good 3-coloring c of $G - (Y \cup Y_1)$. Let c' be a 3-coloring of G such that

$$c'(v) = \begin{cases} c_Y(v) & \text{if } v \in Y, \\ 2 & \text{if } v \in Y_1, \\ c(v) & \text{if } v \notin Y \cup Y_1. \end{cases}$$

No monochromatic component of G with respect to c' can meet both $Y \cup Y_1$ and Y_2 because Y_1 and Y_2 are colored differently. Since c is Y_2 -good, every monochromatic component of G disjoint from $Y \cup Y_1$ contains at most $\eta^2\Delta$ vertices. The union of the monochromatic components of G meeting $Y \cup Y_1$ contains at most $|Y \cup Y_1| \leq (\Delta + 1)|Y|$ vertices. If $(\Delta + 1)|Y| \leq |Y|^2\Delta$, then it implies the condition 1 and condition 2 for monochromatic components meeting $Y \cup Y_1$. If $|Y| \geq 2$, then $|Y \cup Y_1| \leq |Y|^2\Delta$; if $|Y| = 1$, then the monochromatic component of G meeting Y contains exactly one vertex of color 3. Therefore c' is Y -good, contradicting our assumption. \square

Claim 2: There exists no separation (A, B) of G of order less than θ such that $|(V(A) - V(B)) \cap Y| \geq 3\theta$ and $|(V(B) - V(A)) \cap Y| \geq 3\theta$.

Proof of Claim 2: Suppose that G has a separation (A, B) of order less than θ such that $a = |(V(A) - V(B)) \cap Y| \geq 3\theta$ and $b = |(V(B) - V(A)) \cap Y| \geq 3\theta$.

Let $Y_A = (Y \cup V(B)) \cap V(A)$ and $Y_B = (Y \cup V(A)) \cap V(B)$. Then,

$$\begin{aligned} |Y|^2 - (|Y_A|^2 + |Y_B|^2) &\geq (a+b)^2 - (a+\theta)^2 - (b+\theta)^2 \\ &= \frac{2ab}{3} + \frac{2ab}{3} + \frac{2ab}{3} - 2(a+b)\theta - 2\theta^2 \\ &\geq 2a\theta + 2b\theta + 6\theta^2 - 2(a+b)\theta - 2\theta^2 = 4\theta^2 > 0. \end{aligned}$$

Now we shall construct a desired 3-coloring of G . We first color vertices in $Y_A - Y (= Y_B - Y)$ arbitrary. Since $|V(A)| \leq |V(G)| - 3\theta$ and $|Y_A| \leq |Y| - 2\theta$, we can apply the induction hypothesis to the graph A with Y_A precolored. Similarly, we can further apply the induction hypothesis to the graph B with Y_B precolored. So by merging the 3-colorings of A and B , we obtain a 3-coloring of G . Let U be the union of the monochromatic components of G either meeting Y , or meeting both A and B . Note that every component of U meets $Y_A \cup Y_B$. By the induction hypothesis, U contains at most $(|Y_A|^2 + |Y_B|^2)\Delta \leq |Y|^2\Delta$ vertices. On the other hand, the induction hypothesis implies that every monochromatic component of G disjoint from Y contains at most $\eta^2\Delta$ vertices. Therefore, G has a Y -good 3-coloring, contradicting our assumption. \square

We define \mathcal{T} to be the set of all separations (A, B) of G of order less than θ such that $|(V(B) - V(A)) \cap Y| \geq 3\theta$.

Claim 3: \mathcal{T} is a tangle in G of order θ .

Proof of Claim 3: Observe that there exists no separation (A, B) of order less than θ such that $|(V(A) - V(B)) \cap Y| < 3\theta$ and $|(V(B) - V(A)) \cap Y| < 3\theta$, since otherwise $|Y| < 7\theta \leq \frac{\eta}{\Delta^2}$, contradicting Claim 1. So \mathcal{T} satisfies the first tangle axiom.

Suppose that there exist $(A_j, B_j) \in \mathcal{T}$ for $1 \leq j \leq 3$ such that $A_1 \cup A_2 \cup A_3 = G$. By Claim 2, $|(V(A_j) - V(B_j)) \cap Y| < 3\theta$ for $1 \leq j \leq 3$. So $|V(A_j) \cap Y| < 4\theta$ for $1 \leq j \leq 3$. As a result, $|Y| \leq \sum_{j=1}^3 |Y \cap V(A_j)| < 12\theta \leq \frac{\eta}{\Delta^2}$, a contradiction. Hence the second tangle axiom holds.

If $V(A) = V(G)$ for some $(A, B) \in \mathcal{T}$, then $|Y| < 4\theta \leq \frac{\eta}{\Delta^2}$ by Claim 2, a contradiction. Therefore, \mathcal{T} is a tangle of order θ . \square

Claim 4: \mathcal{T} controls no L -minor.

Proof of Claim 4: Suppose that \mathcal{T} controls an L -minor α . Since α is an L -minor in G , by Theorem 3.5, there exists a set X of vertices such that $|X| \leq 8|V(W)|$ such that the unique block U of $G - X$ intersecting all branch vertices of α disjoint from X is bipartite.

Let C_1, C_2, \dots, C_m be the list of induced subgraphs of $G - X - V(U)$ such that $G - X - V(U)$ is the disjoint union of C_1, C_2, \dots, C_m , each component of C_i has the same set of neighbors in U , and for $i \neq j$, the set of neighbors of C_i in U is not equal to that of C_j in U .

As U is a block of $G - X$, each C_i has at most one neighbor in U . For each $i \in \{1, 2, \dots, m\}$, Let A_i be the subgraph of G induced by the union of $X, V(C_i)$, and the set of all neighbors of $V(C_i)$ in U . Let B_i be the subgraph of $G - E(A_i)$ induced on $V(G) - V(C_i)$. Note that (A_i, B_i) is a separation of G such that $|V(A_i) \cap V(B_i)| \leq |X| + 1 \leq 8|V(W)| + 1 < \theta$. As \mathcal{T} is a tangle of G of order θ , $(A_i, B_i) \in \mathcal{T}$ or $(B_i, A_i) \in \mathcal{T}$. Since \mathcal{T} controls an L -minor α and all branch vertices of α disjoint from X intersect U , $(A_i, B_i) \in \mathcal{T}$. By the definition of \mathcal{T} , we deduce that

$$|(V(B_i) - V(A_i)) \cap Y| \geq 3\theta.$$

By Claim 2, $|(V(A_i) - V(B_i)) \cap Y| < 3\theta$.

First we properly color U by colors 1 and 2 and color all vertices in X by color 3. This coloring of $G[V(U) \cup X]$ has the property that each monochromatic component has at most $|X| < \theta$ vertices. Then we recolor vertices in $Y \cap (V(U) \cup X)$ by its given color. By Lemma 3.4, this new coloring of $G[V(U) \cup X]$ has the property that the union of all monochromatic components intersecting $Y \cap (V(U) \cup X)$ has at most $\eta(\Delta\theta + 1) \leq 2\eta\Delta\theta$ vertices.

For each $i \in \{1, 2, \dots, m\}$, let $Y'_i = (Y \cap V(A_i)) \cup (V(A_i) \cap V(B_i))$. Note that $|Y'_i| < 4\theta$. By the induction hypothesis, there exists a Y'_i -good coloring f_i of A_i extending the coloring of $G[V(U) \cup X]$ given in the previous step such that the union of all monochromatic components of A_i in f_i intersecting Y'_i has at most $(4\theta)^2\Delta$ vertices and every monochromatic component of A_i in f_i has at most $\eta^2\Delta$ vertices.

Let f be a 3-coloring of G obtained by combining the coloring of $G[V(U) \cup X]$ and the coloring f_i for each $i \in \{1, 2, \dots, m\}$. This coloring f is well-defined and furthermore the union of all monochromatic components in G intersecting $Y \cup X \cup \bigcup_{i=1}^m (V(A_i) \cap V(U))$ has at most $(2\eta\Delta\theta)(4\theta)^2\Delta^2 \leq |Y|^2\Delta$ vertices. In addition, each monochromatic component in G not intersecting $Y \cup X \cup \bigcup_{i=1}^m (V(A_i) \cap V(U))$ has at most $\eta^2\Delta$ vertices. This completes the proof. \square

Now we may assume that \mathcal{T} controls no L -minor. By Theorem 2.3, there exist $Z \subseteq V(G)$ with $|Z| \leq \xi$ and a maximal $(\mathcal{T} - Z)$ -central segregation

$\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ of $G - Z$ properly arranged by an arrangement α in a surface Σ in which L cannot be drawn, where every $(S, \Omega) \in \mathcal{S}_1$ has the property that $|\bar{\Omega}| \leq 3$, and $|\mathcal{S}_2| \leq \kappa$ and every member (S, Ω) in \mathcal{S}_2 is a ρ -vortex with a vortical decomposition (P_S, \mathcal{X}_S) of adhesion at most ρ such that the extended skeleton of α of \mathcal{S} in Σ with respect to $\{(P_S, \mathcal{X}_S) : (S, \Omega) \in \mathcal{S}_2\}$, denoted by G' , is a d -local outgrowth by $\kappa(2\rho + 3)$ -rings of the skeleton of α of \mathcal{S} in Σ and the maximum degree of G' is at most $\max\{3\Delta, 2\rho + 1\} \leq 3\Delta + 2\rho$.

Let c' be a 3-coloring of G' given by Lemma 3.3 such that every monochromatic component of G' with respect to c' contains at most M vertices. Let G'' be the graph obtained from the disjoint union of G' and $G[Z]$ by adding the edges of G between Z and $V(G') \cap V(G)$. Then we extend c' to a 3-coloring c'' of G'' by coloring every vertex in Z by color 1. Then each monochromatic component in G'' contains at most $\max(|Z|, 1)(M\Delta + 1)$ vertices. Since $|Z| < \theta$, we know that $\max(|Z|, 1)(M\Delta + 1) \leq 2\theta M\Delta$. Note that the maximum degree of G'' is still at most $3\Delta + 2\rho$.

For each $(S, \Omega) \in \mathcal{S}_1$, let $Q_S = G[V(S) \cup Z]$ and

$$Y_S = \bar{\Omega} \cup Z \cup (Y \cap V(S)).$$

Since $|\bar{\Omega} \cup Z| \leq |Z| + 3 < \theta$, $(Q_S, G - (V(S) - \bar{\Omega}) - E(Q_S))$ is a separation of G having order less than θ . Since \mathcal{S} is $(\mathcal{T} - Z)$ -central, $(Q_S, G - (V(S) - \bar{\Omega}) - E(Q_S)) \in \mathcal{T}$ and therefore $|Y \cap V(S)| < 4\theta$. So $|Y_S| \leq |Z| + 3 + |Y \cap V(S)| < \xi + 3 + 4\theta \leq 6\theta\Delta \leq \eta$.

For each $(S, \Omega) \in \mathcal{S}_2$ and $1 \leq i \leq |\bar{\Omega}|$, let $X_{S,i}$ be the i -th bag of \mathcal{X}_S , which contains the i -th vertex $v_{S,i}$ in $\bar{\Omega}$; let $Q_{S,i} = G[X_{S,i} \cup Z]$, $B_{S,i} = Z \cup (X_{S,i} \cap (X_{S,i-1} \cup X_{S,i+1} \cup \{v_{S,i}\}))$ where $X_{S,0} = X_{S,|\bar{\Omega}|+1} = \emptyset$; let

$$Y_{S,i} = B_{S,i} \cup (N_G(B_{S,i}) \cap X_{S,i}) \cup (Y \cap X_{S,i}).$$

Note that there exists $(A, B) \in \mathcal{T}$ with $V(A) \cap V(B) = B_{S,i}$ and $G[X_{S,i}] \subseteq A$, since $|B_{S,i}| \leq 2\rho + 1 + \xi < \theta$ and \mathcal{S} is $(\mathcal{T} - Z)$ -central. Thus, $|Y \cap X_{S,i}| < 4\theta$ and therefore $|Y_{S,i}| \leq |B_{S,i} \cup N_G(B_{S,i}) \cup (Y \cap X_{S,i})| < \theta(\Delta + 1) + 4\theta \leq 6\theta\Delta \leq \eta$.

For $(S, \Omega) \in \mathcal{S}_2$ and $1 \leq i \leq |\bar{\Omega}|$, let $x_{S,i,1}, x_{S,i,2}, x_{S,i,3}$ be the vertices of G' mentioned in the definition of the extended skeleton, and let $W_{S,\Omega}$ be a minimum set with $W_{S,\Omega} \subseteq \{x_{S,i,1}, x_{S,i,2}, x_{S,i,3}\}$ and $|W_{S,\Omega}| \leq \min\{|Y \cap X_{S,i} \cap N_G(B_{S,i})|, 3\}$.

Now we define a new 3-coloring c''' of G'' by the following rule.

- $c'''(v) = c_Y(v)$ if $v \in Y \cap V(G'')$.

- For $(S, \Omega) \in \mathcal{S}_2$ and $1 \leq i \leq |\bar{\Omega}|$, define c''' on $\{x_{S,i,1}, x_{S,i,2}, x_{S,i,3}\}$ such that $c'''(\{x_{S,i,1}, x_{S,i,2}, x_{S,i,3}\}) \supseteq c_Y(Y \cap X_{S,i} \cap N_G(B_{S,i}))$ and $c'''(v) = c''(v)$ for every $v \in \{x_{S,i,1}, x_{S,i,2}, x_{S,i,3}\} - W_{S,\Omega}$.
- $c'''(v) = c''(v)$ for other vertices of G'' .

Let

$$Y' = \{v \in V(G'') : c'''(v) \neq c''(v)\} \cup (Y \cap V(G'')) \cup Z \\ \cup \bigcup_{(S,\Omega) \in \mathcal{S}_1, Y \cap V(S) - \bar{\Omega} \neq \emptyset} \bar{\Omega},$$

$$Y_1 = \{v \in Y : v \in V(S) - \bar{\Omega} \text{ for some } (S, \Omega) \in \mathcal{S}_1\} \\ \cup \{v \in Y : v \in X_{S,i} - (B_{S,i} \cup N_G(B_{S,i})) \text{ for some } (S, \Omega) \in \mathcal{S}_2, 1 \leq i \leq |\bar{\Omega}|\},$$

and $Y_2 = Y - Y_1$. Since $X_{S,i} \cap N_G(B_{S,i})$ are pairwise disjoint for different pairs of $(S, \Omega) \in \mathcal{S}_2$ and i , $|Y'| \leq |Y_2| + \theta + 3|Y_1| \leq 4|Y|$. Hence, the union of the monochromatic components in G'' with respect to c''' intersecting Y' contains at most $4|Y|((3\Delta + 2\rho)(2\theta M\Delta) + 1) \leq 48\rho\theta M|Y|\Delta^2$ by Lemma 3.4. And every monochromatic component in G'' with respect to c''' disjoint from Y' has at most $2\theta M\Delta$ vertices.

For $(S, \Omega) \in \mathcal{S}_1$, let c_S be a 3-coloring of Y_S such that

$$c_S(v) = \begin{cases} c_Y(v) & \text{if } v \in Y, \\ c'''(v) & \text{if } v \in \bar{\Omega} \cup Z, \end{cases}$$

for $v \in Y_S$. As $|Y_S| \leq \eta$, we can apply the induction hypothesis to Q_S with the 3-coloring c_S to obtain a Y_S -good 3-coloring c'_S of Q_S .

For $(S, \Omega) \in \mathcal{S}_2$ and $1 \leq i \leq |\bar{\Omega}|$, let $c_{S,i}$ be a 3-coloring of $Y_{S,i}$ such that

$$c_{S,i}(v) = \begin{cases} c_Y(v) & \text{if } v \in Y, \\ c'''(v) & \text{if } v \in \bar{\Omega} \cup Z \cup (X_{S,i} \cap (X_{S,i-1} \cup X_{S,i+1})), \\ c'''(x_{S,i,1}) & \text{if } v \in (N_G(B_{S,i}) \cap X_{S,i}) - Y, \end{cases}$$

for $v \in Y_{S,i}$. As $|Y_{S,i}| \leq \eta$, we can apply the induction hypothesis to $Q_{S,i}$ with the 3-coloring $c_{S,i}$ to obtain a $Y_{S,i}$ -good 3-coloring $c'_{S,i}$ of $Q_{S,i}$.

Let c be a 3-coloring of G such that

$$c(v) = \begin{cases} c'''(v) & \text{if } v \in V(G''), \\ c'_S(v) & \text{if } v \in V(S) - \bar{\Omega} \text{ for some } (S, \Omega) \in \mathcal{S}_1, \\ c'_{S,i}(v) & \text{if } v \in X_{S,i} - B_{S,i} \\ & \text{for some } (S, \Omega) \in \mathcal{S}_2 \text{ and } 1 \leq i \leq |\bar{\Omega}| \end{cases}$$

for $v \in V(G)$.

We now claim that c is a Y -good 3-coloring of G . We say that a subgraph R of G is *hiding* if either there exists $(S, \Omega) \in \mathcal{S}_1$ such that $V(R) \subseteq V(S) - \bar{\Omega}$, or there exists $(S, \Omega) \in \mathcal{S}_2$ and $1 \leq i \leq |\bar{\Omega}|$ such that $V(R) \subseteq X_{S,i} - (B_{S,i} \cup N_G(B_{S,i}))$.

Let U be the union of monochromatic components of G meeting Y . For the condition 1, we need to show that $|V(U)| \leq |Y|^2 \Delta$.

Firstly let us count the vertices of U that are in hiding components. For each hiding monochromatic component R , R contains a vertex in Y_1 and has at most $25\theta^2 \Delta^3$ vertices by the properties of c'_S and $c'_{S,i}$. Thus, U has at most $25\theta^2 \Delta^3 |Y_1|$ vertices in hiding components.

Secondly let us count vertices of U in non-hiding components. Let U' be the graph obtained from U by deleting $V(U) \cap V(S) - \bar{\Omega}$ and adding edges on $V(U) \cap \bar{\Omega}$ such that $U'[\bar{\Omega}]$ is a complete subgraph for every $(S, \Omega) \in \mathcal{S}_1$, and identifying the vertices in $V(U) \cap (X_{S,i} - B_{S,i})$ of color j in the 3-coloring c into a vertex $u_{S,i,j}$ for each $(S, \Omega) \in \mathcal{S}_2$, $1 \leq i \leq |\bar{\Omega}|$ and $1 \leq j \leq 3$. Note that U' is isomorphic to a subgraph of G'' . Furthermore, for every $(S, \Omega) \in \mathcal{S}_2$, $1 \leq i \leq |\bar{\Omega}|$ and $1 \leq j \leq 3$, whenever $u_{S,i,j}$ exists, there exists k with $1 \leq k \leq 3$ such that $c'''(x_{S,i,k}) = j$, by the definition of c''' . So we may assume that U' is a subgraph of G'' with the coloring c''' . Every component of U' meets Y' , since every non-hiding component of U either meets $(Y \cap V(G'')) \cup Z$, or meets $\bar{\Omega}$ for some $(S, \Omega) \in \mathcal{S}_1$ with $Y \cap V(S) - \bar{\Omega} \neq \emptyset$, or meets both $Y \cap X_{S,i}$ and $X_{S,i} \cap N_G(B_{S,i})$ for some $(S, \Omega) \in \mathcal{S}_2$ and $1 \leq i \leq |\bar{\Omega}|$. Therefore, U' contains at most $48\rho\theta M |Y| \Delta^2$ vertices.

For each vertex v in a non-hiding component of U but not in U' , v is either

- contained in a monochromatic component of Q_S meeting $Y_S \cap V(U')$ with respect to c'_S for some $(S, \Omega) \in \mathcal{S}_1$, or
- contained in a monochromatic component of $Q_{S,i}$ meeting $X_{S,i} \cap N_G(B_{S,i})$

with respect to $c'_{S,i}$ for some $(S, \Omega) \in \mathcal{S}_2$ and $1 \leq i \leq |\bar{\Omega}|$ such that $\{x_{S,i,1}, x_{S,i,2}, x_{S,i,3}\} \cap V(U') \neq \emptyset$.

Since \mathcal{S} is maximal, for every vertex v of $G - Z$, there exist at most Δ societies $(S, \Omega) \in \mathcal{S}_1$ such that $v \in V(S)$, so there are at most $|U'| \Delta$ such societies in \mathcal{S}_1 mentioned in the former case; since $|\bigcup_{(S, \Omega) \in \mathcal{S}_2} \bigcup_{1 \leq i \leq |\bar{\Omega}|} \{x_{S,i,1}, x_{S,i,2}, x_{S,i,3}\} \cap V(U')| \leq |U'|$, so there are at most $|U'|$ such $Q_{S,i}$ mentioned in the latter case. By the properties of c'_S , the union of all monochromatic components mentioned in the former case contains at most $(5\theta\Delta)^2 \cdot |U'| \Delta$ vertices; by the properties of $c'_{S,i}$, the union of all monochromatic components mentioned in the latter case contains at most $(5\theta\Delta)^2 \cdot |U'|$ vertices. Hence, the number of vertices in some non-hiding components of U but not in U' contains at most $25\theta^2 |U'| \Delta^2 (\Delta + 1) \leq 1200\rho\theta^3 M |Y| \Delta^4 (\Delta + 1)$ vertices.

Consequently, U contains at most $25\theta^2 \Delta^3 |Y_1| + 1200\rho\theta^3 M |Y| \Delta^4 (\Delta + 1) \leq 2000\rho\theta^3 M |Y| \Delta^5 \leq |Y|^2 \Delta$ vertices, by Claim 1 and the assumption $\Delta \geq 2$. This proves that c satisfies condition 1.

Let R be a monochromatic component of G not meeting Y with respect to c . For condition 2, it suffices to show that R contains at most $\eta\Delta^2$ vertices. It is clear that R contains at most $\max\{25\theta^2 \Delta^3, \eta\Delta^2\} \leq \eta\Delta^2$ vertices if R is hiding by the properties of c_S and $c_{S,i}$. So we may assume that R is not hiding.

Construct R' from R as we constructed U' from U . That is, let R' be the graph obtained from R by deleting $V(R) \cap V(S) - \bar{\Omega}$ and adding edges on $V(R) \cap \bar{\Omega}$ such that $R'[\bar{\Omega}]$ is a complete subgraph for every $(S, \Omega) \in \mathcal{S}_1$, and identifying the vertices in $V(R) \cap X_{S,i} \cap N_G(B_{S,i})$ of color j in the 3-coloring c into a vertex $u_{S,i,j}$ for each $(S, \Omega) \in \mathcal{S}_2$, $1 \leq i \leq |\bar{\Omega}|$ and $1 \leq j \leq 3$. We may again assume that R' is a subgraph of G'' with the coloring c''' . Since R is connected, R' is connected. Hence, R' is a monochromatic component of G'' with respect to c''' and contains at most $48\rho\theta M |Y| \Delta^2$ vertices.

For each vertex v in R but not in R' , v is either

- contained in a monochromatic component of Q_S meeting $Y_S \cap V(R')$ with respect to c'_S for some $(S, \Omega) \in \mathcal{S}_1$, or
- contained in a monochromatic component of $Q_{S,i}$ meeting $X_{S,i} \cap N_G(B_{S,i})$ with respect to $c'_{S,i}$ for some $(S, \Omega) \in \mathcal{S}_2$ and $1 \leq i \leq |\bar{\Omega}|$ such that $\{x_{S,i,1}, x_{S,i,2}, x_{S,i,3}\} \cap V(R') \neq \emptyset$.

Therefore, the same argument shows that the number of vertices of R but not in R' is at most $25\theta^2 |R'| \Delta^2 (\Delta + 1)$ vertices. As a result, R contains at

most $|R'|(1 + 25\theta^2\Delta^2(\Delta + 1)) \leq 2000\rho\theta^3M|Y_2|\Delta^5 \leq \eta\Delta^2$. This shows that c satisfies condition 2 and completes the proof. \square

4 Concluding remarks

We remark that Theorems 1.2 and 1.5 are best possible in the sense that it is impossible to partition the vertex set into three sets such that each set induces a subgraph of bounded diameter. The following observation is due to Esperet and Joret. Recall that every graph with bounded tree-width does not contain a large grid as a minor.

Theorem 4.1 (Esperet and Joret [private communication, 2015]). *For every positive integers w, d , there exists a graph G of tree-width at most w such that for every w -coloring of G , there exists a monochromatic component of G with diameter greater than d .*

Proof. We shall construct graphs G_i of tree-width at most i for every $i \geq 1$ such that every i -coloring of G has a monochromatic component of diameter greater than d recursively. Define G_1 to be the path on d vertices. Clearly, G_1 has tree-width one and every 1-coloring of G_1 contains a monochromatic component of diameter greater than d .

Assume that we have constructed the graph G_{i-1} of tree-width at most $i-1$ such that every $(i-1)$ -coloring of G has a monochromatic component of diameter greater than d . Let $n = |V(G_{i-1})|$. Let T be the rooted n -ary tree with root r such that every internal node of T has degree n , and the distance between r and any leaf of T is d . For every node t of T , we create a copy H_t of G_{i-1} , and we denote the vertices of H_t by $u_{t,1}, \dots, u_{t,n}$. For every internal node t of T , we denote the children of t by $c_{t,1}, c_{t,2}, \dots, c_{t,n}$. Then we construct G_i from the disjoint union of H_t for all nodes t of T by adding a new vertex v adjacent to all vertices of H_r for the root r of T and adding edges $u_{t,j}u'$ for every non-leaf t of T , $1 \leq j \leq n$ and $u' \in V(H_{c_{t,j}})$.

Now we prove that G_i has the desired property. Suppose that f is a i -coloring of G_i such that every monochromatic component has diameter at most d . As G_{i-1} has the desired property, $V(H_t)$ receives exactly i colors by f for every vertex t of T . In particular, each H_t contains a vertex x_t with $f(x_t) = f(v)$. Since T contains a path $rt_1t_2 \cdots t_d$ of length d , $vx_r x_{t_1} x_{t_2} \cdots x_{t_d}$ is a monochromatic path of length $d+1$, a contradiction.

In addition, every block of G_i is obtained from a copy of G_{i-1} by adding a vertex. So the tree-width of G_i is at most the one more than the tree-width of G_{i-1} . This completes the proof. \square

Note that the graphs G_2 and G_3 mentioned in the proof of Theorem 4.1 are outerplanar and planar, respectively. So Theorem 1.1 cannot be improved in the same direction, either. On the other hand, it is well known that every graph of tree-width at most w contains a vertex of degree at most w and hence can be properly colored by $w + 1$ colors. So Theorem 4.1 is the best possible.

Esperet and Joret [private communication, 2015] also points out that the construction of G_3 disproves the following conjecture of Nešetřil and Ossona de Mendez [14], since long paths have large tree-depth.

Conjecture 4.2 ([14, Conjecture 7.1]). *There exists a constant t such that one can color the vertices of every planar graph by 3 colors in such a way that no monochromatic component will have tree-depth greater than t .*

We also remark that Theorem 1.5 cannot be generalized to graphs with no H -topological minor in general. The following is proved by using an idea of Alon et al. [1].

Theorem 4.3. *For positive integers k, N , there exists a $(4k - 2)$ -regular graph G such that for every partition of $V(G)$ into k sets X_1, X_2, \dots, X_k , there exists i with $1 \leq i \leq k$ such that some component of $G[X_i]$ contains at least N vertices.*

Proof. It was proved by Erdős and Sachs [8] that there exists a $2k$ -regular graph R with girth at least N . Since R contains $k|V(R)|$ edges, for any partition of $E(R)$ into k sets, some set contains at least $|V(R)|$ edges and hence induces a subgraph W of R having a cycle. Since the girth of R is at least N , some component of W contains at least N edges. Therefore, for every partition of $E(R)$ into k sets, there exists a set in the partition such that some component induced by this set contains at least N edges.

Define G to be the line graph of R . So G is $(4k - 2)$ -regular. Furthermore, every partition of $V(G)$ into k sets X_1, X_2, \dots, X_k corresponds to a partition of $E(R)$ into k sets, so there exists i with $1 \leq i \leq k$ such that $G[X_i]$ has a component with at least N vertices. \square

Since every graph of maximum degree at most $4k - 2$ does not contain any graph with maximum degree at least $4k - 1$ as a topological minor, Theorem 4.3 shows that Theorem 1.5 cannot be generalized to topological minor-free graphs in general.

Acknowledgement. The authors would like to thank anonymous referees for their valuable suggestions, and thank Louis Esperet and Gwenaël Joret for bringing Theorem 4.1 and Conjecture 4.2 to our attention and allowing us to include their proof into this paper. The authors would also like to thank Dong Yeap Kang for reading this manuscript carefully and finding a few typos.

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