EXCLUDED VERTEX-MINORS FOR GRAPHS OF LINEAR RANK-WIDTH AT MOST k

JISU JEONG, O-JOUNG KWON, AND SANG-IL OUM

Abstract. Linear rank-width is a graph width parameter, which is a variation of rank-width by restricting its tree to a caterpillar. As a corollary of known theorems, for each k, there is a finite obstruction set $\mathcal{O}_k$ of graphs such that a graph G has linear rank-width at most k if and only if no vertex-minor of G is isomorphic to a graph in $\mathcal{O}_k$. However, no attempts have been made to bound the number of graphs in $\mathcal{O}_k$ for $k \geq 2$. We show that for each k, there are at least $2^{\Omega(3^k)}$ pairwise locally non-equivalent graphs in $\mathcal{O}_k$, and therefore the number of graphs in $\mathcal{O}_k$ is at least double exponential.

To prove this theorem, it is necessary to characterize when two graphs in $\mathcal{O}_k$ are locally equivalent. A graph is a *block graph* if all of its blocks are complete graphs. We prove that if two block graphs without simplicial vertices of degree at least 2 are locally equivalent, then they are isomorphic. This not only is useful for our theorem but also implies a theorem of Bouchet [*Transforming trees by successive local complementations*, J. Graph Theory **12** (1988), no. 2, 195–207] stating that if two trees are locally equivalent, then they are isomorphic.

1. Introduction

Linear rank-width is a width parameter of graphs motivated by rank-width of graphs introduced by Oum and Seymour [16]. A vertex-minor relation is a graph containment relation such that rank-width and linear rank-width cannot increase when taking vertex-minors of a graph. Two graphs G, H are called *locally equivalent* if H is a vertex-minor of G and $|V(H)| = |V(G)|$. The definitions can be found in Section 2.

Oum [15] proved that for every infinite sequence $G_1, G_2, \ldots$ of graphs of bounded rank-width, there exist $i < j$ such that G_i is isomorphic to a vertex-minor of G_j. As a corollary, we immediately obtain the following theorem.

Theorem 1.1 (Oum [15]). *For every class $\mathcal{C}$ of graphs of bounded rank-width, there is a finite list of graphs $G_1, G_2, \ldots, G_m$ such that a graph is in $\mathcal{C}$ if and only if it does not have a vertex-minor isomorphic to G_i for some $i \in \{1, 2, \ldots, m\}$.*

Because rank-width is always less than or equal to linear rank-width, we deduce the following.

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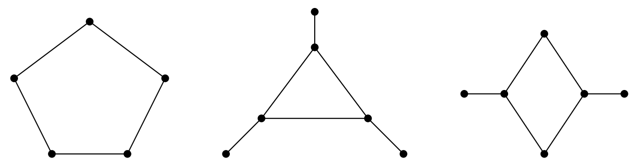


FIGURE 1. Graphs in $\mathcal{O}_1$.

width parameter	relation	type	references
path-width	minor	L, U	[17],[13]
linear-width	minor	L	[18]
tree-width	minor	L, U	[11], [13]
tree-depth	minor, induced subgraph	L, U	[8]
rank-width	vertex-minor	U	[14]
branch-width (graphs, matroids)	minor	U	[10]

TABLE 1. Known lower or upper bound of the size of the obstruction set for graphs of bounded width parameters. In the column of type, L and U mean a lower and upper bound, respectively.

Corollary 1.2. *For a fixed k, there exists a finite set $\mathcal{O}_k$ of graphs $G_1, G_2, \ldots, G_m$ such that a graph has linear rank-width at most k if and only if it does not have a vertex-minor isomorphic to G_i for some $i \in \{1, 2, \ldots, m\}$.*

However, Theorem 1.1 does not produce an explicit upper or lower bound on the number of graphs in $\mathcal{O}_k$ for Corollary 1.2. We aim to prove a lower bound on $|\mathcal{O}_k|$. Our main result is the following.

Theorem 1.3. *Let $k \geq 2$ be an integer. There exist at least $2^{\Omega(3^k)}$ pairwise locally non-equivalent graphs that are vertex-minor minimal with the property that they have linear rank-width larger than k. In other words, $|\mathcal{O}_k| \geq 2^{\Omega(3^k)}$ in Corollary 1.2.*

It is non-trivial to characterize the set of all graphs of linear rank-width at most k in terms of forbidden vertex-minors. So far only one case is known. For $k = 1$, Adler, Farley, and Proskurowski [1] characterized the graphs of linear rank-width at most 1 by a set $\mathcal{O}_1$ of three graphs in Figure 1. Ganian [9] described the structure of graphs of linear rank-width 1.

There have been similar results on the number of forbidden minors for various graph width parameters, see Table 1.

One of the main ingredients is a generalization of a theorem of Bouchet. To show Theorem 1.3, we will construct, for each non-negative integer k, a set Δ_k of vertex-minor minimal graphs with the property that they have linear rank-width larger than k. To obtain the lower bound on $|\mathcal{O}_k|$, it is necessary to understand when two graphs in Δ_k are locally equivalent. We resolve this problem by showing the following stronger theorem. A vertex is *simplicial* if the set of its neighbors is a clique.

Theorem 1.4. *If two block graphs without simplicial vertices of degree at least 2 are locally equivalent, then they are isomorphic.*

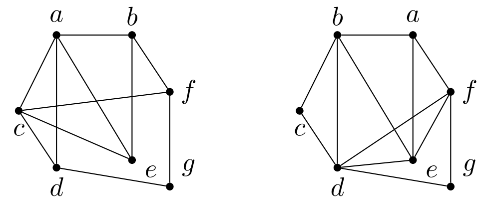


FIGURE 2. Pivoting an edge ab.

All graphs in Δ_k have no simplicial vertices of degree at least 2. Hence, Theorem 1.4 is useful for proving Theorem 1.3. Since trees are block graphs without simplicial vertices of degree at least 2, we deduce the following corollary, originally shown by Bouchet.

Corollary 1.5 (Bouchet [4])**.** *If two trees are locally equivalent, then they are isomorphic.*

The paper is organized as follows. In Section 2, we present necessary definitions. In Section 3, we construct the set Δ_k and prove that the graphs in Δ_k are vertex-minor minimal graphs with the property that they have linear rank-width larger than k. In Section 4, we prove that no two non-isomorphic graphs in Δ_k are locally equivalent by showing Theorem 1.4. In Section 5, we count graphs in Δ_k up to isomorphism, and we conclude that $|\mathcal{O}_k| \geq 2^{\Omega(3^k)}$. Final remarks are made in Section 6.

2. Preliminaries

In this paper, graphs have no loops and parallel edges. Let $G = (V(G), E(G))$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced on S. And for $v \in V(G)$, we denote $N_G(v)$ as the set of the neighbors of v in G. A vertex v in G is a *leaf* if $|N_G(v)| = 1$.

For an $X \times Y$ matrix $M = (m_{i,j})_{i \in X, j \in Y}$ and subsets $A \subseteq X$ and $B \subseteq Y$, $M[A, B]$ denotes the $A \times B$ submatrix $(m_{i,j})_{i \in A, j \in B}$ of M.

Vertex-minors. The *local complementation* at a vertex v of a graph $G = (V, E)$ is an operation to obtain a graph $G * v$ from G by replacing the subgraph $G[N_G(v)]$ with the complementary subgraph of $G[N_G(v)]$. The graph obtained from G by *pivoting* an edge uv is defined by $G \wedge uv = G * u * v * u$.

To see how we obtain the resulting graph by pivoting an edge uv, let $V_1 = N_G(u) \cap N_G(v)$, $V_2 = N_G(u) \setminus N_G(v) \setminus \{v\}$, and $V_3 = N_G(v) \setminus N_G(u) \setminus \{u\}$. One can easily verify that $G \wedge uv$ is identical to the graph obtained from G by complementing adjacency of vertices between distinct sets V_i and V_j, and swapping the vertices u and v [14]. See Figure 2 for an example.

A graph H is a *vertex-minor* of G if H can be obtained from G by applying a sequence of vertex deletions and local complementation. A graph H is *locally equivalent* to G if H can be obtained from G by applying a sequence of local complementation.

A vertex-minor H of G is *elementary* if $|V(H)| = |V(G)| - 1$. For a set $\mathcal{C}$ of graphs closed under taking vertex-minors, a graph G is an *excluded vertex-minor* for C if $G \notin \mathcal{C}$ and $H \in \mathcal{C}$ for every elementary vertex-minor H of G.

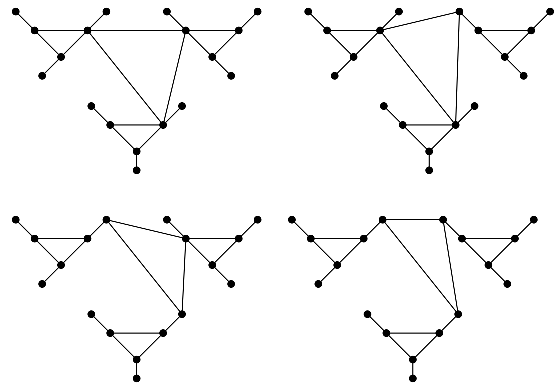


FIGURE 3. All non-isomorphic graphs in Δ_2.

Linear rank-width. The adjacency matrix of a graph G, which is a $(0,1)$-matrix over the binary field, will be denoted by $A(G)$. The *cut-rank* function $\rho_G : 2^V \to \mathbb{Z}$ of a graph G is defined by $\rho_G(X) = \operatorname{rank}(A(G)[X, V\setminus X])$. The cut-rank function satisfies the submodular inequality [16]:

$$\rho_G(X) + \rho_G(Y) \geq \rho_G(X \cap Y) + \rho_G(X \cup Y).$$

This implies that $\rho_G(X \cup \{v\}) \leq \rho_G(X) + 1$ for $X \subseteq V(G)$ and $v \in V(G)$.

A *linear layout* L of G is a sequence $(v_1, v_2, \ldots, v_{|V(G)|})$ of $V(G)$. For a linear layout L of G and $a, b \in V(G)$, we denote $a \leq_L b$ or $b \geq_L a$ if $a = b$ or a appears before b in L. For two sequences $L_1 = (v_1, v_2, \ldots, v_n)$ and $L_2 = (w_1, w_2, \ldots, w_m)$, we define $L_1 \oplus L_2 = (v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_m)$.

The *width* of a linear layout L in G, denoted by $\operatorname{lrw}_L(G)$, is defined as the maximum over all $\rho_G(\{w : w \leq_L v\})$ for $v \in V(G)$. We say that the width of L is 0 if $|V(G)| \leq 1$. The *linear rank-width* of G, denoted by $\operatorname{lrw}(G)$, is the minimum width of all linear layouts of G.

If two graphs are locally equivalent, then they have the same linear rank-width by the following proposition.

Proposition 2.1 (Bouchet [5]; see Oum [14])**.** *Let G be a graph and $v \in V(G)$. Then $\rho_G(X) = \rho_{G*v}(X)$ for all $X \subseteq V(G)$.*

It follows easily that if H is a vertex-minor of G, then $\operatorname{lrw}(H) \leq \operatorname{lrw}(G)$.

3. Excluded vertex-minors for graphs of bounded linear rank-width

To prove Theorem 1.3, we construct a set Δ_k of graphs that are vertex-minor minimal with the property that the linear rank-width is larger than k.

A *delta composition* G of graphs G_1, G_2, and G_3 is a graph obtained from the disjoint union of G_1, G_2, and G_3 by adding a triangle $v_1v_2v_3$ where $v_i \in V(G_i)$ for $i = 1, 2, 3$. We call $v_1v_2v_3$ the *main triangle* of G. For a non-negative integer k, we define Δ_k as follows:

(1) $\Delta_0 = \{(\{x, y\}, \{xy\})\}$. (It is isomorphic to K_2.)
(2) For $i \geq 1$, Δ_i is the set of all delta compositions of three graphs in Δ_{i-1}.

All non-isomorphic graphs in Δ_2 are depicted in Figure 3. Here is the main theorem of this section.

Theorem 3.1. *Let k be a non-negative integer. Every graph in Δ_k is an excluded vertex-minor for graphs of linear rank-width at most k.*

3.1. **Linear rank-width of a graph in Δ_k.** First, we prove that every graph in Δ_k has linear rank-width $k+1$.

Lemma 3.2. *The linear rank-width of a graph in Δ_k is at least $k+1$.*

Proof. We use induction on k. We may assume that $k \geq 1$. Since $G \in \Delta_k$, G is a delta composition of $G_1, G_2, G_3 \in \Delta_{k-1}$ with the main triangle $v_1v_2v_3$ such that $v_i \in V(G_i)$ for $i = 1, 2, 3$.

Suppose that G has linear rank-width at most k. By the induction hypothesis, G_1 has linear rank-width at least k and therefore G has linear rank-width exactly k. Let L be a linear layout of G having width k. For $v \in V(G)$, we define $S_v = \{x \in V(G) : x \leq_L v\}$ and $T_v = V(G) \setminus S_v$. Let a and b be the first and the last vertices in L such that $\rho_G(S_a) = \rho_G(S_b) = k$. Without loss of generality, we may assume that $\{a, b\} \subseteq V(G_2) \cup V(G_3)$. Let L_1 be the subsequence of L whose elements are the vertices of G_1.

For contradiction, we claim that L_1 is a linear layout of G_1 having width at most $k-1$. Let $v \in V(G_1)$. It is sufficient to show that $\rho_{G_1}(S_v \cap V(G_1)) \leq k-1$. Note that $v \neq a$ and $v \neq b$. If $v \leq_L a$ or $v \geq_L b$, then

$$\rho_{G_1}(S_v \cap V(G_1)) \leq \rho_G(S_v) \leq k-1.$$

So we may assume that $a \leq_L v \leq_L b$. Note that one of $S_v \cap V(G_1)$ and $T_v \cap V(G_1)$ does not have a neighbor in $G[V(G) \setminus V(G_1)]$ because v_1 is the unique vertex in G_1 which has a neighbor in $G[V(G) \setminus V(G_1)]$. And since $G[V(G) \setminus V(G_1)]$ is connected and $a \in S_v \setminus V(G_1)$ and $b \in T_v \setminus V(G_1)$, there is an edge u_1u_2 in $G[V(G) \setminus V(G_1)]$ such that $u_1 \in S_v \setminus V(G_1)$ and $u_2 \in T_v \setminus V(G_1)$. So $A(G)[S_v \setminus V(G_1), T_v \setminus V(G_1)]$ is a non-zero matrix. Depending on whether $v_1 \in S_v \cap V(G_1)$ or $v_1 \in T_v \cap V(G_1)$,

$$\begin{aligned}
&\rho_G(S_v) \\
&\quad = \operatorname{rank} \begin{pmatrix} A(G)[S_v \cap V(G_1), T_v \cap V(G_1)] & 0 \\ A(G)[S_v \setminus V(G_1), T_v \cap V(G_1)] & A(G)[S_v \setminus V(G_1), T_v \setminus V(G_1)] \end{pmatrix} \\
&\quad \geq \operatorname{rank}\left(A(G)[S_v \cap V(G_1), T_v \cap V(G_1)]\right) + \operatorname{rank}\left(A(G)[S_v \setminus V(G_1), T_v \setminus V(G_1)]\right),
\end{aligned}$$

or

$$\begin{aligned}
&\rho_G(S_v) \\
&\quad = \operatorname{rank} \begin{pmatrix} A(G)[S_v \cap V(G_1), T_v \cap V(G_1)] & A(G)[S_v \cap V(G_1), T_v \setminus V(G_1)] \\ 0 & A(G)[S_v \setminus V(G_1), T_v \setminus V(G_1)] \end{pmatrix} \\
&\quad \geq \operatorname{rank}\left(A(G)[S_v \cap V(G_1), T_v \cap V(G_1)]\right) + \operatorname{rank}\left(A(G)[S_v \setminus V(G_1), T_v \setminus V(G_1)]\right),
\end{aligned}$$

respectively. Thus, we have

$$\begin{aligned}
\rho_{G_1}(S_v \cap V(G_1)) &= \operatorname{rank}\left(A(G)[S_v \cap V(G_1), T_v \cap V(G_1)]\right) \\
&\leq \rho_G(S_v) - \operatorname{rank}\left(A(G)[S_v \setminus V(G_1), T_v \setminus V(G_1)]\right) \\
&\leq \rho_G(S_v) - 1 \leq k-1.
\end{aligned}$$

So L_1 is a linear layout of G_1 having width at most $k-1$, which is contradiction. Hence, $\operatorname{lrw}(G) \geq k+1$. $\square$

A vertex w is called a *twin* of another vertex v in a graph if no vertex other than v and w is adjacent to exactly one of v and w.

If w is a twin of v in a graph G and $G \setminus w$ has linear rank-width $k+1$ with a linear layout of width $k+1$ starting with v, then clearly G also admits a linear layout of width $k+1$ starting with v because we can easily put w in the second place. But the following lemma claims that we can place w at the end if $G \setminus w \in \Delta_k$.

Lemma 3.3. *Let v be a vertex of a graph G and let w be a twin of v. If $G \setminus w \in \Delta_k$, then G has a linear layout L of width $k+1$ such that the first vertex of L is v and the last vertex of L is w.*

Before proving the lemma, we first show that Lemma 3.3 implies the following proposition determining the exact linear rank-width of a graph in Δ_k.

Proposition 3.4. *Every graph in Δ_k has linear rank-width $k+1$. Moreover, for every vertex v of $G \in \Delta_k$, there exists a linear layout of G having width $k+1$ whose first vertex is v.*

Proof of Proposition 3.4. By Lemma 3.2, the linear rank-width of a graph G in Δ_k is at least $k+1$. Let $v \in V(G)$ and let G' be a graph obtained by adding a twin w of v to G. Then Lemma 3.3 implies that G' has a linear layout L of width $k+1$ starting at v and ending at w. We discard w from L to obtain a linear layout of G starting with v having width $k+1$. □

Proof of Lemma 3.3. We prove by induction on k. If $k = 0$, then G is a connected graph on three vertices and therefore every linear layout of G has width 1. Thus we may assume that $k \geq 1$. Let $G \setminus w$ be a delta composition of $G_1, G_2, G_3 \in \Delta_{k-1}$ with the main triangle $v_1v_2v_3$ such that $v_i \in V(G_i)$ for $i = 1, 2, 3$. We may assume that $v \in V(G_2)$.

By the induction hypothesis, G_1 has a linear layout L_1 of width k whose last vertex is v_1, and G_3 has a linear layout L_3 of width k whose first vertex is v_3.

Let

$$H = \begin{cases} G \setminus (V(G_1) \cup V(G_3)) & \text{if } v \neq v_2, \\ G \setminus (V(G_1) \cup V(G_3)) \setminus vw & \text{if } v = v_2, \text{ and } v,\ w \text{ are adjacent in } G, \\ G \setminus (V(G_1) \cup V(G_3)) + vw & \text{otherwise.} \end{cases}$$

By the induction hypothesis, H has a linear layout $(v) \oplus L_H \oplus (w)$ of width k.

(1) Clearly, $\rho_G(V(G_1) \cup \{v\}) \leq 2 \leq k+1$ and $\rho_G(V(G_3) \cup \{w\}) \leq 2 \leq k+1$.

(2) We claim that for $X \subseteq V(G_1) \setminus \{v_1\}$, if $\rho_{G_1}(X) \leq k$, then $\rho_G(X \cup \{v\}) \leq k+1$. This is because no vertex in X has a neighbor in $V(G) \setminus V(G_1)$ and therefore $\rho_{G_1}(X) = \rho_G(X) \geq \rho_G(X \cup \{v\}) - 1$ by the submodular inequality.

(3) Similar to (2), we deduce that for $X \subseteq V(G_3) \setminus \{v_3\}$, if $\rho_{G_3}(X) \leq k$, then $\rho_G(X \cup \{v\}) \leq k+1$.

(4) We claim that if $v \neq v_2$, $X \subseteq V(H)$, and $\rho_H(X) \leq k$, then $\rho_G(V(G_1) \cup X) \leq k+1$. By symmetry between G_1 and G_3, we may assume that $v_2 \notin X$. By the submodular inequality, $\rho_G(V(G_1) \cup X) \leq \rho_G(X) + \rho_G(V(G_1)) = \rho_H(X) + 1 \leq k+1$.

(5) We claim that if $v = v_2$, $v \in X \subseteq V(H)$, $w \notin X$, and $\rho_H(X) \leq k$, then $\rho_G(V(G_1) \cup X) \leq k+1$. By adding the row of v_1 to that of v_2 in $A(G)[X \cup V(G_1), (V(H) \setminus X) \cup V(G_3)]$, we see that $\rho_G(X \cup V(G_1)) \leq \rho_H(X) + 1 \leq k+1$.

By combining (1), (2), (3), (4), and (5), we conclude that $(v)\oplus L_1\oplus L_H\oplus L_3\oplus(w)$ is a linear layout of G having width at most $k+1$. Clearly it has width $k+1$ because $G\setminus w$ has linear rank-width $k+1$ by Lemma 3.2. □

3.2. **Combining graphs in Δ_k.** The following two lemmas will help us to prove that elementary vertex-minors of graphs in Δ_k have linear rank-width at most k.

Lemma 3.5. *Let k be a positive integer and let $G_1, G_2 \in \Delta_{k-1}$. Let G be a graph obtained from the disjoint union of G_1 and G_2 by adding an edge w_1w_2 for fixed $w_1 \in V(G_1)$ and $w_2 \in V(G_2)$. Then G has linear rank-width k.*

Proof. It is trivial that the linear rank-width of G is at least k because an induced subgraph G_1 of G has linear rank-width k by Proposition 3.4. By Proposition 3.4, there is a linear layout L_1 of G_1 having width k such that the last vertex of L_1 is v_1, and there is a linear layout L_2 of G_2 having width k such that the first vertex of L_2 is v_2. Then obviously $L_1 \oplus L_2$ is a linear layout of G having width at most k. □

Lemma 3.6. *Let k be a positive integer. Let G_1, $G_2 \in \Delta_{k-1}$, and let G_3 be a graph having linear rank-width at most $k-1$. Then every delta composition of G_1, G_2 and G_3 has linear rank-width k.*

Proof. Let G be a delta composition of G_1, G_2 and G_3 with the main triangle $v_1v_2v_3$ such that $v_i \in V(G_i)$ for $i = 1, 2, 3$. Clearly the linear rank-width of G is at least k because an induced subgraph G_1 of G has linear rank-width k by Proposition 3.4.

Since $G_1, G_2 \in \Delta_{k-1}$, by Proposition 3.4, there is a linear layout L_1 of G_1 having width k such that the last vertex of L_1 is v_1, and there is a linear layout L_2 of G_2 having width k such that the first vertex of L_2 is v_2. Let L_3 be a linear layout of G_3 having width at most $k-1$.

We claim that $L = L_1 \oplus L_3 \oplus L_2$ is a linear layout of G having width at most k. Let $v \in V(G)$, $S_v = \{x : x \leq_L v\}$, and $T_v = V(G)\setminus S_v$. We need to show that $\rho_G(S_v) \leq k$ for all $v \in V(G)$. This is clearly true if $v \in V(G_1)\cup V(G_2)$. So let us assume that $v \in V(G_3)$. By symmetry we may assume $v_3 \notin S_v$, because we can swap G_1 and G_2. Then no vertex of G_2 has a neighbor in $S_v \cap V(G_3)$ and therefore

$$\begin{aligned}\rho_G(S_v) &\leq \operatorname{rank}(A(G)[S_v\cap V(G_1), T_v]) + \operatorname{rank}(A(G)[S_v \cap V(G_3), T_v])\\ &= 1 + \rho_{G_3}(S_v \cap V(G_3)) \leq k.\end{aligned}$$

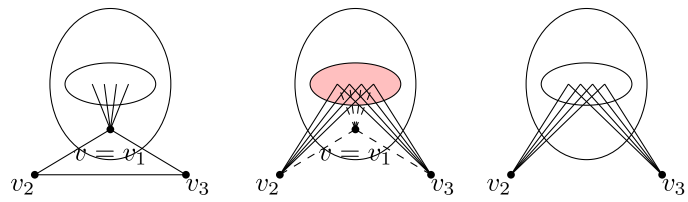
Therefore, G has linear rank-width at most k. □

3.3. **Linear rank-width of elementary vertex-minors of a graph in Δ_k.** We will prove that every elementary vertex-minor of G in Δ_k has linear rank-width at most k. To prove it, we will use the following lemmas.

Lemma 3.7 (Bouchet [3])**.** *Let G be a graph, v be a vertex of G and w be an arbitrary neighbor of v. Then every elementary vertex-minor obtained from G by deleting v is locally equivalent to either $G\setminus v$, $G*v\setminus v$, or $G\wedge vw\setminus v$.*

Lemma 3.8 (Bouchet [3, (8.2)]; see Oum [14])**.** *Let G be a graph and $vv_1, vv_2 \in E(G)$. Then $v_1v_2 \in E(G\wedge vv_1)$ and $G\wedge vv_1\wedge v_1v_2 = G\wedge vv_2$.*

By Lemma 3.7, it is sufficient to prove that $G\setminus v$, $G*v\setminus v$, and $G\wedge vw\setminus v$ has linear rank-width one less than the linear rank-width of G.



$G[V(G_1) \cup \{v_2, v_3\}]$ G_1' $G_1' * v_2$

FIGURE 4. The case $G * v \setminus v$ where $v = v_1$ in the proof of Lemma 3.10.

Lemma 3.9. *Let k be a non-negative integer and $G \in \Delta_k$. Then $G \setminus v$ has linear rank-width at most k for each vertex v.*

Proof. We use induction on k. We may assume $k \geq 1$. So G is a delta composition of three graphs in Δ_{k-1}, say G_1, G_2 and G_3 with the main triangle $v_1v_2v_3$ such that $v_i \in V(G_i)$ for $i = 1, 2, 3$. We may assume that $v \in V(G_1)$. By the induction hypothesis, $G_1 \setminus v$ has linear rank-width at most $k - 1$.

If $v = v_1$, then $G \setminus v$ is obtained from the disjoint union of three graphs $G_1 \setminus v$, G_2, G_3 by adding an edge v_2v_3 and so $G \setminus v$ has linear rank-width k by Lemma 3.5.

If $v \neq v_1$, then $G \setminus v$ is a delta composition of two graphs in Δ_{k-1} and one graph having linear rank-width at most $k - 1$. Thus by Lemma 3.6, $\operatorname{lrw}(G \setminus v) = k$. $\square$

Lemma 3.10. *Let k be a non-negative integer and $G \in \Delta_k$. Then $G * v \setminus v$ has linear rank-width at most k for each vertex v.*

Proof. We use induction on k. We may assume $k \geq 1$. Let G be a delta composition of $G_1, G_2, G_3 \in \Delta_{k-1}$ with the main triangle $v_1v_2v_3$ such that $v_i \in V(G_i)$ for $i = 1$, 2, 3. We may assume that $v \in V(G_1)$.

If $v \neq v_1$, then $G * v \setminus v$ is a delta composition of $G_1 * v \setminus v$, G_2 and G_3 where $G_1 * v \setminus v$ has linear rank-width at most $k - 1$ by the induction hypothesis. Thus by Lemma 3.6, $G * v \setminus v$ has linear rank-width k.

So we may assume $v = v_1$. let $G_1' = G * v \setminus v \setminus (V(G_2) \cup V(G_3) \setminus \{v_2, v_3\})$. Since v_3 is a twin of v_2 and v_3 is not adjacent to v_2 in $G_1' * v_2$ and $G_1' * v_2 \setminus v_3$ is isomorphic to G_1 (see Figure 4), by Lemma 3.3, G_1' has a linear layout $(v_2) \oplus L_1 \oplus (v_3)$ of width k.

By Proposition 3.4, G_2 has a linear layout L_2 of width k whose last vertex is v_2, and G_3 has a linear layout L_3 of width k whose first vertex is v_3.

It follows easily that $L = L_2 \oplus L_1 \oplus L_3$ is a linear layout of $G * v \setminus v$ having width k because $(G * v \setminus v)[V(G_2)] = G_2$, $(G * v \setminus v)[V(G_3)] = G_3$, and $(G * v \setminus v)[V(G_1) \cup \{v_2, v_3\}] = G_1'$. $\square$

Lemma 3.11. *Let k be a non-negative integer and $G \in \Delta_k$. Then $G \wedge vw \setminus v$ has linear rank-width at most k for each edge vw.*

Proof. For each vertex v, it is enough to prove it for one neighbor w of v by Proposition 2.1 and Lemma 3.8.

We use induction on k. We may assume $k \geq 1$. Let G be a delta composition of $G_1, G_2, G_3 \in \Delta_{k-1}$ with the main triangle $v_1v_2v_3$ such that $v_i \in V(G_i)$ for $i = 1$, 2, 3. We may assume that $v \in V(G_1)$.

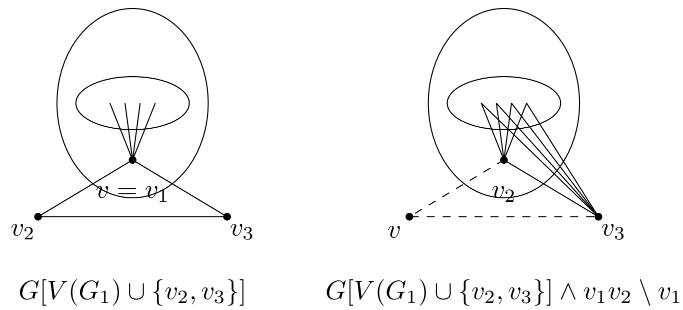


FIGURE 5. The case $G \wedge v_1v_2 \setminus v$ in the proof of Lemma 3.11.

If v has only one neighbor w, then $G \wedge vw \setminus v$ is isomorphic to $G \setminus w$ and by Lemma 3.9 we know that $G \setminus w$ has linear rank-width at most k. So we may assume that v has at least two neighbors.

If $v \neq v_1$, then we choose a neighbor w of v such that $w \neq v_1$. It is easy to observe that $G \wedge vw \setminus v$ is a delta composition of $G_1 \wedge vw \setminus v$, G_2, G_3 where $G_1 \wedge vw \setminus v$ has linear rank-width at most $k-1$ by the induction hypothesis. Hence, by Lemma 3.6, $G \wedge vw \setminus v$ has linear rank-width k.

Thus we may assume $v = v_1$. Since $G[V(G_1) \cup \{v_2, v_3\}] \wedge vv_2 \setminus v$ is isomorphic to a graph obtained from G_1 by adding a twin of v (see Figure 5), by Proposition 3.3, $G[V(G_1) \cup \{v_2, v_3\}] \wedge vv_2 \setminus v$ has a linear layout $(v_2) \oplus L_1 \oplus (v_3)$ of width k.

Let w be a neighbor of v in G_1 and let $G_1' = G[V(G_1) \cup \{v_2, v_3\}] \wedge vw \setminus v$. By Lemma 3.8, $G_1' \wedge v_2w = G[V(G_1) \cup \{v_2, v_3\}] \wedge vw \wedge v_2w \setminus v = G[V(G_1) \cup \{v_2, v_3\}] \wedge vv_2 \setminus v$ and therefore $(v_2) \oplus L_1 \oplus (v_3)$ is also a linear layout of G_1' having width k.

By Proposition 3.4, G_2 has a linear layout L_2 of width k whose last vertex is v_2, and G_3 has a linear layout L_3 of width k whose first vertex is v_3.

It is now easy to see that $L = L_2 \oplus L_1 \oplus L_3$ is a linear layout of $G \wedge vw \setminus v$ having width at most k because $(G \wedge vw \setminus v)[V(G_2)] = G_2$, $(G \wedge vw \setminus v)[V(G_3)] = G_3$, and $(G \wedge vw \setminus v)[V(G_1) \cup \{v_2, v_3\}] = G_1'$. □

Finally we are ready to prove the main theorem of this section.

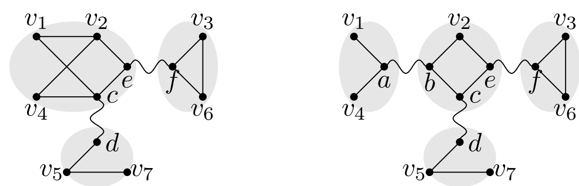
Proof of Theorem 3.1. Let $G \in \Delta_k$. By Proposition 3.4, G has linear rank-width $k+1$. And by lemmas 3.9, 3.10, and 3.11, every elementary vertex-minor of G has linear rank-width at most k. Therefore, G is an excluded vertex-minor for graphs of linear rank-width at most k. □

4. LOCALLY EQUIVALENT GRAPHS IN Δ_k ARE ISOMORPHIC

In this section, we will prove that if two graphs in Δ_k are locally equivalent, then they are isomorphic. We will prove the theorem for a more general class of graphs containing Δ_k.

A *block* in a graph G is a maximal connected subgraph of G having no cut-vertices. A graph is a *block graph* if every block of it is a complete graph. It is easy to see that every induced subgraph of a block graph is a block graph.

A partition (A, B) of $V(G)$ is a *split* of a graph G if $|A| \geq 2$, $|B| \geq 2$, and $\rho_G(A) \leq 1$.



($\Longrightarrow$) Replacing a bag with a simple decomposition of it
($\Longleftarrow$) Recomposing along a marked edge ab

FIGURE 6. Two operations on a split decomposition.

We first show that every graph in Δ_k is a block graph without simplicial vertices of degree at least 2.

Lemma 4.1. *Every graph in Δ_k is a block graph without simplicial vertices of degree at least* 2.

Proof. Let G be a graph in Δ_k. From the construction of Δ_k, every vertex of G has odd degree and each block of G is isomorphic to K_2 or K_3. Therefore G is a block graph and has no simplicial vertex of degree at least 2. $\square$

We will prove that if two block graphs without simplicial vertices of degree at least 2 are locally equivalent, then they are isomorphic. We will use the *canonical decomposition* of a graph, a useful tool introduced by Cunningham [7].

4.1. **Canonical decompositions of a connected graph.** In this subsection, we will define the canonical decompositions of a connected graph, following the presentation by Bouchet [4], and discuss the canonical decompositions of locally equivalent graphs.

A *marked graph* is a graph with a set of *marked* edges, and for a marked graph D, let $M(D)$ be the set of all marked edges of D. A *marked vertex* of a marked graph is a vertex incident with some marked edges.

A graph without splits is called *prime*. If G has a split (A, B), then a marked graph G' is called a *simple decomposition* of G if G' is obtained from the disjoint union of $G[A]$ and $G[B]$ by adding two new vertices a and b, adding a marked edge ab, making a adjacent to all vertices in A having neighbors in B in G, and making b adjacent to all vertices in B having neighbors in A in G.

A *split decomposition* of a connected graph G is recursively defined to be either G with no marked edges or a marked graph obtained from a split decomposition D by replacing a component H of $D \setminus M(D)$ with a simple decomposition of H.

Let D be a split decomposition of a connected graph G. Clearly D is connected. Each component of $D \setminus M(D)$ is called a *bag* of D. If ab is a marked edge in a split decomposition D, then $D \wedge ab \setminus a \setminus b$ is called a split decomposition obtained by *recomposing* ab. See Figure 6 for an example. Given a split decomposition D, we can recover the graph G by recomposing all marked edges. Note that the set of vertices of G is exactly the set of all unmarked vertices of D.

Note that every marked vertex in a bag represent at least one vertex in the original graph. It is easy to observe the following.

Lemma 4.2. *Let D be a split decomposition of a connected graph G. If B is a bag of D, then G has an induced subgraph isomorphic to B.*

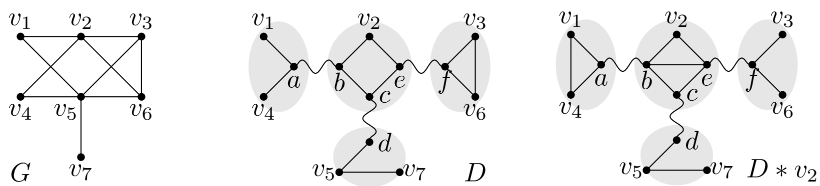


FIGURE 7. From a split decomposition D of a graph G, we obtain a split decomposition $D * v_2$ of $G * v_2$. Note that v_2 is represented by v_2, a, f in D.

A bag is called *star* if it is isomorphic to $K_{1,n}$ for some $n \geq 2$ and it is called *complete* if it is isomorphic to K_n for some $n \geq 1$. A non-leaf vertex of a star bag is called the *center*. Two bags C_1 and C_2 of D are *neighbors* if there exist $v_1 \in V(C_1)$, $v_2 \in V(C_2)$ such that $v_1v_2 \in M(D)$. A split decomposition D of a connected graph is called the *canonical decomposition* if it satisfies the following:

(i) each bag of D is prime, star, or complete,
(ii) no two complete bags are neighbors,
(iii) if two star bags are neighbors and e is the marked edge connecting them, then two end vertices of e are both centers or both leaves of the bags.

The conditions (ii) and (iii) can be justified as follows. If there are two complete bags that are neighbors, then we can recompose them to create a bigger complete bag. If there are two star bags having a marked edge joining a center of one to a leaf of another, we can also recompose them to make a bigger star bag. Thus the conditions (ii) and (iii) ensure that we do not decompose a complete or star bag. It turns out that each connected graph has a unique canonical decomposition.

Lemma 4.3 (Cunningham [7])**.** *Every connected graph has a unique canonical decomposition.*

In Appendix A. we present the canonical decompositions of graphs in Δ_k.

A path in a marked graph is *alternating* if every second edge is marked and other edges are unmarked. Let D be a split decomposition of a connected graph G. Two unmarked vertices x and y are *linked* in D if D has an alternating path from x to y. The proof of the following lemma is an easy induction on the number of bags of the decomposition.

Lemma 4.4. *Let D be a split decomposition of a connected graph G and let v, w be two distinct unmarked vertices of D. Then v and w are linked in D if and only if $vw \in E(G)$.*

For $v \in V(G)$, we say a vertex w in D *represents* v if D has an alternating path from v to w with even length (possibly 0).

For $v \in V(G)$, let $D * v$ be a marked graph obtained from D by replacing B with $B * w$ for each bag B of D having a vertex w representing v. See Figure 7 for an example. It is easy to observe that $D * v$ is a split decomposition of $G * v$. Moreover, the following lemma is known.

Lemma 4.5 (Bouchet [4])**.** *If D is the canonical decomposition of a connected graph G and $v \in V(G)$, then $D * v$ is the canonical decomposition of $G * v$.*

A graph G is *distance-hereditary* [2] if for each connected induced subgraph H of G and two distinct vertices x, y in H, their distance in H is the same as in G. It is known that connected distance-hereditary graphs are exactly the graphs having the canonical decomposition whose bags are either star or complete [4]. It is easy to see that every block graph is distance-hereditary [2].

4.2. **Canonical decompositions of block graphs.** A *diamond graph* is the graph obtained from K_4 by removing one edge. By definition, neither a diamond graph nor C_k for $k \geq 4$ is a block graph. Actually Bandelt and Murder [2] showed that a graph is a block graph if and only if it has no induced subgraph isomorphic to a diamond graph or C_k for $k \geq 4$.

In the following proposition, we will characterize block graphs from their canonical decompositions.

Proposition 4.6. *Let D be the canonical decomposition of a connected graph G. Then G is a block graph if and only if every bag of D is either star or complete, and the center of each star bag of D is unmarked.*

Proof. We may assume that G is distance-hereditary because otherwise D has a bag that is neither star nor complete, and G is not a block graph.

We first suppose that D has a star bag B having a marked center w. There exists a marked edge ww' joining B with a bag B'. Since D is a canonical decomposition, B' is either complete or star with the center w'. If B' is complete, then by recomposing ww' we obtain a bag which has an induced subgraph isomorphic to a diamond graph. Thus G has an induced subgraph isomorphic to a diamond graph by Lemma 4.2. Since a diamond graph is not a block graph, we deduce that G is not a block graph. If B' is a star bag with the center w', then by recomposing ww', we obtain a bag which has an induced subgraph isomorphic to C_4. By Lemma 4.2, G should have an induced subgraph isomorphic to C_4, and therefore G is not a block graph.

To prove the converse, we claim a stronger statement: if D is a *split* decomposition of a connected graph G whose bags are star or complete and no center of a star bag in D is marked, then G is a block graph. We proceed by induction on $|V(D)|$. We may assume that D has a star bag B because otherwise G is a complete graph. Let v be the center of B. If B has another unmarked vertex w, then let G' be a graph obtained by recomposing all marked edges in $D \setminus w$. Here G is obtained from G' by adding a pendant vertex w to v. By the induction hypothesis, G' is a block graph and so is G. We may now assume that every vertex in B other than v is marked. Let $B = \{v, v_1, v_2, \ldots, v_n\}$ and let $v_1w_1, v_2w_2, \ldots, v_nw_n$ be the marked edges incident with B. Let D_i be the component of $D \setminus V(B)$ containing w_i. By the induction hypothesis, the graph G_i obtained by recomposing all marked edges in D_i is a block graph. The graph G is obtained from $G_1, G_2, \ldots, G_n$ by identifying $w_1, w_2, \ldots, w_n$ with a new vertex v. Since each block of G is a block of G_i for some i, we deduce that G is a block graph. □

We now characterize block graphs without simplicial vertices of degree at least 2 in terms of their canonical decompositions.

Proposition 4.7. *Let D be the canonical decomposition of a connected block graph G. Then G has a simplicial vertex of degree at least 2 if and only if D has a complete bag B having more than 2 vertices containing an unmarked vertex.*

Proof. Suppose that $v \in V(G)$ is a simplicial vertex of degree at least 2 in G. Clearly v is not a center of a star bag of D by Lemma 4.2. Because the center of a star bag is unmarked by Proposition 4.6 and v has degree at least 2, v cannot belong to a star bag. So v is in a complete bag of D.

Conversely suppose that D has a complete bag B having more than 2 vertices containing an unmarked vertex v. By Lemma 4.2, the degree of v is at least 2. Since all neighboring bags of B are star bags whose centers are unmarked by Proposition 4.6, v is a simplicial vertex of G. $\square$

4.3. **Generalizing a theorem of Bouchet.** Now we are ready to prove Theorem 1.4. This theorem is best possible for block graphs, because if v is a simplicial vertex of a block graph G, then $G * v$ is also a block graph.

Theorem 1.4. *If two block graphs without simplicial vertices of degree at least* 2 *are locally equivalent, then they are isomorphic.*

Proof. Suppose that two block graphs G and H without simplicial vertices of degree at least 2 are locally equivalent. Let D_G and D_H be the canonical decompositions of G and H, respectively. We may assume that $|V(G)| = |V(H)| \geq 3$ and therefore each bag of D_G or D_H has at least 3 vertices.

Since G and H are locally equivalent, by Lemma 4.5 we assume that D_H is obtained from D_G by a sequence of local complementation. Note that applying local complementation in a split decomposition does not change the number of marked vertices and unmarked vertices in each bag.

Suppose that a bag B of D_G corresponds to a bag $B' = D_H[V(B)]$ of D_H. If B is a complete bag in D_G, then by Proposition 4.7, B has no unmarked vertex in D_G and therefore B' has no unmarked vertex in D_H. Since every star bag of D_H should have at least one unmarked vertex by Proposition 4.6, B' is a complete bag in D_H. Similarly, if B' is a complete bag in D_H, then B is a complete bag in D_G.

Thus B is a star bag of D_G if and only if B' is a star bag of D_H. By Proposition 4.6, the center of a star bag in D_G or D_H is an unmarked vertex. Since a bag B in D_G and B' in D_H have the same number of neighbor bags and unmarked vertices in each canonical decomposition, the unmarked vertices of B in D_G must be mapped to the unmarked vertices of B' in D_H. Therefore, D_G is isomorphic to D_H and so G is isomorphic to H. $\square$

5. The number of non-isomorphic graphs in Δ_k

In this section, we will prove that Δ_k has at least $2^{\Omega(3^k)}$ pairwise non-isomorphic graphs. A *rooted graph* is a pair of a graph and a specified vertex called a *root.* Two rooted graphs (G, v) and (G', v') are *isomorphic* if there exists a graph isomorphism ϕ from G to G' that maps v to v'. Let us write $\mathrm{Aut}(G)$ to denote the automorphism group of a graph G. For a rooted graph (G, v), we write $\mathrm{Aut}(G, v)$ to denote the automorphism group of (G, v). In other words, $\mathrm{Aut}(G, v) = \{\phi \in \mathrm{Aut}(G) : \phi(v) = v\}$.

First we show that each graph in Δ_k has a unique main triangle.

Lemma 5.1. *Let $k \geq 1$ and $G \in \Delta_k$. Then G has a unique cycle $v_1v_2v_3$ of length* 3 *such that $G \setminus v_1v_2 \setminus v_2v_3 \setminus v_3v_1$ has exactly three components G_1, G_2, G_3, each of which is in Δ_{k-1}.*

Proof. Clearly there is at least one such cycle because of the construction. Suppose there are two such cycles $T = v_1v_2v_3$ and $T' = v_1'v_2'v_3'$. Let H be a component of $G \setminus v_1v_2 \setminus v_2v_3 \setminus v_3v_1$ having no vertex of T'. By the condition, $H \in \Delta_{k-1}$ and so H has exactly $2 \cdot 3^{k-1}$ vertices. We may assume $v_1 \in V(H)$. The component J of $G \setminus v_1'v_2' \setminus v_2'v_3' \setminus v_3'v_1'$ intersecting $V(H)$ should be equal to H because T' does not intersect H and $|V(J)| = |V(H)|$. Thus $v_2, v_3 \in T'$ and so v_2 and v_3 have a common neighbor other than v_1. However, this contradicts our assumption that $G \setminus v_1v_2 \setminus v_2v_3 \setminus v_3v_1$ has exactly three components. □

Let $k \geq 2$ and let G be a graph in Δ_k. By the construction, G is a delta composition of three graphs $G_1, G_2, G_3 \in \Delta_{k-1}$ with the main triangle $v_1v_2v_3$ such that $v_i \in V(G_i)$ for $i = 1, 2, 3$. We call $G \in \Delta_k$

- *Type-A* if (G_1, v_1), (G_2, v_2), and (G_3, v_3) are pairwise isomorphic,
- *Type-B* if exactly two of (G_1, v_1), (G_2, v_2), (G_3, v_3) are isomorphic,
- *Type-C* otherwise.

Lemma 5.2. *Let $k \geq 1$ and G be a delta composition of three graphs $G_1, G_2, G_3 \in \Delta_{k-1}$ with the main triangle $v_1v_2v_3$ such that $v_i \in V(G_i)$ for all $i = 1, 2, 3$. Then,*

(1) $\mathrm{Aut}(G) \simeq S_3 \times \mathrm{Aut}(G_1, v_1) \times \mathrm{Aut}(G_2, v_2) \times \mathrm{Aut}(G_3, v_3)$ *if G is Type-A.*
(2) $\mathrm{Aut}(G) \simeq S_2 \times \mathrm{Aut}(G_1, v_1) \times \mathrm{Aut}(G_2, v_2) \times \mathrm{Aut}(G_3, v_3)$ *if G is Type-B.*
(3) $\mathrm{Aut}(G) \simeq \mathrm{Aut}(G_1, v_1) \times \mathrm{Aut}(G_2, v_2) \times \mathrm{Aut}(G_3, v_3)$ *if G is Type-C.*

Proof. Let $g \in \mathrm{Aut}(G)$. By Lemma 5.1, $g(\{v_1, v_2, v_3\}) = \{v_1, v_2, v_3\}$ and therefore $g(V(G_1)), g(V(G_2)), g(V(G_3)) \in \{V(G_1), V(G_2), V(G_3)\}$. So $\mathrm{Aut}(G)$ induces a subgroup Γ of S_3 on $\{v_1, v_2, v_3\}$ based on the type of G. It is clear that $\mathrm{Aut}(G)/\Gamma$ is a composition of automorphism groups of three rooted graphs (G_1, v_1), (G_2, v_2) and (G_3, v_3). □

For a graph G and $x \in V(G)$, we define the *orbit* of x in G as the set

$$\{w \in V(G) : w = f(x) \text{ for some automorphism } f \text{ of } G\},$$

and we denote $\#\mathrm{Orb}(G)$ as the number of all distinct orbits of G. For a rooted graph (G, v) and $x \in V(G)$, we define the *orbit* of x in (G, v) as the set

$$\{w \in V(G) : w = f(x) \text{ for some automorphism } f \text{ of } (G, v)\},$$

and we denote $\#\mathrm{Orb}(G, v)$ as the number of all distinct orbits of (G, v).

Lemma 5.3. *Let $k \geq 1$ and G be a delta composition of three graphs $G_1, G_2, G_3 \in \Delta_{k-1}$ with the main triangle $v_1v_2v_3$ such that $v_i \in V(G_i)$ for all $i = 1, 2, 3$. If $v \in V(G_1)$, then*

$$\#\mathrm{Orb}(G, v) \geq \#\mathrm{Orb}(G_1, v_1) + \#\mathrm{Orb}(G_2, v_2).$$

Proof. By Lemma 5.1, no vertex in G_1 can be mapped to a vertex in G_2 or G_3 by an automorphism of G fixing v. Thus orbits of (G, v) intersecting $V(G_1)$ cannot contain a vertex in G_2 or G_3. The number of orbits of (G, v) intersecting $V(G_1)$ is equal to the number of automorphisms of G_1 fixing both v and v_1 and this number is at least $\#\mathrm{Orb}(G_1, v_1)$. The number of orbits of (G, v) not intersecting $V(G_1)$ is at least $\#\mathrm{Orb}(G_2, v_2)$ by Lemma 5.2. Thus, we obtain the desired inequality. □

Lemma 5.4. *Let k be a non-negative integer and $G \in \Delta_k$ and $v \in V(G)$. Then (G, v) has at least 2^{k+1} orbits.*

Proof. Trivial if $k = 0$. It follows easily by induction from Lemma 5.3. $\square$

Lemma 5.5. *Let k be a positive integer and $G \in \Delta_k$.*

(1) If G is Type-A, then G has at least 2^k orbits.
(2) If G is Type-B, then G has at least $2 \cdot 2^k$ orbits.
(3) If G is Type-C, then G has at least $3 \cdot 2^k$ orbits.

Proof. Let G be a delta composition of $G_1, G_2, G_3 \in \Delta_{k-1}$ with the main triangle $v_1v_2v_3$ such that $v_i \in V(G_i)$ for all $i = 1, 2, 3$. By Lemma 5.2,

(1) $\#\mathrm{Orb}(G) = \#\mathrm{Orb}(G_1, v_1)$ if G is Type-A,
(2) $\#\mathrm{Orb}(G) = \#\mathrm{Orb}(G_1, v_1) + \#\mathrm{Orb}(G_2, v_2)$ if G is Type-B and (G_1, v_1) is isomorphic to (G_3, v_3),
(3) $\#\mathrm{Orb}(G) = \#\mathrm{Orb}(G_1, v_1) + \#\mathrm{Orb}(G_2, v_2) + \#\mathrm{Orb}(G_3, v_3)$ if G is Type-C.

By Lemma 5.4, we deduce the lemma. $\square$

Let p_k be the number of non-isomorphic rooted graphs (G, v) with $G \in \Delta_k$. Then $p_0 = 1$, $p_1 = 2$, and $p_2 = 24$ (see Figure 3). We can easily verify that Δ_k has

- exactly p_{k-1} non-isomorphic Type-A graphs,
- exactly $p_{k-1}(p_{k-1} - 1)$ non-isomorphic Type-B graphs,
- exactly $\binom{p_{k-1}}{3}$ non-isomorphic Type-C graphs.

We are now ready to provide a lower bound on the number of non-isomorphic graphs in Δ_k.

Proposition 5.6. *Let $k \geq 2$ be an integer. Then Δ_k has at least $2^{\Omega(3^k)}$ non-isomorphic graphs.*

Proof. Let a_k, b_k, c_k be the number of non-isomorphic graphs in Δ_k that is Type-A, Type-B, and Type-C respectively. By Lemma 5.5,

$$p_k \geq 2^k a_k + 2 \cdot 2^k b_k + 3 \cdot 2^k c_k.$$

Since $a_k = p_{k-1}$, $b_k = p_{k-1}(p_{k-1} - 1)$ and $c_k = \binom{p_{k-1}}{3}$, we obtain the following recurrence relation;

$$a_{k+1} = p_k \geq 2^k a_k + 2 \cdot 2^k b_k + 3 \cdot 2^k c_k = 2^{k-1} a_k^2 (a_k + 1) \geq 2^{k-1} a_k^3$$

and $a_2 = 2$. We deduce that $a_k \geq 2^{(1-2k)/4 + 7 \cdot 3^k/36} = 2^{\Omega(3^k)}$. $\square$

Now we can combine all to prove our main theorem.

Proof of Theorem 1.3. By Theorems 3.1, $\mathcal{O}_k$ must contain a graph locally equivalent to each graph in Δ_k. Proposition 5.6 states that Δ_k has at least $2^{\Omega(3^k)}$ non-isomorphic graphs. Lemma 4.1 and Theorem 1.4 show that two non-isomorphic graphs in Δ_k cannot be locally equivalent. Therefore, $|\mathcal{O}_k| \geq 2^{\Omega(3^k)}$. $\square$

6. Concluding remarks

We present $2^{\Omega(3^k)}$ lower bound of the number of pairwise locally non-equivalent vertex-minor minimal graphs with the property that they have linear rank-width larger than k.

A question naturally arises in the context.

Question 1. Find an explicit upper bound on the number of vertices in a graph that is vertex-minor minimal with the property having linear rank-width larger than k.

So far, we do not know any explicit upper bound; its existence is given by Corollary 1.2. The only known fixed-parameter algorithm to decide linear rank-width at most k is based on this list; it uses the modulo-2 counting monadic second-order logic formula to decide whether a given graph has linear rank-width at most k by using the existence of forbidden vertex-minors. However, no explicit methods are known to construct such a list of forbidden vertex-minors and so perhaps we can say "we know such an algorithm exists but we do not know what it is."

A similar trouble appears in the problem of deciding rank-width at most k in Courcelle and Oum [6]. But for rank-width, there is an explicit upper bound on the number of vertices of forbidden vertex-minors [14] and therefore in theory, one can enumerate all graphs up to that bound and construct the list of forbidden vertex-minors. If we resolve the above question, then we will be able to *construct* a fixed parameter algorithm to decide linear rank-width at most k.

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Appendix A. Canonical decompositions of graphs in Δ_k

We now aim to describe the canonical decomposition D_G of each graph G in Δ_k for $k \geq 1$ explicitly. Let us call the edges of the graph in Δ_0 *thick*. In graphs in Δ_k, the edges originated from Δ_0 are *thick* and all other edges introduced by a delta composition are *thin*. Observe the set of thick edges of $G \in \Delta_k$ is a perfect matching and therefore we deduce the following.

Lemma A.1. *For graphs in Δ_k, each leaf is incident only with a thick edge and no two leaves have a common neighbor.*

For $G \in \Delta_k$, let $\mathcal{C}(G)$ be the set of triangles in G. First let us describe the set $\Theta(G)$ of marked vertices of D_G. For each thick edge uv joining two non-leaf vertices, we have two new vertices $m(u, v)$ and $m(v, u)$ in $\Theta(G)$ and for each pair of a vertex v and a triangle C containing v, we have two new vertices $m(C, v)$ and $m(v, C)$. We will construct D_G so that $V(D_G)$ is the disjoint union of $V(G)$ and $\Theta(G)$. For convenience, if w is a leaf incident with an (thick) edge vw, then $m(v, w) := w$.

Now we describe all bags of D_G. For each vertex v in G of degree $n > 1$, if w is the unique neighbor of v joined by a thick edge, then let $B(v)$ be the graph isomorphic to $K_{1,(n-1)/2+1}$ on the vertex set

$$\{v, m(v, w)\} \cup \{m(v, C) : C \in \mathcal{C}(G), v \in V(C)\}$$

with the center v. For each triangle C of G, let $B(C)$ be the graph isomorphic to K_3 on the vertex set $\{m(C, v) : v \in V(C)\}$.

Let D_G be the marked graph on the vertex set $\Theta(G) \cup V(G)$ such that all bags of D_G are

$$\{B(v) : v \text{ is a non-leaf vertex in } G\} \cup \{B(C) : C \in \mathcal{C}(G)\}$$

and the set $M(D_G)$ of all marked edges is exactly

$$\begin{aligned}&\{m(v, C)m(C, v) : C \in \mathcal{C}(G), v \in V(C)\}\\ &\qquad \cup \{m(v, w)m(w, v) : vw \text{ is the thick edge joining two non-leaf vertices}\}\end{aligned}$$

For a graph G in Δ_2, the marked graph D_G is depicted in Figure 8.

We now show that if $G \in \Delta_k$, then D_G is the canonical decomposition of G.

Proposition A.2. *For each graph $G \in \Delta_k$ with $k \geq 1$, the marked graph D_G is the canonical decomposition of G.*

Proof. We first prove that D_G is a split decomposition of G. We use induction on k. We may assume that $k \geq 2$ and let C be the main triangle $v_1v_2v_3$ of G. For each $1 \leq i \leq 3$, let G_i be the component of $G \setminus v_1v_2 \setminus v_2v_3 \setminus v_3v_1$ such that $v_i \in V(G_i)$, and let D_i be the component of $D_G[V(D_G) \setminus \{m(C, v_1), m(C, v_2), m(C, v_3)\}]$ such that $v_i \in V(D_i)$. Let w_i be the neighbor of v_i such that v_iw_i is thick.

If v_i is not a leaf in G_i, then by construction, $D_i \setminus m(v_i, C) = D_{G_i}$. If v_i is a leaf of G_i, then D_{G_i} is obtained from $D_i \setminus m(v_i, C)$ by recomposing a marked edge joining $m(v_i, w_i)$ and $m(w_i, v_i)$. By the induction hypothesis, D_{G_i} is a split decomposition of G_i and therefore in both cases, $D_i \setminus m(v_i, C)$ is a split decomposition of G_i because we obtain G_i from $D_i \setminus m(v_i, C)$ by recomposing all marked edges of $D_i \setminus m(v_i, C)$.

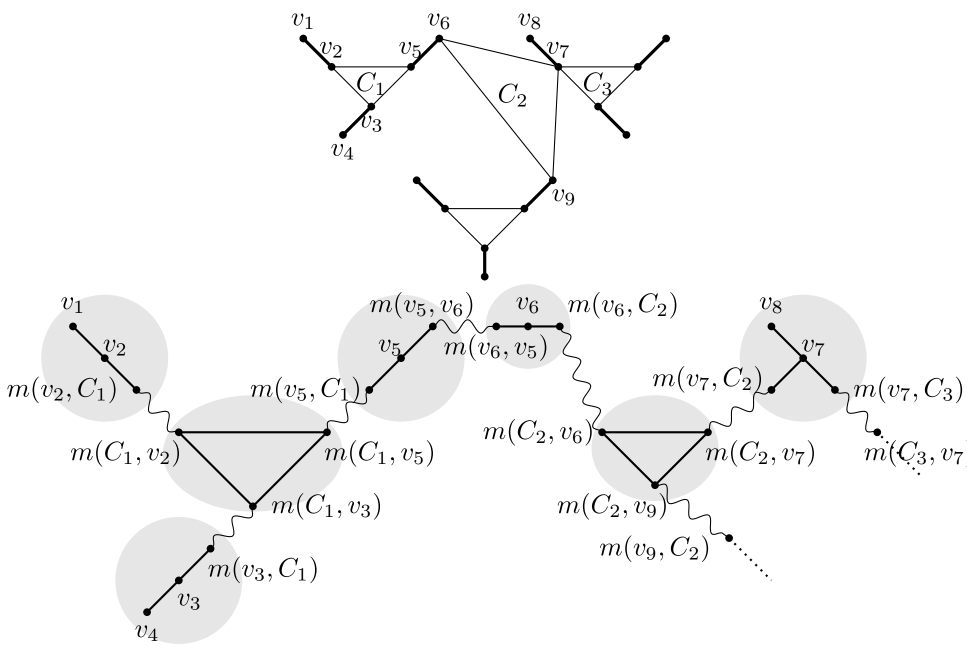

FIGURE 8. A graph $G \in \Delta_2$ with thick edges, and a part of D_G.

Let G_i' be the graph obtained from D_i by recomposing all marked edges of D_i. Then $m(v_i, C)$ is a leaf of G_i' and $G_i' \setminus m(v_i, C) = G_i$.

If we recompose all marked edges of D_G except three marked edges associated with C, then we obtain a marked graph obtained from the disjoint union of G_1', G_2', G_3', and $B(C)$ by adding three marked edges in $\{m(v_i, C), m(C, v_i)\}_{1\le i\le 3}$. It is then clear that G is obtained from this graph by recomposing three marked edges in $\{m(v_i, C), m(C, v_i)\}_{1\le i\le 3}$ from this graph. This proves that D_G is a split decomposition of G.

It remains to check that D_G is a canonical decomposition. From the construction, every bag of D_G is a complete bag or a star bag, and every star bag has marked vertices only on its leaves and no two complete bags are neighbors. This proves the lemma. □

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