

EVEN-CYCLE DECOMPOSITIONS OF GRAPHS WITH NO ODD- K_4 -MINOR

TONY HUYNH, SANG-IL OUM, AND MARYAM VERDIAN-RIZI

ABSTRACT. An *even-cycle decomposition* of a graph G is a partition of $E(G)$ into cycles of even length. Evidently, every Eulerian bipartite graph has an even-cycle decomposition. Seymour (1981) proved that every 2-connected loopless Eulerian planar graph with an even number of edges also admits an even-cycle decomposition. Later, Zhang (1994) generalized this to graphs with no K_5 -minor.

Our main theorem gives sufficient conditions for the existence of even-cycle decompositions of graphs in the absence of *odd minors*. Namely, we prove that every 2-connected loopless Eulerian odd- K_4 -minor-free graph with an even number of edges has an even-cycle decomposition.

This is best possible in the sense that ‘odd- K_4 -minor-free’ cannot be replaced with ‘odd- K_5 -minor-free.’ The main technical ingredient is a structural characterization of the class of odd- K_4 -minor-free graphs, which is due to Lovász, Seymour, Schrijver, and Truemper.

1. INTRODUCTION

A graph G is *even-cycle decomposable* if its edge set can be partitioned into even cycles. Note that if G is even-cycle decomposable, then necessarily G is Eulerian, loopless, and $|E(G)|$ is even. For bipartite graphs, these conditions are also sufficient, since every cycle is even.

Proposition 1.1 (Euler). *Every Eulerian bipartite graph is even-cycle decomposable.*

One motivation to study even-cycle decompositions is their connection to the four colour theorem [1, 9]. For example, as noted by Seymour [11], one consequence of the four colour theorem is that every 2-connected cubic planar graph has a set of *even cycles* in which each edge occurs exactly twice. In the same paper, Seymour also proves that planar graphs (which satisfy the obvious necessary conditions) are always even-cycle decomposable.

Theorem 1.2 (Seymour [11]). *Every 2-connected Eulerian loopless planar graph with an even number of edges is even-cycle decomposable.*

Date: April 24, 2017.

Key words and phrases. cycle, even-cycle decomposition, Eulerian graph, odd minor.

Supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2012-0004119). S. O. is also supported by TJ Park Junior Faculty Fellowship.

T. H., S. O., and M. V.-R. are supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Science, ICT & Future Planning (2011-0011653). T. H. was also supported by the NWO (The Netherlands Organization for Scientific Research) free competition project “Matroid Structure - for Efficiency” led by Bert Gerards.

Note that the 2-connected condition is with little loss of generality, since a graph G is even-cycle decomposable if and only if each block of G is even-cycle decomposable.

Later, Zhang [15] generalized Theorem 1.2 to graphs with no K_5 -minor.

Theorem 1.3 (Zhang [15]). *Every 2-connected Eulerian loopless K_5 -minor-free graph with an even number of edges is even-cycle decomposable.*

In this paper we consider sufficient conditions for the existence of even-cycle decompositions in graphs with no *odd- K_t -minor* (definitions are deferred until the next section). For further information on even-cycle decomposable graphs and related results we refer the reader to the surveys of Jackson [7] or Fleischner [3].

Our main result is the following.

Theorem 1.4. *Every 2-connected Eulerian loopless odd- K_4 -minor-free graph with an even number of edges is even-cycle decomposable.*

Theorem 1.4 is best possible in the sense that ‘odd- K_4 -minor-free’ cannot be replaced with ‘odd- K_5 -minor-free.’

Theorem 1.5. *There exists a 2-connected Eulerian loopless odd- K_5 -minor-free graph with an even number of edges which is not even-cycle decomposable.*

In a previous version of this paper, we conjectured that all 2-connected Eulerian loopless odd- K_5 -minor-free graphs with an even number of edges are even-cycle decomposable. Note that if true, this conjecture would simultaneously imply both Proposition 1.1 and Theorem 1.3. By a celebrated theorem of Guenin [6], a signed graph is odd- K_5 -minor-free if and only if it is weakly bipartite (as defined by Grötschel and Pulleyblank [5]). See also Naserasr, Rollová, and Sopena [8] for connections between odd- K_5 -minor-free signed graphs and the 4-colour theorem.

However, Cheolwon Heo disproved our conjecture. We would like to thank him for graciously allowing us to include the proof of Theorem 1.5 in this paper. The (graph version of the) example used in the proof of Theorem 1.5 also appears in [11], where it is used as a counterexample to the claim that all 2-connected Eulerian loopless graphs with an even number of edges is even-cycle decomposable. Note that K_5 is another counterexample to this claim.

For the proof of Theorem 1.4, a potentially useful inductive tool is the following nice theorem of Conlon [2].

Theorem 1.6 (Conlon [2]). *Let G be a simple 3-connected graph of minimum degree 4. If G is not isomorphic to K_5 , then G contains an even cycle C such that $G \setminus E(C)$ is 2-connected.*

Unfortunately the assumptions in Theorem 1.6 are much too strong for our purposes. Instead, our result relies on a structural description of the signed graphs with no odd- K_4 -minor. According to Gerards [4], this structure theorem is due to Lovász, Seymour, Schrijver, and Truemper. The proof of the structure theorem makes use of the regular matroid decomposition theorem of Seymour [10]. It is also a special instance of a decomposition theorem for binary matroids with no F_7 -minor using a fixed element due to Truemper and Tseng [12, 13] (F_7 denotes the Fano matroid).

Our strategy for proving Theorem 1.4 is to first prove it for the two basic classes given by the structure theorem. We then show how to combine even-cycle

decompositions across low order separations. This last step contains some technical difficulties. Indeed, we end up having to apply the structure theorem *twice* (once on the original graph and then also on an auxiliary graph).

The graphs we consider are not necessarily simple, and we do not know an easy proof of our theorem for simple graphs. For example, the decomposition step will introduce parallel edges even if the original graph is simple. Our proof would be slightly shorter if we used Theorem 1.2, but we do a bit of extra work so that we avoid using Theorem 1.2.

The rest of the paper is organized as follows. In Section 2, we define signed graphs and prove Theorem 1.5. In Section 3, we present the structure theorem for signed graphs with no odd- K_4 -minor. In Section 4 and Section 5 we prove our main theorem for the two basic classes of signed graphs appearing in the structure theorem. Finally, in Section 6, we prove our main theorem.

2. SIGNED GRAPHS, RE-SIGNING, AND ODD MINORS

A *signed graph* is a pair (G, Σ) consisting of a graph G together with a *signature* $\Sigma \subseteq E(G)$. The edges in Σ are *negative* and the other edges are *positive*. A cycle (or path) is *balanced* if it contains an even number of negative edges; otherwise it is *unbalanced*. We say that a signed graph (G, Σ) is *balanced-cycle decomposable*, if $E(G)$ can be partitioned into balanced cycles of (G, Σ) .

For a signed graph (H, Σ) define $\mathbf{graph}(H, \Sigma)$ to be the graph obtained from H by subdividing every positive edge once. Then it is easy to observe the following lemma, because a cycle in (H, Σ) is balanced if and only if the corresponding cycle in $\mathbf{graph}(H, \Sigma)$ is even. This lemma will be used later as we will frequently reduce signed graphs to graphs.

Lemma 2.1. *A signed graph (H, Σ) is balanced-cycle decomposable if and only if $\mathbf{graph}(H, \Sigma)$ is even-cycle decomposable.*

For $X \subseteq V(G)$, we let $\delta_G(X)$ be the set of edges with exactly one end in X . We say that $\delta_G(X)$ is the *cut induced by X* . Two signatures $\Sigma_1, \Sigma_2 \subseteq E(G)$ are *equivalent* if their symmetric difference is a cut. The operation of changing to an equivalent signature is called *re-signing*. A key observation is that if $\Sigma_1, \Sigma_2 \subseteq E(G)$ are equivalent signatures, then (G, Σ_1) and (G, Σ_2) have exactly the same set of balanced cycles. Thus, for equivalent signatures Σ_1 and Σ_2 , (G, Σ_1) is balanced-cycle decomposable if and only if (G, Σ_2) is balanced-cycle decomposable.

We will require the following well-known lemma, first proved by Zaslavsky [14].

Lemma 2.2 (Zaslavsky [14]). *Let (G, Σ) be a signed graph. For every forest F which is a subgraph of G , there exists a signature which is disjoint from $E(F)$ and equivalent to Σ .*

A *minor* of a signed graph (G, Σ) is a signed graph that can be obtained from (G, Σ) by any of the following operations: re-signing, deleting edges or vertices, and contracting positive edges. For a graph H , *odd- H* is the signed graph $(H, E(H))$. A signed graph is *odd- H -minor-free* if it has no minor that is isomorphic to an odd- H . A graph G is *odd- H -minor-free* if the odd- G is odd- H -minor-free.

Now that our terms have been defined, we restate our main theorem.

Theorem 2.3. *Every 2-connected Eulerian loopless odd- K_4 -minor-free signed graph with an even number of negative edges is balanced-cycle decomposable.*

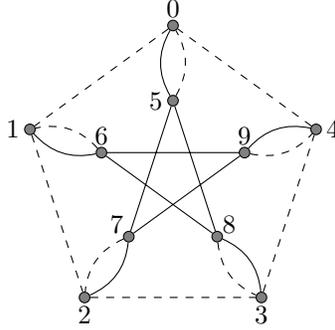


FIGURE 1. A 2-connected Eulerian signed graph with no odd- K_5 -minor and no balanced-cycle decompositions. (Solid lines denote negative edges.)

As we discussed in Section 1, K_4 cannot be replaced with K_5 in the above theorem. For that, we state an equivalent formulation of Theorem 1.5 and present a proof which is due to Cheolwon Heo.

Theorem 2.4. *There exists a 2-connected Eulerian loopless odd- K_5 -minor-free signed graph with an even number of negative edges which is not balanced-cycle decomposable.*

Proof. We claim that the signed graph (G, Σ) in Figure 1 is such a signed graph. Evidently, G is 2-connected, Eulerian and loopless. We claim that (G, Σ) is not balanced-cycle decomposable. Let C be an arbitrary cycle in a balanced-cycle decomposition of (G, Σ) . Note that G only contains cycles of length 2, 5, 6, 8, or 9. Since all 2-cycles are unbalanced, C has length 5, 6, 8, or 9. On the other hand, it is easy to check that if C has length 5 or 6, then some block of $(G \setminus E(C), \Sigma \setminus E(C))$ is an unbalanced 2-cycle, which is a contradiction. Thus, C has length 8 or 9. But $|E(G)| = 20$ and $8a + 9b \neq 20$ for all non-negative integers a, b .

It remains to show that (G, Σ) does not contain an odd- K_5 -minor. By degree considerations, the only way to obtain a K_5 -minor from the Petersen graph is to contract a perfect matching. Thus, to obtain odd- K_5 from (G, Σ) we must delete exactly one edge from each 2-cycle and then contract a perfect matching M . Let C be the balanced 5-cycle 57968. It is easy to see that $|M \cap E(C)| \in \{0, 2\}$. Thus, our odd- K_5 -minor either contains a balanced 5-cycle or a balanced 3-cycle, which is a contradiction. \square

3. STRUCTURE THEOREM FOR SIGNED GRAPHS WITH NO ODD- K_4 -MINOR

In this section we describe the structure of signed graphs with no odd- K_4 -minor. We begin by describing the basic building blocks in the decomposition theorem.

Almost balanced. A signed graph is *balanced* if every cycle is balanced. Note that Lemma 2.2 easily implies the following alternate definition of balanced signed graphs.

Lemma 3.1. *A signed graph is balanced if and only if we can re-sign so that all its edges are positive.*

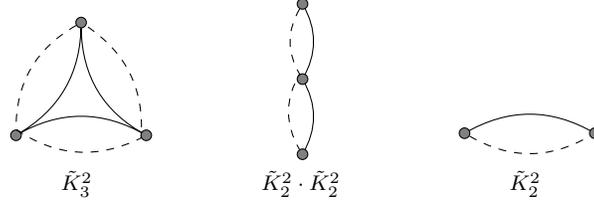


FIGURE 2. The signed graphs \tilde{K}_3^2 , $\tilde{K}_2^2 \cdot \tilde{K}_2^2$, and \tilde{K}_2^2 . (Solid lines denote negative edges.)

Since re-signing an Eulerian signed graph (G, Σ) does not change the parity of $|\Sigma|$, it follows that balanced Eulerian signed graphs always contain an even number of negative edges. We say that a signed graph is *almost balanced* if there exists a vertex whose deletion yields a balanced signed graph. Since balanced signed graphs have no odd- K_3 -minors, it follows that almost balanced signed graphs have no odd- K_4 -minors.

Planar with two unbalanced faces. A signed graph (G, Σ) is *planar* if the underlying graph G is planar. A face F of a planar embedding of (G, Σ) is *balanced* if the facial walk corresponding to F contains an even number of negative edges, otherwise F is *unbalanced*. We say that (G, Σ) is *planar with at most two unbalanced faces* if (G, Σ) has a planar embedding with at most two unbalanced faces. Notice that if every face of a planar embedding of (G, Σ) is balanced, then (G, Σ) is balanced.

Observe that the property of being planar with at most two unbalanced faces is preserved under taking minors and that odd- K_4 does not have this property. Therefore, signed graphs that are planar with at most two unbalanced faces do not have odd- K_4 -minors.

The signed graph \tilde{K}_3^2 . We define \tilde{K}_3^2 to be the signed graph (G, Σ) where G is a triangle with doubled edges and Σ is the edge-set of a triangle (see Figure 2). Evidently, \tilde{K}_3^2 has no odd- K_4 -minor, but it is neither almost balanced nor planar with two unbalanced faces. Two more small signed graphs that turn up in our proofs are \tilde{K}_2^2 (two vertices connected by a positive and a negative edge) and $\tilde{K}_2^2 \cdot \tilde{K}_2^2$ (two \tilde{K}_2^2 's joined at a vertex). See Figure 2 for pictures of \tilde{K}_3^2 , $\tilde{K}_2^2 \cdot \tilde{K}_2^2$, and \tilde{K}_2^2 .

These turn out to be essentially the only signed graphs with no odd- K_4 -minor. To define what we mean by ‘essentially’, we introduce the notion of separations. A *separation* of a graph G is a pair (G_1, G_2) where G_1 and G_2 are edge-disjoint subgraphs of G , such that $G_1 \cup G_2 = G$. The *boundary* of (G_1, G_2) is $V(G_1) \cap V(G_2)$, and its *order* is $|V(G_1) \cap V(G_2)|$. We say that (G_1, G_2) is a *proper separation* if both $V(G_1) \setminus V(G_2)$ and $V(G_2) \setminus V(G_1)$ are non-empty. A separation of order k is called a *k-separation*. A *separation* of a signed graph (G, Σ) is simply a separation of G .

We can now state the structure theorem.

Theorem 3.2 (Lovász, Seymour, Schrijver, and Truemper in Gerards [4, Theorem 3.2.4]). *Let (G, Σ) be a signed graph with no odd- K_4 -minor. Then at least one of the following holds.*

- (i) (G, Σ) is almost balanced, or planar with two unbalanced faces, or isomorphic to \tilde{K}_3^2 .
- (ii) (G, Σ) is not 2-connected.
- (iii) (G, Σ) has a 2-separation (G_1, G_2) where each $(G_i, E(G_i) \cap \Sigma)$ is connected and not a signed subgraph of \tilde{K}_2^2 .
- (iv) (G, Σ) has a 3-separation (G_1, G_2) where $(G_2, E(G_2) \cap \Sigma)$ is balanced, connected, and has at least 4 edges.

4. ALMOST BALANCED SIGNED GRAPHS

We begin by proving Theorem 2.3 for almost balanced signed graphs. We require the following lemma.

Lemma 4.1. *If G is a connected graph and X is a set of $2k$ vertices of G , then there is a collection of k pairwise edge-disjoint paths in G whose set of ends is precisely X .*

Proof. Let (G, X) be a counterexample with $|E(G)|$ minimum. Note that G must be a tree, since every spanning tree of G is also a counterexample. Next, observe that each leaf of G is in X , otherwise deleting such a leaf contradicts minimality. Let ℓ be a leaf and let w be the unique neighbour of ℓ . If $w \in X$, then $(G - \ell, X \setminus \{\ell, w\})$ is a smaller counterexample since we can link ℓ and w via the edge ℓw . On the other hand, if $w \notin X$, then deleting ℓ and adding w to X yields a smaller counterexample. \square

Proposition 4.2. *Every 2-connected Eulerian loopless almost balanced signed graph with an even number of negative edges is balanced-cycle decomposable.*

Proof. Let (H, Σ) be a 2-connected Eulerian loopless almost balanced signed graph with an even number of negative edges. Call a graph G *almost bipartite* if $G - v$ is bipartite for some $v \in V(G)$. By replacing (H, Σ) with $\text{graph}(H, \Sigma)$, it suffices to show that every 2-connected Eulerian loopless almost bipartite *graph* with an even number of edges is even-cycle decomposable. Let G be such a graph and let v be a vertex of G such that $G - v$ is bipartite. We may assume that there are at most two parallel edges between every pair of vertices, else we can remove a 2-cycle and apply induction.

Let (A, B) be a bipartition of $G - v$. Let X be the set of neighbours of v in A and partition X as $X_1 \cup X_2$, where $x \in X_i$ if and only if there are i edges between x and v . Since $|E(G)|$ is even, it follows that $|X_1|$ is even because

$$|E(G)| = \sum_{u \in B} \deg_G(u) + 2|X_2| + |X_1|.$$

Now, since G is 2-connected, the graph $G - v$ is connected. Therefore, by Lemma 4.1 there is a collection \mathcal{P} of $|X_1|/2$ pairwise edge-disjoint paths in $G - v$ whose set of ends is precisely X_1 . Note that each path $P \in \mathcal{P}$ has even length since $G - v$ is bipartite. Thus we may combine the paths in \mathcal{P} with edges between v and X_1 to obtain a family \mathcal{C}_1 of $|X_1|/2$ pairwise edge-disjoint even cycles. Evidently, the edges between v and X_2 can be partitioned into a family \mathcal{C}_2 of 2-cycles. Let \mathcal{E} be the set of edges in $\mathcal{C}_1 \cup \mathcal{C}_2$. Observe that the graph $G - \mathcal{E}$ is bipartite since in $G - \mathcal{E}$, the vertex v is only adjacent to vertices in B . Hence, $G - \mathcal{E}$ is even-cycle decomposable and we are done. \square

5. PLANAR SIGNED GRAPHS WITH TWO UNBALANCED FACES

We now prove that planar signed graphs with two unbalanced faces are balanced-cycle decomposable. Note that this follows from Theorem 1.2, but we give a short proof in order to keep our proof of Theorem 2.3 self-contained.

Proposition 5.1. *Every 2-connected Eulerian loopless planar signed graph with an even number of negative edges and exactly two unbalanced faces is balanced-cycle decomposable.*

Proof. Let (H, Σ) be a 2-connected Eulerian loopless planar signed graph with an even number of negative edges and exactly two unbalanced faces. By passing to $\text{graph}(H, \Sigma)$, it suffices to show that every 2-connected Eulerian loopless planar graph with an even number of edges and exactly two odd-length faces is even-cycle decomposable. Let G be such a graph and let F_1 and F_2 be the two odd-length faces of G . Since G is Eulerian, the dual graph G^* of G is bipartite. Let (A, B) be a bipartition of $V(G^*)$. Since G (and hence also G^*) has an even number of edges, F_1 and F_2 must be on the same side of the bipartition, say $F_1, F_2 \in A$. Since B is both an independent set and a vertex cover of G^* , the faces in G corresponding to the vertices in B form an even-cycle decomposition of G . \square

6. PROOF OF THEOREM 2.3

In this section we prove Theorem 2.3, thus proving our main theorem, Theorem 1.4. We start with a simple parity lemma.

Lemma 6.1. *Let G be an Eulerian graph and let (G_1, G_2) be a separation. Then*

$$\sum_{v \in V(G_1) \cap V(G_2)} \deg_{G_1}(v) \equiv \sum_{v \in V(G_1) \cap V(G_2)} \deg_{G_2}(v) \equiv 0 \pmod{2}.$$

Proof. Observe that

$$\sum_{v \in V(G_1)} \deg_G(v) = 2|E(G_1)| + \sum_{v \in V(G_1) \cap V(G_2)} \deg_{G_2}(v).$$

The lemma then follows easily from the above equation. \square

A 2-separation (G_1, G_2) of an Eulerian graph G is *odd* if $\deg_{G_1}(v)$ is odd for all vertices $v \in V(G_1) \cap V(G_2)$. It is *even* if $\deg_{G_1}(v)$ is even for all vertices $v \in V(G_1) \cap V(G_2)$. By Lemma 6.1, every 2-separation of an Eulerian graph is either odd or even.

Our next lemma asserts that 2-connected Eulerian graphs with at least one even 2-separation can be decomposed into a ‘necklace structure’ of Eulerian subgraphs.

Lemma 6.2. *Let G be a 2-connected Eulerian loopless graph having an even 2-separation (G_1, G_2) such that G_1 and G_2 are connected. Then there exist pairwise edge-disjoint connected Eulerian subgraphs B_1, B_2, \dots, B_n of G with $n \geq 2$ satisfying the following.*

- (i) $\bigcup_{i=1}^n E(B_i) = E(G)$.
- (ii) For each i , either B_i is 2-connected or B_i has two vertices.
- (iii) (a) Either $n = 2$ and (B_1, B_2) is a 2-separation of G ,

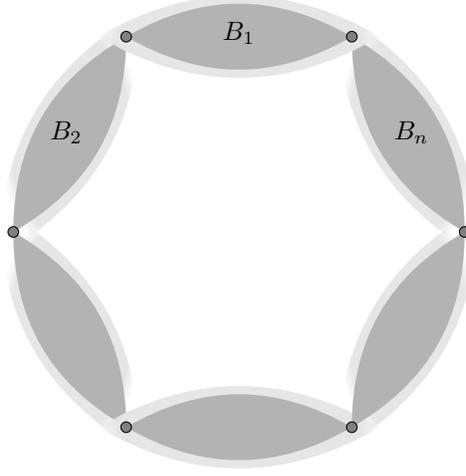


FIGURE 3. A necklace decomposition into beads B_1, \dots, B_n .

(b) or $n \geq 3$ and for all $1 \leq i < j \leq n$,

$$|V(B_i) \cap V(B_j)| = \begin{cases} 1 & \text{if } i - j \equiv \pm 1 \pmod{n}, \\ 0 & \text{otherwise.} \end{cases}$$

(iv) There exists k such that $B_1 \cup B_2 \cup \dots \cup B_k = G_1$ and $B_{k+1} \cup \dots \cup B_n = G_2$.

Proof. We choose pairwise edge-disjoint connected Eulerian subgraphs B_1, B_2, \dots, B_n with $n \geq 2$ satisfying (i), (iii), and (iv) so that n is maximized. Such a choice must exist because the sequence G_1, G_2 satisfies (i), (iii), and (iv). Let $B_{n+1} = B_1$ and $B_0 = B_n$. Note that $|V(B_i)| > 1$ because otherwise either $V(B_{i-1}) \cap V(B_{i+1}) \neq \emptyset$ when $n > 2$ or $|V(B_1) \cap V(B_2)| \leq 1$ when $n = 2$.

Suppose that (ii) is false. By symmetry, we may assume that $|V(B_1)| \geq 3$ and B_1 has a 1-separation (F_1, F_2) such that both F_1 and F_2 have at least two vertices. Let v be the vertex on the boundary of (F_1, F_2) .

Suppose that $n = 2$. Since v is not a cut vertex of G , $v \notin V(B_1) \cap V(B_2)$ and $|V(B_2) \cap V(F_1)| = |V(B_2) \cap V(F_2)| = 1$. Since B_1 is an Eulerian subgraph, both F_1 and F_2 are Eulerian. Then a sequence F_1, F_2, B_2 satisfies (i), (iii), and (iv) and therefore it contradicts our assumption that n is maximum.

Thus $n > 2$. If F_2 meets both B_n and B_2 , then v is a cut vertex of G , contradicting the assumption that G is 2-connected. Thus F_2 meets at most one of B_n and B_2 . Similarly F_1 meets at most one of B_n and B_2 . Since each B_i is Eulerian, $(F_1, F_2 \cup \bigcup_{i=2}^n B_i)$ is an even 2-separation and therefore F_1 is Eulerian. Similarly F_2 is Eulerian because $(F_2, F_1 \cup \bigcup_{i=2}^n B_i)$ is an even 2-separation.

We may assume that F_1 meets B_n and F_2 meets B_2 . Then we consider a sequence $F_1, F_2, B_2, \dots, B_n$ of edge-disjoint connected Eulerian subgraphs satisfying (i), (iii), and (iv). This contradicts the assumption that n is chosen to be maximum. \square

We call any B_1, \dots, B_n given by Lemma 6.2 a necklace decomposition of (G, Σ) (See Figure 3). Each B_i is called a bead of the necklace decomposition.

FIGURE 4. $\tilde{K}_2^2 \cdot \tilde{K}_2^2 \cdot \tilde{K}_2^2$. (Solid lines denote negative edges.)

We now proceed to prove a sequence of lemmas concerning a counterexample (G, Σ) to Theorem 2.3 with $|E(G)|$ minimum.

Lemma 6.3. *A minimum counterexample (G, Σ) cannot have an odd 2-separation (G_1, G_2) where each $(G_i, E(G_i) \cap \Sigma)$ is connected and not a signed subgraph of \tilde{K}_2^2 .*

Proof. Suppose that (G, Σ) has an odd 2-separation (G_1, G_2) such that each $(G_i, E(G_i) \cap \Sigma)$ is connected and not a signed subgraph of \tilde{K}_2^2 . Let u and v be the vertices on the boundary of (G_1, G_2) . By assumption, $\deg_{G_1}(u)$ and $\deg_{G_1}(v)$ are odd.

We first handle the subcase that one of the $(G_i, E(G_i) \cap \Sigma)$, say $(G_2, E(G_2) \cap \Sigma)$ is balanced. In this case, by Lemma 3.1, we may assume that all edges in G_2 are positive and thus $\Sigma \subseteq E(G_1)$. Let (G'_1, Σ) be the signed graph obtained from (G_1, Σ) by adding a positive edge between u and v . Note that G'_1 is 2-connected since G is 2-connected. Moreover, (G'_1, Σ) is a proper minor of (G, Σ) since there is a balanced path between u and v in (G_2, \emptyset) . (Note that every path in (G_2, \emptyset) is balanced because no edge is negative.) By assumption, (G'_1, Σ) is not a counterexample and therefore (G'_1, Σ) has a balanced-cycle decomposition. It follows that $E(G_1)$ can be decomposed as $\mathcal{C}_1 \cup \{P_1\}$, where \mathcal{C}_1 is a family of balanced cycles and P_1 is a balanced path between u and v . Since (G_2, \emptyset) has no negative edges, and u and v are the only odd degree vertices in G_2 , we can decompose $E(G_2)$ as $\mathcal{C}_2 \cup \{P_2\}$ where \mathcal{C}_2 is a family of balanced cycles, and P_2 is a balanced path between u and v . Therefore $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \{P_1 \cup P_2\}$ is a balanced-cycle decomposition of (G, Σ) .

The other subcase is if neither $(G_1, E(G_1) \cap \Sigma)$ nor $(G_2, E(G_2) \cap \Sigma)$ is balanced. Let (G'_i, Σ_i) be the signed graph obtained from $(G_i, E(G_i) \cap \Sigma)$ by adding a positive edge between u and v if $|\Sigma \cap E(G_i)|$ is even, and adding a negative edge between u and v if $|\Sigma \cap E(G_i)|$ is odd. Since $(G_1, E(G_1) \cap \Sigma)$ and $(G_2, E(G_2) \cap \Sigma)$ are not balanced and G is 2-connected, by Menger's theorem we can find two vertex-disjoint paths from u and v to an unbalanced cycle in $(G_i, E(G_i) \cap \Sigma)$ for each $i \in \{1, 2\}$. Therefore, both (G'_1, Σ_1) and (G'_2, Σ_2) are proper minors of (G, Σ) . By the minimality assumption, $E(G_i)$ can be decomposed into $\mathcal{C}_i \cup \{P_i\}$ where \mathcal{C}_i is a family of balanced cycles and P_i is a path between u and v . Again, $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \{P_1 \cup P_2\}$ is a balanced-cycle decomposition of (G, Σ) . \square

We call a signed subgraph (H, Γ) of (G, Σ) an *albatross* if $(H, \Gamma) \cong \tilde{K}_2^2 \cdot \tilde{K}_2^2$ (see Figure 2) and the degree-4 vertex in H also has degree 4 in G .

Lemma 6.4. *In a minimum counterexample (G, Σ) , all albatrosses are edge-disjoint. Moreover, if G has maximum degree 4, then all albatrosses are vertex-disjoint.*

Proof. Towards a contradiction, let (H_1, Σ_1) and (H_2, Σ_2) be distinct albatrosses of (G, Σ) that are not edge-disjoint. Since G is 2-connected, it is not possible that $(H_1 \cup H_2, \Sigma_1 \cup \Sigma_2) \cong \tilde{K}_3^2$ (see Figure 2). Thus $(H_1 \cup H_2, \Sigma_1 \cup \Sigma_2) \cong \tilde{K}_2^2 \cdot \tilde{K}_2^2 \cdot \tilde{K}_2^2$ (see Figure 4) and $H_1 \cup H_2$ attaches to the rest of G at the degree-2 vertices of $H_1 \cup H_2$.

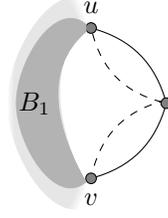


FIGURE 5. A *Bermuda triangle*. Note that the right half is an albatross. (Solid lines denote negative edges.)

Let (G', Σ') be the signed graph obtained from (G, Σ) by replacing $(H_1 \cup H_2, \Sigma_1 \cup \Sigma_2)$ with a single \tilde{K}_2^2 . Note that (G', Σ') is a 2-connected proper minor of (G, Σ) . By the minimality assumption, we have that (G', Σ') has a balanced-cycle decomposition \mathcal{C}' , which we can lift to a balanced-cycle decomposition \mathcal{C} of (G, Σ) . The second part of the lemma follows in the same way. \square

We call (G, Σ) a *Bermuda triangle* if all necklace decompositions B_1, \dots, B_n of (G, Σ) satisfy $n = 3$ with two $(B_i, E(B_i) \cap \Sigma)$ isomorphic to \tilde{K}_2^2 , see Figure ???. Note that every Bermuda triangle contains an albatross.

Lemma 6.5. *A minimum counterexample (G, Σ) cannot have an even 2-separation (G_1, G_2) such that each $(G_i, E(G_i) \cap \Sigma)$ is connected and not isomorphic to \tilde{K}_2^2 , unless (G, Σ) is a Bermuda triangle.*

Proof. Suppose that (G, Σ) has an even 2-separation (G_1, G_2) such that each $(G_i, E(G_i) \cap \Sigma)$ is connected and not isomorphic to \tilde{K}_2^2 and that (G, Σ) is not a Bermuda triangle. By Lemma 6.2, (G, Σ) has a necklace decomposition B_1, \dots, B_n . If each bead B_i contains an even number of negative edges, we contradict the fact that (G, Σ) is a minimum counterexample. Therefore, there are at least two beads with an odd number of negative edges. Call such a bead an *odd bead*. Moreover, since (G, Σ) is not a Bermuda triangle, we may assume that $n \neq 3$ or $n = 3$ and at most one $(B_i, E(B_i) \cap \Sigma) \cong \tilde{K}_2^2$. It follows that we may choose an even 2-separation (H_1, H_2) of (G, Σ) such that both $(H_1, E(H_1) \cap \Sigma)$ and $(H_2, E(H_2) \cap \Sigma)$ contain an odd bead and are not isomorphic to \tilde{K}_2^2 . This also implies that $|E(H_1)|, |E(H_2)| > 2$ because odd beads that are not isomorphic to \tilde{K}_2^2 contain at least three edges.

Let x and y be the vertices on the boundary of (H_1, H_2) . We first handle the subcase that H_1 and H_2 contain an odd number of negative edges. In this case we define (H'_i, Σ_i) to be the signed graph obtained from $(H_i, E(H_i) \cap \Sigma)$ by adding a positive edge and a negative edge between x and y . Since each H_i contains an odd bead, there exist an unbalanced x - y -path and a balanced x - y -path in $(H_i, E(H_i) \cap \Sigma)$. Therefore, (H'_i, Σ_i) is odd- K_4 -minor-free and 2-connected. By the minimality assumption, $E(H_i)$ can be decomposed as $\mathcal{C}_i \cup \{E_i\} \cup \{O_i\}$, where \mathcal{C}_i is a family of balanced cycles, E_i is a balanced x - y -path and O_i is an unbalanced x - y -path. But then $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \{E_1 \cup E_2\} \cup \{O_1 \cup O_2\}$ is a balanced-cycle decomposition of (G, Σ) .

The other remaining subcase is when $(H_1, E(H_1) \cap \Sigma)$ and $(H_2, E(H_2) \cap \Sigma)$ contain an even number of negative edges. In this case we let (H'_i, Σ_i) be the graph obtained from $(H_i, E(H_i) \cap \Sigma)$ by adding two positive edges between x

and y . Again note that (H'_i, Σ_i) is 2-connected and odd- K_4 -minor-free. By the minimality assumption, (H'_i, Σ_i) is balanced-cycle decomposable. Moreover, since $(H_i, E(H_i) \cap \Sigma)$ is *not* balanced-cycle decomposable (by virtue of containing an odd bead), the balanced 2-cycle formed by the two newly added edges cannot be used as a cycle in the decomposition. Therefore, $E(H_i)$ can be decomposed as $\mathcal{C}_i \cup \{E_{i,1}, E_{i,2}\}$ where \mathcal{C}_i is a family of balanced cycles and $E_{i,1}$ and $E_{i,2}$ are balanced x - y -paths. But then $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \{E_{1,1} \cup E_{2,1}, E_{1,2} \cup E_{2,2}\}$ is a balanced-cycle decomposition of (G, Σ) . \square

Lemma 6.6. *A minimum counterexample (G, Σ) cannot contain two parallel edges of the same sign.*

Proof. Let G_2 be a connected subgraph of G having exactly two parallel edges of the same sign. Let $G_1 = G \setminus E(G_2)$. Since G is 2-connected, G_1 is connected. By Lemma 6.2, there is a necklace decomposition B_1, B_2, \dots, B_n extending the even 2-separation (G_1, G_2) . We may assume that $B_n = G_2$. Since $|\Sigma|$ is even, G_1 is not isomorphic to \tilde{K}_2^2 . Lemma 6.5 implies that G is a Bermuda triangle and therefore $n = 3$ and B_1, B_2 are isomorphic to \tilde{K}_2^2 . But then, (G, Σ) is easily seen to be decomposable into two balanced 3-cycles. \square

We say that a signed graph (G, Σ) is *almost 3-connected* if for all proper 2-separations (G_1, G_2) of G , $(G_1, E(G_1) \cap \Sigma)$ or $(G_2, E(G_2) \cap \Sigma)$ is isomorphic to $\tilde{K}_2^2 \cdot \tilde{K}_2^2$.

Lemma 6.7. *A minimum counterexample (G, Σ) is almost 3-connected.*

Proof. Let (G_1, G_2) be a proper 2-separation of G . Since G is 2-connected, $|E(G_1)|, |E(G_2)| > 2$ and G_1, G_2 are connected. By Lemma 6.3, (G_1, G_2) must be an even 2-separation. By Lemma 6.5, G is a Bermuda triangle. Let B_1, B_2, B_3 be a necklace decomposition of G extending (G_1, G_2) given by Lemma 6.2. We may assume that B_1, B_2 are isomorphic to \tilde{K}_2^2 and $G_1 = B_1 \cup B_2$. This implies that $(G_1, E(G_1) \cap \Sigma)$ is isomorphic to $\tilde{K}_2^2 \cdot \tilde{K}_2^2$. \square

Lemma 6.8. *A minimum counterexample (G, Σ) cannot have a 3-separation (G_1, G_2) , where $(G_2, E(G_2) \cap \Sigma)$ is balanced, connected, and has at least 4 edges.*

Proof. Choose such a 3-separation (G_1, G_2) with $|E(G_2)|$ minimum. By Lemma 3.1, we may assume that all edges in $(G_2, E(G_2) \cap \Sigma)$ are positive and so $\Sigma \subseteq E(G_1)$. In particular, all paths contained in G_2 are balanced and $\tilde{K}_2^2 \cdot \tilde{K}_2^2$ is not a signed subgraph of (G_2, \emptyset) . Also note that G_2 does not contain parallel edges by Lemma 6.6. Let x, y and z be the vertices on the boundary of (G_1, G_2) .

We first prove that G_2 is 2-connected. Suppose not and let (H_1, H_2) be a proper 1-separation of G_2 with $V(H_1) \cap V(H_2) = \{w\}$. If $w \in \{x, y, z\}$ then H_1 and H_2 both induce 2-separations in G . Since (G, Σ) is almost 3-connected and contains no parallel edges of the same sign, it follows that H_1 and H_2 are each just a single edge. This contradicts $|E(G_2)| \geq 4$. Thus, $w \notin \{x, y, z\}$ and we may assume $V(H_1) \cap \{x, y, z\} = \{x\}$. Thus, H_1 induces a 2-separation in G and must just be the single edge xw . Hence $G_2 - x$ induces a 3-separation in G . This contradicts the minimality of G_2 unless, $G_2 - x$ has exactly three edges. On the other hand, recall that G_2 contains no parallel edges. Furthermore, Lemma 6.7 implies that no vertex of G has degree 2 (such a vertex gives a proper 2-separation with neither side equal to $\tilde{K}_2^2 \cdot \tilde{K}_2^2$). Since G is Eulerian, G has minimum degree at least 4.

Since G_2 contains only four edges, we conclude that $V(G_2) \setminus \{x, y, z\} = \{w\}$. Since $\deg_{G_2}(w) \geq 4$, G_2 must contain parallel edges, which is a contradiction.

Now by Lemma 6.1,

$$\deg_{G_1}(x) + \deg_{G_1}(y) + \deg_{G_1}(z) \equiv 0 \pmod{2}.$$

There are two possibilities to consider: either two of x, y , and z have odd degree in G_1 or none of x, y , and z have odd degree in G_1 .

We handle the former possibility first. By symmetry, suppose that $\deg_{G_1}(x)$ and $\deg_{G_1}(y)$ are odd. Let $G_1^{e,f}$ be the graph obtained from G_1 by adding an edge e between z and y and an edge f between z and x . We claim that $G_1^{e,f}$ is 2-connected. Suppose not and let (H_1, H_2) be a proper 1-separation of $G_1^{e,f}$, with $V(H_1) \cap V(H_2) = \{w\}$. Note that $\{x, y, z\}$ cannot be a subset of $V(H_i)$, else w is a cut-vertex of G . Now, if $z \in V(H_i) \setminus \{w\}$, then $\{x, y\} \subseteq V(H_i)$, since zx and zy are edges of $G_1^{e,f}$, a contradiction. Hence $w = z$. By symmetry we may assume $x \in V(H_1)$ and $y \in V(H_2)$. Thus, $H_1 \setminus f$ and $H_2 \setminus e$ induce 2-separations in G , with boundary vertices $\{x, z\}$ and $\{y, z\}$ respectively. Each of these 2-separations is odd since $\deg_{H_1 \setminus f}(x) = \deg_{G_1}(x)$ and $\deg_{H_2 \setminus e}(y) = \deg_{G_1}(y)$. By Lemma 6.3, $H_1 \setminus f$ and $H_2 \setminus e$ are each just a single edge. But now, $G - z$ is a subgraph of G_2 , and is hence balanced as a signed graph. This contradicts Proposition 4.2. Thus, $G_1^{e,f}$ is 2-connected as claimed.

Since G_2 is also 2-connected, there are two internally disjoint paths P_1 and P_2 in G_2 from $\{z\}$ to $\{x, y\}$. Let $H := G_1 \cup E(P_1) \cup E(P_2)$. Observe that H is 2-connected since it is a subdivision of $G_1^{e,f}$. Evidently, H is Eulerian and contains an even number of negative edges since $\Sigma \cap E(G_2) = \emptyset$. Moreover, $P_1 \cup P_2 \neq G_2$ since G has minimum degree 4 by Lemma 6.7. Hence $H \neq G$. We are done since $(H, \Sigma \cap E(H))$ is balanced-cycle decomposable by induction and $(G \setminus E(H), \Sigma \setminus E(H))$ is balanced-cycle decomposable since it is balanced.

We now consider the second possibility that each of $\deg_{G_1}(x)$, $\deg_{G_1}(y)$, and $\deg_{G_1}(z)$ is even. In this case we let (G_1^Δ, Σ) be the signed graph obtained from (G_1, Σ) by adding three positive edges $e = xy$, $f = yz$, and $g = xz$. Evidently, G_1^Δ is Eulerian, 2-connected and contains an even number of negative edges.

We claim that (G_1^Δ, Σ) is a minor of (G, Σ) . First observe that G_2 contains a cycle C as it is Eulerian. Recall that G_2 contains no parallel edges; hence C is not a 2-cycle. Next notice that G has no proper 2-separation (H_1, H_2) with $V(H_1) \cap V(H_2) \subseteq V(G_2)$ because G is almost 3-connected and G_2 has no negative edges. By Menger's theorem, there are three vertex-disjoint paths from $\{x, y, z\}$ to $V(C)$. Now it is easy to obtain (G_1^Δ, Σ) as a minor of (G, Σ) ; we begin by contracting edges in those three paths.

By the minimality assumption, (G_1^Δ, Σ) has a balanced-cycle decomposition \mathcal{C}^Δ . If $\{e, f, g\} \in \mathcal{C}^\Delta$, then (G_1, Σ) is balanced-cycle decomposable. But (G_2, \emptyset) is also balanced-cycle decomposable since it is Eulerian and balanced. Thus, (G, Σ) is balanced-cycle decomposable.

If $e \in C_1 \in \mathcal{C}^\Delta$ and $\{f, g\} \subset C_2 \in \mathcal{C}^\Delta$ for $C_1 \neq C_2$, then it suffices to find two edge-disjoint x - y paths in G_2 , at least one of which avoids z . Since G_2 is 2-connected, $G - z$ is connected, so there does exist an x - y path P in G_2 that avoids z . But now the second path exists since x and y are the only odd degree vertices in $G_2 \setminus E(P)$.

By symmetry, the only remaining possibility is if e, f , and g are in different cycles of \mathcal{C}^Δ . In this case it suffices to show that there are pairwise edge-disjoint paths $Q_{x,y}, Q_{y,z}$, and $Q_{x,z}$, where $Q_{i,j}$ is an i - j -path in G_2 such that $|V(Q_{i,j}) \cap \{x, y, z\}| = 2$. We may assume that G_2 has no cycle containing x, y , and z , else we are done. Since G_2 is 2-connected, G_2 has a cycle C containing y and z . Since G_2 has no cycles containing x, y , and z , there do not exist three vertex-disjoint paths from the neighbours of x in G_2 to $V(C)$. By Menger's theorem, there is a proper 2-separation (H_1, H_2) of G_2 such that $x \in V(H_1) \setminus V(H_2)$ and $y, z \in V(C) \subseteq V(H_2)$. Since H_1 induces a 3-separation in G , we have that $|E(H_1)| \leq 3$. In particular, x has degree 2 in H_1 (and hence also in G_2). Since G_2 is 2-connected, we can find two vertex-disjoint paths $Q_{x,y}$ and $Q_{x,z}$. Note that in $G_2 \setminus E(Q_{x,y} \cup Q_{x,z})$, y and z are the only vertices of odd degree and x is an isolated vertex. Thus, $G_2 \setminus E(Q_{x,y} \cup Q_{x,z})$ has a path $Q_{y,z}$ from y to z avoiding x . This completes the proof. \square

We are now ready to prove our main theorem.

Theorem 2.3. *Every 2-connected Eulerian loopless odd- K_4 -minor-free signed graph with an even number of negative edges is balanced-cycle decomposable.*

Proof. Suppose not. Let (G, Σ) be a counterexample with $|E(G)|$ minimum. By Lemma 6.6, there are at most two parallel edges between every pair of vertices. If x is a degree-2 vertex, we can contract an edge incident with x , if necessary by re-signing, to obtain a smaller counterexample. Thus, G has minimum degree at least 4.

By Theorem 3.2, one of the following holds.

- (i) (G, Σ) is almost balanced or planar with two unbalanced faces.
- (ii) (G, Σ) has a 2-separation (G_1, G_2) where each $(G_i, E(G_i) \cap \Sigma)$ is connected and not a signed subgraph of \tilde{K}_2^2 .
- (iii) (G, Σ) has a 3-separation (G_1, G_2) where $(G_2, E(G_2) \cap \Sigma)$ is balanced, connected, and has at least 4 edges.

By Propositions 4.2 and 5.1, (i) is not possible. Lemma 6.8 implies that (iii) is impossible.

Thus (ii) holds and let (G_1, G_2) be such a 2-separation. By Lemma 6.3, (G_1, G_2) is an even separation. By Lemma 6.5, (G, Σ) is a Bermuda triangle. Let B_1, B_2, B_3 be a necklace decomposition of G with $(B_2, E(B_2) \cap \Sigma) \cong (B_3, E(B_3) \cap \Sigma) \cong \tilde{K}_2^2$. If B_1 has only two vertices, then B_1 is just two edges of the same sign, which contradicts Lemma 6.6. So B_1 is 2-connected. Since B_1 contains an even number of negative edges, by the minimality assumption, B_1 has a balanced-cycle decomposition \mathcal{C} .

We first claim that G is 4-regular. Let u and v be the vertices in $V(B_1) \cap (V(B_2) \cup V(B_3))$. Choose a shortest chain of balanced cycles C_1, C_2, \dots, C_k ($k \geq 1$) in \mathcal{C} such that $u \in V(C_1)$, $v \in V(C_k)$ and $V(C_i) \cap V(C_{i+1}) \neq \emptyset$ for all $i \in \{1, 2, \dots, k-1\}$. Since this chain is shortest, $V(C_i) \cap V(C_j) = \emptyset$ whenever $j > i+1$. Let $W = E(B_1) - \bigcup_{i=1}^k E(C_i)$. If W is nonempty, then $\mathcal{C} - \{C_1, C_2, \dots, C_k\}$ is a balanced-cycle decomposition of $(B_1 - \bigcup_{i=1}^k E(C_i), W \cap \Sigma)$. Moreover by the minimality assumption, the signed graph $(G_0, \Sigma_0) = (G - W, (E(G) - W) \cap \Sigma)$ must have a balanced-cycle decomposition \mathcal{D} . Then $(\mathcal{C} - \{C_1, C_2, \dots, C_k\}) \cup \mathcal{D}$ is a balanced-cycle decomposition of (G, Σ) , which is a contradiction. Therefore W

is empty and hence $B_1 = \bigcup_{i=1}^k C_i$. Consequently, every vertex of G has degree at most 4 (and hence exactly 4). Since $|V(B_1)| > 2$, we have $k > 1$ and u is not adjacent to v in G .

Let (G', Σ') be the signed graph obtained from (G, Σ) by replacing each albatross with a \tilde{K}_2^2 . Since all albatrosses are vertex-disjoint by Lemma 6.4, (G', Σ') is well-defined. By applying Theorem 3.2 again we conclude that one of the following holds.

- (1) (G', Σ') is almost balanced.
- (2) (G', Σ') is planar with two unbalanced faces.
- (3) (G', Σ') has a 2-separation (G'_1, G'_2) where each $(G'_i, E(G'_i) \cap \Sigma')$ is connected and not a signed subgraph of \tilde{K}_2^2 .
- (4) (G', Σ') has a 3-separation (G'_1, G'_2) where $(G'_2, E(G'_2) \cap \Sigma')$ is balanced, connected, and has at least 4 edges.

By Lemma 6.7, (G, Σ) is almost 3-connected, and hence (G', Σ') is 3-connected. Let x, y, z be the vertices of an arbitrary albatross A in (G, Σ) , where $\deg_A(y) = 4$. Note that x and z are not adjacent in G since (G, Σ) is almost 3-connected. Therefore, since (G, Σ) does not contain any parallel edges of the same sign, (G', Σ') also does not contain any parallel edges of the same sign. It follows that (3) is impossible.

We next exclude possibility (1). By re-signing in G' , we may assume that there exists a vertex $t \in V(G')$ such that all edges of G' are positive except possibly those incident with t . Since all albatrosses of (G, Σ) are vertex-disjoint, it follows that $B_2 \cup B_3$ is the only albatross of (G, Σ) , and that $t \in \{u, v\}$. Thus, (G', Σ') is obtained from (G, Σ) by replacing $B_2 \cup B_3$ with a \tilde{K}_2^2 between u and v . Let w be the vertex of $B_2 \cup B_3$ not in $\{u, v\}$. By performing the same re-signings in G as we performed in G' we may assume that all edges in G are positive except possibly those incident with t or w .

As G is 4-regular and contains an even number of negative edges, exactly 1 or 3 edges incident with t are negative. By re-signing at t in G , we may assume that exactly one edge incident with t is negative. This negative edge is also necessarily incident with w . Therefore, $G - w$ is balanced as a signed graph, which contradicts Proposition 4.2.

We next handle possibility (4). Let (G'_1, G'_2) be such a 3-separation. Since $(G'_2, E(G'_2) \cap \Sigma')$ is balanced, it evidently cannot contain any \tilde{K}_2^2 subgraphs. Thus, by putting back the albatrosses, we obtain a 3-separation (G_1, G_2) of (G, Σ) where $(G_2, E(G_2) \cap \Sigma)$ is balanced, connected, and has at least 4 edges. This is impossible by Lemma 6.8.

We finish by ruling out possibility (2). In this case, we claim (G', Σ') has a balanced Eulerian signed subgraph $(H', \Sigma' \cap E(H'))$ such that $E(H') \neq \emptyset$ and $G' \setminus E(H')$ is 2-connected after removing isolated vertices. We can then lift $(H', \Sigma' \cap E(H'))$ to a balanced Eulerian signed subgraph $(H, \Sigma \cap E(H))$ of (G, Σ) as follows. Suppose that an albatross A in (G, Σ) has been replaced with $K = \tilde{K}_2^2$ in (G', Σ') . Note that H' cannot use both edges of K , since $(H', \Sigma' \cap E(H'))$ is balanced. If H' uses a positive edge e of K , we replace e by a balanced path in A . If H' uses a negative edge e of K , we replace e by an unbalanced path in A . If H' uses no edges of K , then H uses no edge of A .

Note that $G \setminus E(H)$ is 2-connected after removing isolating vertices, since $G' \setminus E(H')$ is 2-connected after removing isolated vertices. By Lemma 3.1, H

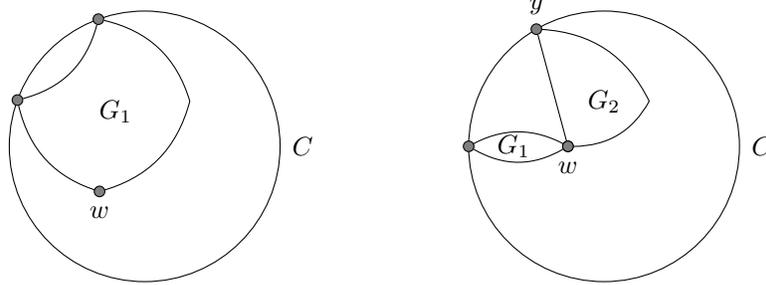


FIGURE 6. On the left, $G_1 \not\cong \tilde{K}_2^2$ and shares at least two vertices with C . On the right, $G_1 \cong \tilde{K}_2^2$ and shares at least one vertex with C .

(and hence $G \setminus E(H)$) has an even number of negative edges. We are then finished since $(G \setminus E(H), \Sigma \setminus E(H))$ is balanced-cycle decomposable by induction. We now proceed to show that such an H' exists.

We again avoid using Theorem 1.2 to keep our proof self-contained. Consider a fixed planar embedding of (G', Σ') with at most two unbalanced faces. Note that each \tilde{K}_2^2 must bound a face of (G', Σ') , else its two vertices would form a vertex-cut of size two. Since (G', Σ') contains no parallel edges of the same sign, it follows that G' contains at most two pairs of parallel edges.

Recall that u and v are the vertices of a contracted albatross in (G', Σ') . Let F and F' be the two faces adjacent to the unbalanced face given by u and v . We may assume F is the outer face and balanced. Let C be the boundary cycle of F . Let $G'' = G' \setminus E(C)$, and $\{G_1, \dots, G_k\}$ be its set of blocks. We may assume $k \geq 2$, else let $H' = C$. Note that each G_i is an Eulerian plane subgraph of G' , and is 2-connected unless it is a \tilde{K}_2^2 . We may further assume at least one G_i , say G_1 , is not balanced else let $H' = G''$.

Since each $(G_i, \Sigma' \cap E(G_i))$ inherits all of its finite faces from (G', Σ') , and C contains an edge from a finite unbalanced face of (G', Σ') , there is at most one finite unbalanced face left in $(G'', \Sigma' \cap E(G''))$. Therefore, for $i \geq 2$, every finite face of $(G_i, \Sigma' \cap E(G_i))$ is balanced and hence $(G_i, \Sigma' \cap E(G_i))$ is balanced. We may assume for $i \geq 2$, $(G_i, \Sigma' \cap E(G_i))$ is a balanced cycle of length at least three, else let $H' = G_i \setminus E(C_i)$ where C_i is the boundary cycle of the outer face of G_i .

Since G' is 3-connected, every face of G' is bounded by a non-separating chordless cycle. Moreover G_1 has an unbalanced finite face and therefore no edge of C has a parallel edge except the edge joining u and v . As C is non-separating, G'' is connected, so its block graph T is a tree. Let G_i be a leaf in T for some $i \neq 1$. Let x be the cut vertex of G'' belonging to G_i . Since G_i is a cycle, every vertex in $V(G_i) \setminus \{x\}$ must belong to C because G' is 4-regular. Since C is chordless, every edge of $G_i - x$ must have endpoints u and v . In particular, G_i must be a triangle and G_i is the only leaf in $\{G_2, \dots, G_k\}$. Thus, T is a path with leaves G_1 and G_i . We re-label so that the G_j 's are labelled according to their order in T . Let w be the cut-vertex of G'' belonging to G_1 .

Suppose $(G_1, \Sigma' \cap E(G_1)) \not\cong \tilde{K}_2^2$. Since G_1 has at least three vertices and G' is 3-connected, G_1 must share at least two vertices with C , otherwise $(V(G_1) \cap V(C)) \cup$

$\{w\}$ is a vertex-cut in G' of size at most two. Therefore $G_1 \cup C$ is 2-connected, see Figure 6. We are done since we may take $H' = G_2 \cup \dots \cup G_k$.

So we may assume $(G_1, \Sigma' \cap E(G_1)) \cong \tilde{K}_2^2$. Since the two cut-vertices of G'' in G_2 cannot be a vertex-cut in G' , G_2 shares at least one vertex y with C . Also, $y \neq w$, since G' is 4-regular. Hence, $G_1 \cup G_2 \cup C$ has an ear decomposition starting from G_2 (see Figure 6), and is thus 2-connected. We are finished by letting $H' = G_3 \cup \dots \cup G_k$, unless $k = 2$. If $k = 2$, then $(G_1, \Sigma' \cap E(G_1)) \cong \tilde{K}_2^2$ and G_2 and C are triangles. Hence G' is obtained from K_4 by doubling a pair of independent edges. Since each set of parallel edges is a \tilde{K}_2^2 , there is a balanced 4-cycle which passes through both sets of parallel edges. Let H' be such a balanced 4-cycle. We are done since $G' \setminus E(H')$ is just a 4-cycle and evidently 2-connected. \square

Acknowledgements. The authors would like to thank Cheolwon Heo for permission to include the proof of Theorem 1.5. The authors would also like to thank anonymous referees for their valuable suggestions.

REFERENCES

- [1] K. Appel and W. Haken. *Every planar map is four colorable*, volume 98 of *Contemporary Mathematics*. American Mathematical Society, Providence, RI, 1989. With the collaboration of J. Koch.
- [2] J. G. Conlon. Even cycles in graphs. *J. Graph Theory*, 45(3):163–223, 2004.
- [3] H. Fleischner. (Some of) the many uses of Eulerian graphs in graph theory (plus some applications). *Discrete Math.*, 230(1-3):23–43, 2001. Paul Catlin memorial collection (Kalamazoo, MI, 1996).
- [4] A. M. H. Gerards. *Graphs and polyhedra. Binary spaces and cutting planes*, volume 73 of *CWI Tract*. Stichting Mathematisch Centrum Centrum voor Wiskunde en Informatica, Amsterdam, 1990. <http://oai.cwi.nl/oai/asset/12714/12714A.pdf>.
- [5] M. Grötschel and W. R. Pulleyblank. Weakly bipartite graphs and the max-cut problem. *Oper. Res. Lett.*, 1(1):23–27, 1981/82.
- [6] B. Guenin. A characterization of weakly bipartite graphs. *J. Combin. Theory Ser. B*, 83(1):112–168, 2001.
- [7] B. Jackson. On circuit covers, circuit decompositions and Euler tours of graphs. In *Surveys in combinatorics, 1993 (Keele)*, volume 187 of *London Math. Soc. Lecture Note Ser.*, pages 191–210. Cambridge Univ. Press, Cambridge, 1993.
- [8] R. Naserasr, E. Rollová, and E. Sopena. Homomorphisms of planar signed graphs to signed projective cubes. *Discrete Math. Theor. Comput. Sci.*, 15(3):1–11, 2013.
- [9] N. Robertson, D. P. Sanders, P. Seymour, and R. Thomas. A new proof of the four-colour theorem. *Electron. Res. Announc. Amer. Math. Soc.*, 2(1):17–25, 1996.
- [10] P. D. Seymour. Decomposition of regular matroids. *J. Combin. Theory Ser. B*, 28(3):305–359, 1980.
- [11] P. D. Seymour. Even circuits in planar graphs. *J. Combin. Theory Ser. B*, 31(3):327–338, 1981.
- [12] F. T. Tseng and K. Truemper. A decomposition of the matroids with the max-flow min-cut property. *Discrete Appl. Math.*, 15(2-3):329–364, 1986. Applications of combinatorial methods in mathematical programming (Gainesville, Fla., 1985).
- [13] F. T. Tseng and K. Truemper. Addendum: “A decomposition of the matroids with the max-flow min-cut property” [Discrete Appl. Math. **15** (1986), no. 2-3, 329–364; MR0865011 (88b:05046)]. *Discrete Appl. Math.*, 20(1):87–88, 1988.
- [14] T. Zaslavsky. Signed graphs. *Discrete Applied Mathematics*, 4(1):47–74, 1982.
- [15] C. Q. Zhang. On even circuit decompositions of Eulerian graphs. *J. Graph Theory*, 18(1):51–57, 1994.

(Huynh) DEPARTMENT OF MATHEMATICS, UNIVERSITÉ LIBRE DE BRUXELLES, BRUSSELS, BELGIUM

(Oum) DEPARTMENT OF MATHEMATICAL SCIENCES, KAIST, DAEJEON, SOUTH KOREA AND
SCHOOL OF MATHEMATICS, KIAS, SEOUL, SOUTH KOREA

(Verdian-Rizi) DEPARTMENT OF MATHEMATICAL SCIENCES, KAIST, DAEJEON, SOUTH KOREA
E-mail address: tony.bourbaki@gmail.com
E-mail address: sangil@kaist.edu
E-mail address: mverdian@gmail.com