Rank-width of Random Graphs*

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Abstract

Rank-width of a graph G, denoted by $\mathbf{rw}(G)$, is a width parameter of graphs introduced by Oum and Seymour (2006). We investigate the asymptotic behavior of rank-width of a random graph $G(n,p)$. We show that, asymptotically almost surely, (i) if $p \in (0,1)$ is a constant, then $\mathbf{rw}(G(n,p)) = \lceil \frac{n}{3} \rceil - O(1)$, (ii) if $\frac{1}{n} \ll p \leq \frac{1}{2}$, then $\mathbf{rw}(G(n,p)) = \lceil \frac{n}{3} \rceil - o(n)$, (iii) if $p = c/n$ and $c > 1$, then $\mathbf{rw}(G(n,p)) \geq rn$ for some $r = r(c)$, and (iv) if $p \leq c/n$ and $c < 1$, then $\mathbf{rw}(G(n,p)) \leq 2$. As a corollary, we deduce that $G(n,p)$ has linear tree-width whenever $p = c/n$ for each $c > 1$, answering a question of Gao (2006).

Keywords: rank-width, tree-width, clique-width, random graph, sharp threshold.

1 Introduction

Rank-width of a graph G, denoted by $\mathbf{rw}(G)$, is a graph width parameter introduced by Oum and Seymour [10] and measures the complexity of decomposing G into a

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tree-like structure. The precise definition will be given in the following section. One fascinating aspect of this parameter lies in its computational applications, namely, if a class of graphs has bounded rank-width, then many NP-hard problems are solvable on this class in polynomial time; for example, see [2].

We consider the Erdős-Rényi random graph $G(n,p)$. In this model, a graph $G(n,p)$ on a vertex set $\{1,2,\cdots,n\}$ is chosen randomly as follows: for each unordered pair of vertices, they are adjacent with probability p independently at random. Given a graph property $\mathcal{P}$, we say that $G(n,p)$ possesses $\mathcal{P}$ *asymptotically almost surely*, or a.a.s. for brevity, if the probability that $G(n,p)$ possesses $\mathcal{P}$ converges to 1 as n goes to infinity. A function $f:\mathbb{N}\to[0,1]$ is called the *sharp threshold* of $G(n,p)$ with respect to having $\mathcal{P}$ if the following hold: if $p\geq cf(n)$ for a constant $c>1$, then $G(n,p)$ a.a.s. satisfies $\mathcal{P}$ and otherwise if $p\leq cf(n)$ and $c<1$, then $G(n,p)$ a.a.s. does not satisfy $\mathcal{P}$.

The following is our main result.

Theorem 1.1. *For a random graph $G(n,p)$, the following holds asymptotically almost surely:*

(i) if $p\in(0,1)$ is a constant, then $\mathbf{rw}(G(n,p))=\lceil\frac{n}{3}\rceil-O(1)$,
(ii) if $\frac{1}{n}\ll p\leq\frac{1}{2}$, then $\mathbf{rw}(G(n,p))=\lceil\frac{n}{3}\rceil-o(n)$,
(iii) if $p=c/n$ and $c>1$, then $\mathbf{rw}(G(n,p))\geq rn$ for some $r=r(c)$, and
(iv) if $p\leq c/n$ and $c<1$, then $\mathbf{rw}(G(n,p))\leq 2$.

Since $\mathbf{rw}(G)\leq\lceil\frac{|V(G)|}{3}\rceil$ for every graph G, (i) and (ii) of this theorem give a narrow range of rank-width. Note that this theorem also gives a bound when $p\geq\frac{1}{2}$, since the rank-width of $G(n,p)$ in this range can be obtained from the inequality $\mathbf{rw}(\overline{G})\leq\mathbf{rw}(G)+1$.

Clique-width of a graph G, denoted by $\mathbf{cw}(G)$, is a width parameter introduced by Courcelle and Olariu [3]. It is strongly related to rank-width by the following inequality by Oum and Seymour [10].

$$\mathbf{rw}(G)\leq\mathbf{cw}(G)\leq 2^{\mathbf{rw}(G)+1}-1. \tag{1}$$

Tree-width, introduced by Robertson and Seymour [11], is a width parameter measuring how similar a graph is to a tree and is closely related to rank-width. We will denote the tree-width of a graph G as $\mathbf{tw}(G)$. The following inequality was proved by Oum [9]: for every graph G, we have

$$\mathbf{rw}(G)\leq\mathbf{tw}(G)+1. \tag{2}$$

There have been works on tree-width of random graphs. Kloks [8] proved that $G(n,p)$ with $p=c/n$ has linear tree-width whenever $c>2.36$. Gao [6] improved this constant to 2.162 and even conjectured that c can be improved to a constant less than 2. We improve the above constant to the best possible number, 1, by the following corollary, stating that there is the sharp threshold $p=1/n$ of $G(n,p)$ with respect to having linear tree-width.

Corollary 1.2. *Let c be a constant and let $G=G(n,p)$ with $p=c/n$. Then the following holds asymptotically almost surely:*

(i) If $c > 1$, then rank-width, clique-width,and tree-width of G are at least $c'n$ for some constant c' depending only on c.

(ii) If $c < 1$, then rank-width and tree-width of G are at most 2 and clique-width of G is at most 5.

Proof. (i) follows Theorem 1.1 with (1) and (2). (ii) follows easily due to the theorem by Erdős and Rényi [4, 5] stating that asympototically almost surely, each component of $G(n, p)$ with $p = c/n$, $c < 1$ has at most one cycle. It is straightforward to see that such graphs have small tree-width, clique-width, and rank-width. □

2 Preliminaries

All graphs in this paper have neither loops nor parallel edges. Let $\Delta(G), \delta(G)$ be the maximum degree and the minimum degree of a graph G respectively. For two subsets X and Y of $V(G)$, let $E_G(X, Y)$ be the set of ordered pairs (x, y) of adjacent vertices $x \in X$ and $y \in Y$. Let $e_G(X, Y) = |E_G(X, Y)|$. We will omit subscripts if it is not ambiguous.

Let $\mathbb{F}_2 = \{0, 1\}$ be the binary field. For disjoint subsets V_1 and V_2 of $V(G)$, let N_{V_1,V_2} be a 0-1 $|V_1| \times |V_2|$ matrix over $\mathbb{F}_2$ whose rows are labeled by V_1 and columns labeled by V_2, and the entry (v_1, v_2) is 1 if and only if $v_1 \in V_1$ and $v_2 \in V_2$ are adjacent. We define the *cutrank* of V_1 and V_2, denoted by $\rho_G(V_1, V_2)$, to be $\mathbf{rank}(N_{V_1,V_2})$.

A tree T is said to be *subcubic* if every vertex has degree 1 or 3. A *rank-decomposition* of a graph G is a pair (T, L) of a subcubic tree T and a bijection L from $V(G)$ to the set of all leaves of T. Notice that deleting an edge uv of T creates two components C_u and C_v containing u and v respectively. Let $A_{uv} = L^{-1}(C_u)$ and $B_{uv} = L^{-1}(C_v)$. Under these notations, *rank-width* of a graph G, denoted by $\mathbf{rw}(G)$, is defined as

$$\mathbf{rw}(G) = \min_{(T,L)} \max_{uv \in E(T)} \rho_G(A_{uv}, B_{uv}),$$

where the minimum is taken over all possible rank-decompositions. We assume $\mathbf{rw}(G) = 0$ if $|V(G)| \leq 1$.

The following lemma will be used later.

Lemma 2.1. *Let $G = (V, E)$ be a graph with at least two vertices. If rank-width of G is at most k, then there exist two disjoint subsets V_1, V_2 of V such that*

$$|V_1| = \left\lceil \frac{n}{2} \right\rceil, \ |V_2| = \left\lceil \frac{n}{3} \right\rceil, \ \textit{and } \rho_G(V_1, V_2) \leq k.$$

Proof. Let $k = \mathbf{rw}(G)$. Let (T, L) be a rank-decomposition of width k. We claim that there is an edge e of T such that $T \setminus e$ gives a partition (A, B) of $V(G)$ satisfying $|A| \geq n/3$, $|B| \geq n/3$ and $\rho_G(A, B) \leq k$. Assume the contrary. Then for each edge e in T, $T \setminus e$ has a component C_e of $T \setminus e$ containing less than $n/3$ leaves of T. Direct each edge $e = uv$ from u to v if C_e contains u. Since this directed tree is acyclic, there is a vertex t in $V(T)$ such that every edge incident with t is directed toward

t. Then there are at most 3 components in $T \setminus t$ and each component has less than $n/3$ leaves of T, a contradiction. This proves the claim.

Given sets A, B as above, we may assume $|A| \geq n/2$. Take $V_1 \subseteq A$ and $V_2 \subseteq B$ of size $\lceil \frac{n}{2} \rceil$ and $\lceil \frac{n}{3} \rceil$, respectively. Then $\rho_G(V_1, V_2) \leq \rho_G(A, B) \leq k$. $\square$

3 Rank-width of dense random graphs

In this section we will show that if $\frac{1}{n} \ll \min(p, 1-p)$, then the rank-width of $G(n,p)$ is a.a.s. $\lceil \frac{n}{3} \rceil - o(n)$. Moreover, for a constant $p \in (0,1)$, rank-width of $G(n,p)$ is a.a.s. $\lceil \frac{n}{3} \rceil - O(1)$. This bound is achieved by investigating the rank of random matrices. The following proposition provides an exponential upper bound to the probability of a random vector falling into a fixed subspace.

Proposition 3.1. *For $0 < p < 1$, let $\eta = \max(p, 1-p)$. Let $v \in \mathbb{F}_2^n$ be a random 0-1 vector whose entries are 1 or 0 with probability p and $1-p$ respectively. Then for each k-dimensional subspace U of $\mathbb{F}_2^n$,*

$$\mathbf{P}(v \in U) \leq \eta^{n-k}$$

Proof. Let B be a $k \times n$ matrix whose row vectors form a basis of U. By permuting the columns if necessary, we may assume that the first k columns are linearly independent. For a vector $v \in \mathbb{F}_2^n$, let $v^{(k)}$ be the first k entries of v, and note that

$$\mathbf{P}(v \in U) = \sum_{w \in \mathbb{F}_2^k} \mathbf{P}(v \in U | v^{(k)} = w) \mathbf{P}(v^{(k)} = w). \tag{3}$$

Let $u_1, u_2, \cdots, u_k$ be the row vectors of B. Observe that $\{u_j^{(k)}\}_{j=1}^k$ is a basis of $\mathbb{F}_2^k$. Thus, given $v^{(k)} = w = \sum_{i=1}^k c_i u_i^{(k)}$, we have $v \in U$ if and only if $v = \sum_{i=1}^k c_i u_i$. This implies that given each first k entries of v, there is a unique choice of remaining entries yielding $v \in U$. Thus for every $w \in \mathbb{F}_2^k$, $\mathbf{P}(v \in U | v^{(k)} = w) \leq \eta^{n-k}$. Combining with (3), we obtain

$$\mathbf{P}(v \in U) \leq \eta^{n-k} \sum_{w \in \mathbb{F}_2^k} \mathbf{P}(v^{(k)} = w) = \eta^{n-k},$$

and this concludes the proof. $\square$

Let $M(k_1, k_2; p)$ be a random $k_1 \times k_2$ matrix whose entries are mutually independent and take value 0 or 1 with probability $1-p$ and p respectively. Using Proposition 3.1, we can bound the probability that the rank of $M(\lceil \frac{n}{3} \rceil, \lceil \frac{n}{2} \rceil; p)$ deviates from $\lceil \frac{n}{3} \rceil$.

Lemma 3.2. *For $0 < p < 1$, let $\eta = \max(p, 1-p)$. Then for every $C > 0$,*

$$\mathbf{P}\left(\mathbf{rank}\left(M\left(\left\lceil \frac{n}{3} \right\rceil, \left\lceil \frac{n}{2} \right\rceil; p\right)\right) \leq \left\lceil \frac{n}{3} \right\rceil - \frac{C}{\log_2 \frac{1}{\eta}}\right) < 2^{(\frac{1}{2} - \frac{1}{6}C)n}.$$

Proof. Let $M = M(\lceil \frac{n}{3} \rceil, \lceil \frac{n}{2} \rceil; p)$, $\alpha = \lceil \frac{C}{\log_2 \frac{1}{\eta}} \rceil$, and $\mathrm{row}(M)$ be the linear space spanned by the rows of M. We may assume $\lceil \frac{n}{3} \rceil - \alpha \geq 0$. Denote row vectors of M by $v_1, v_2, \cdots, v_{\lceil \frac{n}{3} \rceil}$. Note that $\mathbf{rank}(M)$ is at most $\lceil \frac{n}{3} \rceil - \alpha$ if and only if there are $\lceil \frac{n}{3} \rceil - \alpha$ rows of M spanning $\mathrm{row}(M)$. Thus

$$\mathbf{P}\left(\mathbf{rank}(M) \leq \left\lceil \frac{n}{3} \right\rceil - \alpha\right) \leq \sum_I \mathbf{P}\left(\{v_i\}_{i \in I} \text{ spans } \mathrm{row}(M)\right)$$

where the sum is taken over all $I \subseteq \{1, 2, \cdots, \lceil \frac{n}{3} \rceil\}$ with cardinality $\lceil \frac{n}{3} \rceil - \alpha$. Let U_I be the vector space spanned by row vectors $\{v_i\}_{i \in I}$. By Proposition 3.1, we get

$$\mathbf{P}\left(\{v_i\}_{i \in I} \text{ spans } \mathrm{row}(M)\right) = \mathbf{P}(\{v_j : j \notin I\} \subseteq U_I) \leq (\eta^{\lceil \frac{n}{2} \rceil - \lceil \frac{n}{3} \rceil + \alpha})^{\alpha},$$

since rows are mutually independent random vectors. Combining these inequalities, we conclude that

$$\mathbf{P}\left(\mathbf{rank}(M) \leq \left\lceil \frac{n}{3} \right\rceil - \alpha\right) \leq 2^{\lceil \frac{n}{2} \rceil - 1} (\eta^{\alpha})^{\lceil \frac{n}{2} \rceil - \lceil \frac{n}{3} \rceil + \alpha} \leq 2^{\frac{n}{2}} 2^{-\frac{n}{6} C} = 2^{(\frac{1}{2} - \frac{1}{6} C)n}$$

because $\lceil \frac{n}{2} \rceil - \lceil \frac{n}{3} \rceil + \alpha \geq \frac{n}{6}$ and $\binom{\lceil \frac{n}{2} \rceil}{k} \leq 2^{\lceil \frac{n}{2} \rceil - 1}$. □

Proposition 3.3. *Let $\eta = \max(p, 1-p)$ and $n \geq 2$. Then*

$$\mathbf{P}\left(\mathbf{rw}(G(n,p)) \leq \left\lceil \frac{n}{3} \right\rceil - \frac{12.6}{\log_2 \frac{1}{\eta}}\right) < 2^{-0.015n}.$$

Proof. Let $G = G(n,p)$, $\mathcal{S} = \{N_{V_1, V_2} : |V_1| = \lceil \frac{n}{2} \rceil, |V_2| = \lceil \frac{n}{3} \rceil \text{ for disjoint } V_1, V_2 \subseteq V(G)\}$ and let $\mu = \min_{N \in \mathcal{S}} \mathbf{rank}(N)$. By Lemma 2.1, we have $\mu \leq \mathbf{rw}(G)$. Thus it suffices to show that

$$\mathbf{P}\left(\mu \leq \left\lceil \frac{n}{3} \right\rceil - \frac{12.6}{\log_2 \frac{1}{\eta}}\right) < 2^{-0.015n}.$$

For each $N \in \mathcal{S}$, let A_N be the event that $\mathbf{rank}(N) \leq \lceil \frac{n}{3} \rceil - \frac{12.6}{\log_2 \frac{1}{\eta}}$. Note that

$$\mathbf{P}\left(\mu \leq \left\lceil \frac{n}{3} \right\rceil - \frac{12.6}{\log_2 \frac{1}{\eta}}\right) = \mathbf{P}(\bigcup_{N \in \mathcal{S}} A_N) \leq \sum_{N \in \mathcal{S}} \mathbf{P}(A_N).$$

By Lemma 3.2, we have $\mathbf{P}(A_N) \leq 2^{-1.6n}$. Notice also that $|\mathcal{S}| \leq 3^n$. Therefore,

$$\mathbf{P}\left(\mu \leq \left\lceil \frac{n}{3} \right\rceil - \frac{12.6}{\log_2 \frac{1}{\eta}}\right) \leq 3^n 2^{-1.6n} < 2^{-0.015n}.$$ □

The main theorem directly follows from this proposition.

Theorem 3.4. *Asymptotically almost surely, $G = G(n,p)$ satisfies the following:*

(i) if $p \in (0,1)$ is a constant, then $\lceil \frac{n}{3} \rceil - O(1) \leq \mathbf{rw}(G) \leq \lceil \frac{n}{3} \rceil$, and
(ii) if $\frac{1}{n} \ll \min(p, 1-p)$, then $\lceil \frac{n}{3} \rceil - o(n) \leq \mathbf{rw}(G) \leq \lceil \frac{n}{3} \rceil$.

4 Rank-width of sparse random graphs

In this section we investigate the rank-width of $G(n,p)$ when $p = c/n$ for some constant $c > 0$. Note that Proposition 3.3 does not give any information when $p = c/n$ and c is close to 1. As mentioned in the introduction, the linear lower bound of rank-width in this range of p is closely related to a sharp threshold with respect to having linear tree-width. We show that, when $p = c/n$,

(i) if $c < 1$, then rank-width is a.a.s. at most 2,

(ii) if $c = 1$, then rank-width is a.a.s. at most $O(n^{\frac{2}{3}})$ and,

(iii) if $c > 1$, then there exists $r = r(c)$ such that rank-width is a.a.s. at least rn.

Erdős and Rényi [4, 5] proved that if $c < 1$ then $G(n,p)$ a.a.s. consists of trees and unicyclic (at most one edge added to a tree) components and if $c = 1$ then the largest component has size at most $O(n^{\frac{2}{3}})$. Therefore, (i) and (ii) follow easily because trees and unicyclic graphs have rank-width at most 2.

Thus, (iii) is the only interesting case. When $c > 1$, $G(n,p)$ has a unique component of linear size, called the *giant component*. Hence, in order to prove a lower bound on the rank-width of $G(n,p)$, it is enough to find a lower bound of the rank-width of the giant component.

We need some definitions to describe necessary structures. Let $G = (V,E)$ be a connected graph. For a non-empty proper subset S of $V(G)$, let $d_G(S) = \sum_{v\in S} \deg_G(v)$. The *(edgewise) Cheeger constant* of a connected graph G is

$$\Phi(G) = \min_{\emptyset\neq S\subsetneq V(G)} \frac{e_G(S, V(G)\setminus S)}{\min(d_G(S), d_G(V(G)\setminus S))}.$$

Remark. In [1], the following alternative definition of the Cheeger constant of a connected graph G is used. For a vertex v, let $\pi_v = \frac{\deg_G(v)}{2|E(G)|}$ and for vertices v and w of G, define

$$p_{vw} = \begin{cases} 1/\deg_G(v) & \text{if } v \text{ and } w \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

For a subset S of $V(G)$, let $\pi_G(S) = \sum_{v\in S} \pi_v$. Thus $d_G(S) = 2|E(G)|\pi_G(S)$. In [1], the Cheeger constant of a graph G is defined alternatively as

$$\min_{0<\pi_G(S)\le\frac{1}{2}} \frac{1}{\pi_G(S)} \sum_{i\in S, j\notin S} \pi_i p_{ij}.$$

We can easily see that these definitions are equivalent as follows:

$$\begin{aligned}\Phi(G) = \min_{\emptyset\neq S\subsetneq V(G)} \frac{e_G(S, V(G)\setminus S)}{\min(d_G(S), d_G(V(G)\setminus S))} &= \min_{0<\pi_G(S)\le\frac{1}{2}} \frac{e_G(S, V(G)\setminus S)}{d_G(S)} \\ &= \min_{0<\pi_G(S)\le\frac{1}{2}} \frac{1}{\pi_G(S)} \sum_{i\in S, j\notin S} \pi_i p_{ij},\end{aligned}$$

where the second equality follows from the fact that $\pi_G(S) + \pi_G(V(G)\setminus S) = 1$.

Benjamini, Kozma and Wormald [1] proved the following theorem.

Theorem 4.1 (Benjamini, Kozma and Wormald [1])**.** *Let $c > 1$ and $p = c/n$. Then there exist $\alpha, \delta > 0$ such that $G(n,p)$ a.a.s. contains a connected subgraph H such that $\Phi(H) \geq \alpha$ and $|V(H)| \geq \delta n$.*

Remark. The above theorem is a consequence of [1, Theorem 4.2]. The graph H in Theorem 4.1 is the graph $R_N(G)$ in [1, Theorem 4.2], which proves that $R_N(G)$ is a.a.s. an α-strong core of G. This means that $R_N(G)$ is a subgraph of G with $\Phi(R_N(G)) \geq \alpha$ by the definitions given in Section 2.2 and Section 3 of [1]. The condition $|V(H)| \geq \delta n$ is not explicit in [1, Theorem 4.2]. However this fact follows from [1, Lemma 4.7], because $R_N(G)$ must have more vertices than its kernel $K(R_N(G))$ (the definition of kernel is given in [1, Section 4]). Note that $\hat{n}$ in [1, Lemma 4.7] satisfies $\hat{n} = \Omega(n)$ by the remark following [1, Lemma 4.1]. The proof of Theorem 4.2 given in [1, Section 5] also mentioned this fact explicitly.

A graph H with the property as in Theorem 4.1 is called an *expander graph*. The simple restriction of $\Phi(H)$ being bounded away from 0 provides a strikingly rich structure to the graph as in Theorem 4.1. Interested readers are referred to the survey paper [7].

By using this expander subgraph H, we will show that $G(n,p)$ must have large rank-width when $p = c/n$ and $c > 1$. Before proving this, we need a technical lemma which allows us to control the maximum degree of a random graph $G(n,p)$.

Lemma 4.2. *Let $c > 1$ be a constant and $p = c/n$. Then for every $\varepsilon > 0$, there exists $M = M(c,\varepsilon)$ such that $G = G(n,p)$ a.a.s. has the following property: Let X be the collection of vertices which have degree at least M. Then the number of edges incident with X is at most εn.*

Proof. Let $V = V(G)$. Let M be a large number satisfying

$$\sum_{k=M}^{\infty} k\frac{c^k}{(k-1)!} < \frac{\varepsilon}{2}. \tag{4}$$

For each $v \in V$, define a random variable $Y_v = \deg(v)$ if $\deg(v) \geq M$ and $Y_v = 0$ otherwise. Then by (4),

$$\begin{aligned}\mathbb{E}[Y_v^2] &= \sum_{k=M}^{n-1} k^2 \mathbf{P}(\deg(v) = k) \\ &\leq \sum_{k=M}^{n-1} k^2 \binom{n-1}{k} \left(\frac{c}{n}\right)^k \leq \sum_{k=M}^{\infty} k\frac{c^k}{(k-1)!} < \frac{\varepsilon}{2}.\end{aligned} \tag{5}$$

Since $Y_v \leq Y_v^2$, we also have $\mathbb{E}[Y_v] \leq \varepsilon/2$. Note that the number of edges incident with X is at most $\sum_{v\in V} Y_v$. Hence, it is enough to prove a.a.s. $Y = \sum_{v\in V} Y_v \leq \varepsilon n$. Observe that $\mathbb{E}[Y] \leq \frac{\varepsilon}{2}n$. Moreover, the variance of Y can be computed as

$$\begin{aligned}\mathbb{E}[(Y - \mathbb{E}[Y])^2] &= \sum_{v\in V} \left(\mathbb{E}[Y_v^2] - \mathbb{E}[Y_v]^2\right) + \sum_{v\neq w\in V} \left(\mathbb{E}[Y_v Y_w] - \mathbb{E}[Y_v]\mathbb{E}[Y_w]\right) \\ &\leq \varepsilon n + \sum_{v\neq w\in V} \left(\mathbb{E}[Y_v Y_w] - \mathbb{E}[Y_v]\mathbb{E}[Y_w]\right),\end{aligned} \tag{6}$$

where for each $v, w \in V, v \neq w$,

$$\begin{aligned}
&\mathbb{E}[Y_v Y_w] - \mathbb{E}[Y_v]\mathbb{E}[Y_w] \\
&\qquad = \sum_{k,l=M}^{n-1} kl\big(\mathbf{P}(\deg(v) = k, \deg(w) = l) - \mathbf{P}(\deg(v) = k)\mathbf{P}(\deg(w) = l)\big).
\end{aligned}$$

Let $q_k = \mathbf{P}(\deg(v) = k | vw \notin E(G)) = \mathbf{P}(\deg(v) = k + 1 | vw \in E(G))$, for distinct vertices v, w in $G(n, p)$. Notice that, given either $vw \in E(G)$ or $vw \notin E(G)$, Y_v and Y_w are independent. Thus, we deduce the following:

$$\begin{aligned}
&\mathbb{E}[Y_v Y_w] - \mathbb{E}[Y_v]\mathbb{E}[Y_w] \\
&= \sum_{k,l=M}^{n-1} kl\big(pq_{k-1}q_{l-1} + (1-p)q_k q_l - (pq_{k-1} + (1-p)q_k)(pq_{l-1} + (1-p)q_l)\big) \\
&\leq p \sum_{k,l=M}^{n-1} kl(q_{k-1}q_{l-1} + q_k q_l) \\
&\leq 2p \sum_{k=M-1}^{n-1} (k+1)q_k \sum_{l=M-1}^{n-1} (l+1)q_l \quad \leq \frac{\varepsilon^2}{n}.
\end{aligned}$$

Last inequality follows from (4), since similarly as done in (5) we get

$$\sum_{k=M-1}^{n-1} (k+1)q_k = \sum_{k=M-1}^{n-1} (k+1)\binom{n-2}{k}\left(\frac{c}{n}\right)^k \leq \sum_{k=M}^{\infty} k\frac{c^{k-1}}{(k-1)!} < \frac{\varepsilon}{2c}$$

and $c > 1$. Thus, by (6), we proved that the variance σ^2 of Y is at most $(1+\varepsilon)\varepsilon n$. Finally, using Chebyshev's inequality and the fact $\mathbb{E}[Y_v] \leq \varepsilon/2$, we show that

$$\mathbf{P}\big(Y > \varepsilon n\big) \leq \mathbf{P}\left(Y \geq \mathbb{E}[Y] + \frac{\varepsilon n}{2}\right) \leq \frac{\sigma^2}{\varepsilon^2 n^2/4} \leq \frac{1+\varepsilon}{\varepsilon n/4},$$

which concludes the proof. □

The following lemma will be used in the proof of the main theorem.

Lemma 4.3. *Let A be a matrix over $\mathbb{F}_2$ with at least n non-zero entries. If each row and column contains at most M non-zero entries, then* $\mathbf{rank}(A) \geq \frac{n}{M^2}$.

Proof. We apply induction on n. We may assume $n > M^2$. Pick a non-zero row w of A. We may assume that the first entry of w is non-zero, by permuting columns if necessary. Now remove all rows w' whose first entry is 1. Since the first column has at most M non-zero entries, we remove at most M rows including w itself. Hence, we get a submatrix A' with at least $n - M^2$ non-zero entries. By induction hypothesis,

$$\mathbf{rank}(A') \geq \frac{n - M^2}{M^2} \geq \frac{n}{M^2} - 1.$$

By construction, w does not belong to the row-space of A' and therefore

$$\mathbf{rank}(A) \geq \mathbf{rank}(A') + 1 \geq \frac{n}{M^2}.$$ □

Theorem 4.4. *For $c > 1$, let $p = c/n$. Then there exists $r = r(c)$ such that a.a.s.* $\mathbf{rw}(G(n,p)) \geq rn$.

Proof. Denote $G(n,p)$ by G. Let α, δ be constants from Theorem 4.1, and H be the expander subgraph also given by Theorem 4.1. Let $W = V(H)$ and let (W_1, W_2) be an arbitrary partition of W such that $|W_1|, |W_2| \geq |W|/3$. Then since $\Phi(H) \geq \alpha$ and H is connected, we have

$$\alpha \leq \frac{e_H(W_1, W_2)}{\min(d_H(W_1), d_H(W_2))} \leq \frac{e_H(W_1, W_2)}{\min(|W_1|, |W_2|)} \leq \frac{e_G(W_1, W_2)}{|W|/3}.$$

Thus $e_G(W_1, W_2) \geq \frac{\alpha\delta}{3} n$. By Lemma 4.2, there exists M such that the number of edges incident with vertices of degree greater than M is at most $\frac{\alpha\delta}{6} n$. Let X be the set of vertices of degree greater than M. Let $W_1' = W_1 \setminus X$ and $W_2' = W_2 \setminus X$. Since $e_G(W_1', W_2') \geq \frac{\alpha\delta}{6} n$, $N_{W_1', W_2'}$ has at least $\frac{\alpha\delta}{6} n$ entries with value 1. Moreover, $N_{W_1', W_2'}$ has at most M entries of value 1 in each row and column. Hence, we can use Lemma 4.3 to obtain

$$\frac{\alpha\delta}{6M^2} n \leq \rho_G(W_1', W_2') \leq \rho_G(W_1, W_2).$$

Since W_1, W_2 are arbitrary subsets satisfying $|W_1|, |W_2| \geq |W|/3$, this implies that the induced subgraph $G[W]$ has rank-width at least $\frac{\alpha\delta}{6M^2} n$ by Lemma 2.1. Therefore, rank-width of G is at least $\frac{\alpha\delta}{6M^2} n$. $\square$

Corollary 4.5. *Let $c > 1$ and $p = c/n$. Then there exists $t = t(c)$ such that a.a.s.* $\mathbf{tw}(G(n,p)) \geq tn$.

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References

[1] I. Benjamini, G. Kozma, and N. Wormald. The mixing time of the giant component of a random graph. preprint, 2006, arxiv:math/0610459v1.

[2] B. Courcelle, J. A. Makowsky, and U. Rotics. Linear time solvable optimization problems on graphs of bounded clique-width. *Theory Comput. Syst.*, 33(2):125–150, 2000.

[3] B. Courcelle and S. Olariu. Upper bounds to the clique width of graphs. *Discrete Appl. Math.*, 101(1-3):77–114, 2000.

[4] P. Erdős and A. Rényi. On the evolution of random graphs. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 5:17–61, 1960.

[5] P. Erdős and A. Rényi. On the evolution of random graphs. *Bull. Inst. Internat. Statist.*, 38:343–347, 1961.

[6] Y. Gao. On the threshold of having a linear treewidth in random graphs. In *Computing and combinatorics*, volume 4112 of *Lecture Notes in Comput. Sci.*, pages 226–234. Springer, Berlin, 2006.

[7] S. Hoory, N. Linial, and A. Wigderson. Expander graphs and their applications. *Bull. Amer. Math. Soc. (N.S.)*, 43(4):439–561 (electronic), 2006.

[8] T. Kloks. *Treewidth*, volume 842 of *Lecture Notes in Computer Science*. Springer-Verlag, Berlin, 1994. Computations and approximations.

[9] S. Oum. Rank-width is less than or equal to branch-width. *J. Graph Theory*, 57(3):239–244, 2008.

[10] S. Oum and P. Seymour. Approximating clique-width and branch-width. *J. Combin. Theory Ser. B*, 96(4):514–528, 2006.

[11] N. Robertson and P. Seymour. Graph minors. III. Planar tree-width. *J. Combin. Theory Ser. B*, 36(1):49–64, 1984.