HYPERBOLIC SURFACE SUBGROUPS OF ONE-ENDED DOUBLES OF FREE GROUPS

SANG-HYUN KIM AND SANG-IL OUM

ABSTRACT. Gromov asked whether every one-ended word-hyperbolic group contains a hyperbolic surface group. We prove that every one-ended double of a free group has a hyperbolic surface subgroup if (1) the free group has rank two, or (2) every generator is used the same number of times in the amalgamating words. To prove this, we formulate a stronger statement on Whitehead graphs and prove its specialization by combinatorial induction for (1) and the characterization of perfect matching polytopes by Edmonds for (2).

1. INTRODUCTION

A *hyperbolic surface group* is the fundamental group of a closed surface with negative Euler characteristic. We will denote by F_n the free group of rank n with a fixed basis $\mathcal{A}_n = \{a_1, \ldots, a_n\}$. A *double of a free group* is the fundamental group of a graph of groups where there are two free vertex groups and at least one infinite cyclic edge group; here, each edge group is amalgamated along the copies of some word in the free group (Figure 1). If U is a list of words in F_n, we denote by $D(U)$ the double of F_n where a cyclic edge group is glued along the copies of each word in U.

We study the existence of hyperbolic surface subgroups in doubles of free groups. This is motivated by the following remarkable question due to Gromov.

Question 1 (Gromov [14, p. 277]). *Does every one-ended word-hyperbolic group have a hyperbolic surface subgroup?*

Question 1 has been answered affirmatively for the following cases.

(1) Coxeter groups [11].
(2) Graphs of free groups with infinite cyclic edge groups with nontrivial second rational homology [6].
(3) The fundamental groups of closed hyperbolic 3-manifolds [16].

A basic, but still captivating case is when the group is given as a double of a free group. Using (2), Gordon and Wilton [12] were able to construct explicit families of examples of doubles that contain hyperbolic surface groups; they showed that those families virtually have nontrivial second rational homology. The existence of a hyperbolic surface subgroup is not known for doubles with trivial virtual second rational homology. This leads us to the next question.

Question 2. *Does every one-ended double of a nonabelian free group have a hyperbolic surface subgroup?*

Our first main result resolves Question 2 for rank-two case:

Theorem 1. *A double of a rank-two free group is one-ended if and only if it has a hyperbolic surface subgroup.*

Our second main result on Question 2 is on the free groups in which every generator appears the same number of times in the amalgamating words. More precisely, let U be a list of words in F_n. When approaching Question 2 for $D(U)$, one can always assume that U is *minimal* in the sense that no automorphism of F_n reduces the sum of the lengths of the words in U. This is because the isomorphism type of $D(U)$ is invariant under the automorphisms of F_n. We say U is *k-regular* if each generator in $\mathcal{A}_n$ appears exactly k times in U. Our second main result answers Question 2 affirmatively for a minimal, k-regular list of words.

Theorem 2. *Suppose U is a minimal, k-regular list of words in F_n when $n > 1$. If $D(U)$ is one-ended, then $D(U)$ contains a hyperbolic surface group.*

Here is an overview of our proof. We first explain why Tiling Conjecture [19, 17] implies an affirmative answer for Question 2 in Section 2. And then we reformulate Tiling Conjecture into a purely graph theoretic conjecture in Section 3. We resolve this graph theoretic conjecture in two special cases. In Section 4, we prove it for regular graphs and deduce Theorem 2. Here we use the characterization of perfect matching polytopes of graphs by Edmonds [10]. In Section 5, we prove it for 4-vertex graphs and deduce Theorem 1.

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2. Polygonality and Doubles of Free Groups

Kim and Wilton [19] proved that the double along a *polygonal* word contains a hyperbolic surface group (Theorem 7). Their proof relied on the subgroup separability of free groups and the normal form theorem for graphs of groups. In this section, we give a self-contained geometric proof. Then we describe Tiling Conjecture and its implication.

2.1. **Basic definitions and notations.** Each word in F_n can be written as $w = x_1 x_2 \cdots x_l$ where $x_i \in \mathcal{A}_n \cup \mathcal{A}_n^{-1}$; each x_i is called as a *letter* of w, and the subscript of x_i is taken modulo l. We say that w is *cyclically reduced* if $x_{i+1} \neq x_i^{-1}$ for each $i = 1, 2, \ldots, l$. With respect to the given basis $\mathcal{A}_n$, we denote the Cayley graph of F_n by $\mathrm{Cayley}(F_n)$. There is a natural free action of F_n on $\mathrm{Cayley}(F_n)$, so that $\mathrm{Cayley}(F_n)/F_n$ is a bouquet of circles. Let $\alpha_1, \ldots, \alpha_n$ denote the oriented circles in $\mathrm{Cayley}(F_n)/F_n$ corresponding to $a_1, \ldots, a_n$. The loop obtained by a concatenation $\alpha_i^p \alpha_j^q \cdots \alpha_k^r$ where $p, q, \ldots, r \in \mathbb{Z}$ is said to *read* the word $a_i^p a_j^q \cdots a_k^r$.

Given a list U of nontrivial words $u_1, u_2, \ldots, u_r$ in F_n, take two copies Γ and Γ' of $\mathrm{Cayley}(F_n)/F_n$. To Γ and Γ', we glue a cylinder along the copies of the closed curve reading u_i, for each i. Let $X(U)$ be the resulting space and let $D(U) = \pi_1(X(U))$ be the fundamental group of $X(U)$; see Figure 1. In the literature, $D(U)$ is called a *double of F_n along U*, or simply a *double* [1]. If we let $\mathcal{B}_n$ and $V = \{v_1, \ldots, v_r\}$ denote the copies of $\mathcal{A}_n$ and U respectively, then a presentation of $D(U)$ is given as:

$$D(U) \cong \langle \mathcal{A}_n, \mathcal{B}_n, t_2, t_3, \ldots, t_r \mid u_1 = v_1, u_i^{t_i} = v_i \text{ for } i = 2, \ldots, r \rangle.$$

Since the isomorphism type of $D(U)$ does not change if some words in U are replaced by their conjugates, we may always assume that every word in U is cyclically reduced.

2.2. **Non-positively curved cubical complexes.** We briefly summarize elementary facts on CAT(0)-spaces; a standard reference for this subject is [4]. We denote by $\mathbb{E}^2$ the Euclidean plane. Let X be a geodesic metric space. For a geodesic triangle $\Delta \subseteq X$, there is a geodesic triangle $\Delta' \subseteq \mathbb{E}^2$ of the same side-lengths and a length-preserving map $f \colon \Delta \to \Delta'$. We say that X is a CAT(0)-*space* if $d_X(x, x') \leq d_{\mathbb{E}^2}(f(x), f(x'))$ for every choice of Δ, f and $x, x' \in \Delta$. A metric space X is *non-positively curved* if each point in X has a neighborhood which is a CAT(0)-space. We will need the following.
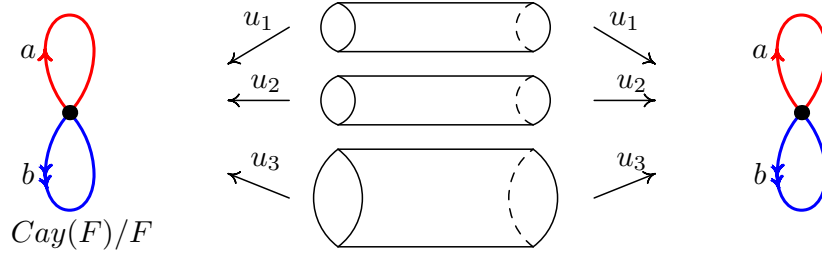
FIGURE 1. A construction of $X(U)$, where $\pi_1(X(U)) = D(U)$.

Proposition 3 (see [4, p. 201]). *Let X and Y be complete geodesic spaces. If X is non-positively curved and $f\colon Y \to X$ is locally an isometric embedding, then Y is non-positively curved and $f_*\colon \pi_1(Y) \to \pi_1(X)$ is injective.*

Let I denote the unit interval. A *cube complex* is a piecewise-Euclidean cell complex X inductively defined as follows: for all k, the k-skeleton $X^{(k)}$ is obtained from $X^{(k-1)}$ by attaching k-dimensional unit cubes I^k such that the restriction of each attaching map to a $(k-1)$-face of I^k is a $(k-1)$-dimensional attaching map. If $X = X^{(2)}$, we say that X is a *square* complex. A finite-dimensional cube complex is known to be a complete geodesic metric space [3]. For a cube complex X and $v \in X^{(0)}$, $\mathrm{Link}_X(x)$ is defined to be the set of unit vectors from v toward X; in particular, a link is naturally equipped with a piecewise-spherical metric. We will only consider cube complexes that are finite-dimensional and locally compact. Moreover, we always assume that given cube complexes are *simple*, in the sense that no vertex has a link containing a bigon; hence, each link will be a simplicial complex [15]. A simplicial complex L is a *flag complex* if every complete subgraph of $L^{(1)}$ is the 1-skeleton of some simplex in L. Gromov gave a combinatorial formulation of non-positive curvature for a cube complex.

Proposition 4 (Gromov [13]). *A cube complex X is non-positively curved if and only if the link of each vertex is a flag complex.*

Recall that for a simplicial complex L and a set of vertices S in L, a *full subcomplex* L' on S is the maximal subcomplex of L whose vertex set is S. A map $f\colon Y \to X$ between cube complexes is *cubical* if f maps each cube to a cube of the same dimension. Locally an isometric cubical map has a combinatorial characterization as follows.

Proposition 5 ([8, 9])**.** *Let X and Y be cube complexes and $f \colon Y \to X$ be a cubical map. Then f is locally an isometric embedding if the following are true for each vertex $y \in Y^{(0)}$.*

   (i) *The induced map on the links* $\mathrm{Link}(f; y) \colon \mathrm{Link}_Y(y) \to \mathrm{Link}_X(f(y))$ *is injective.*

  (ii) *The image of* $\mathrm{Link}(f; y)$ *is a full subcomplex of* $\mathrm{Link}_X(f(y))$.

2.3. **Polygonality.** We let U be a list of cyclically reduced words $u_1, \ldots, u_r$ in F_n. For a word $w = x_1 x_2 \ldots x_l \in F_n$, $x_1 x_2, x_2 x_3, \ldots, x_{l-1} x_l, x_l x_1$ are called *length-2 cyclic subwords* of w. The *Whitehead graph $W(U)$ of U* is constructed as follows [30]:

   (i) the vertex set of $W(U)$ is $\mathcal{A}_n \cup \mathcal{A}_n^{-1}$;

  (ii) For each length-2 cyclic subword xy of a word in U, we add an edge joining x and y^{-1} to $W(U)$.

A *polygonal disk* means a topological 2-disk P equipped with a graph structure on the boundary $\partial P \approx S^1$. We let $Z(U)$ denote the presentation 2-complex of $F_n / \langle\!\langle U \rangle\!\rangle$. This means, $Z(U)$ is obtained from its 1-skeleton $\mathrm{Cayley}(F_n)/F_n$ by attaching a polygonal disk D_i along the loop reading u_i for each $i = 1, 2, \ldots, r$. Here, ∂D_i is regarded as a $|u_i|$-gon. Let α_j denote the oriented loop in $Z(U)^{(1)} = \mathrm{Cayley}(F_n)/F_n$ reading a_j. The link of the unique vertex in $Z(U)$ is seen to be the Whitehead graph of U, by identifying the incoming (outgoing, respectively) portion of α_j with the vertex a_j (a_j^{-1}, respectively) in $W(U)$.

Let us fix a point d_i in the interior of D_i and triangulate D_i so that each triangle contains d_i and one edge of ∂D_i. Remove a small open neighborhood of d_i for each i, to get a square complex Z'; see Figure 2 (a). We obtain a square complex structure on $X(U)$ by taking two copies of Z' and gluing the circles corresponding to the boundary of the neighborhood of each d_i. The unique vertex of $Z(U)$ gives two special vertices of $X(U)$. Note that the link of each special vertex is the barycentric subdivision $W(U)'$ of $W(U)$. Since $W(U)$ has no loops, $W(U)'$ is a bipartite graph without parallel edges. It follows from Proposition 4 that $X(U)$ is non-positively curved.

A *side-pairing* on polygonal disks $P_1, \ldots, P_m$ is an equivalence relation on the sides of $P_1, \ldots, P_m$ such that each equivalence class consists of two sides, along with a choice of a homeomorphism between the two sides of each equivalence class. For a given side-pairing $\sim$ on polygonal disks $P_1, \ldots, P_m$, one gets a closed surface $S = \coprod_i P_i / \sim$ by identifying the sides of P_i by $\sim$. The surface S is naturally equipped with a two-dimensional CW-structure. A graph map $\phi \colon G \to \mathrm{Cayley}(F_n)/F_n$ induces an orientation and a label by $\mathcal{A}_n$ on each edge e of G, so that

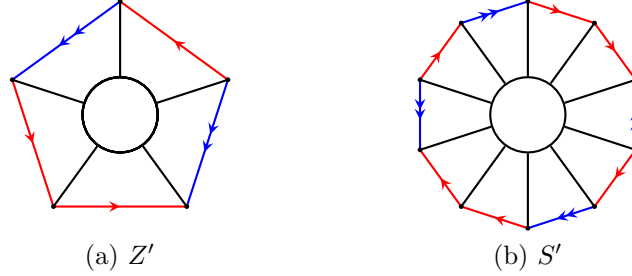(a) Z' (b) S'

FIGURE 2. Square complex structures on Z' and on S'.
A single and a double arrow denote the generators a and
b, respectively. Figure (a) shows a punctured D_i in Z',
divided into squares. Figure (b) is a punctured P_i in S',
where $\partial P_i \to \mathrm{Cayley}(F)/F$ reads $(b^{-1}aba^2)^2$.

the oriented loop $\phi(e)$ reads the label of e. An edge labeled by a_i is
called an a_i-edge. An *immersion* is a locally injective graph map.

Definition 6 ([19, 17])**.** Let U be a list of cyclically reduced words in
F_n. We say U is *polygonal* if there exist a side-pairing $\sim$ on some polyg-
onal disks $P_1, P_2, \ldots, P_m$ and an immersion $S^{(1)} \to \mathrm{Cayley}(F_n)/F_n$
where $S = \coprod_i P_i/\sim$ such that the following hold:

(i) the composition $\partial P_i \to S^{(1)} \to \mathrm{Cayley}(F_n)/F_n$ reads a nontrivial
    power of a word in U for each i;
(ii) the Euler characteristic $\chi(S)$ of S is less than m.

In this case, we call S a *U-polygonal surface*.

Remark.   (1) Polygonality has been defined for a set of words [19,
             17], but we generalize to a (possibly redundant) list of words.
             The main implication of polygonality still holds, as described
             in Theorem 7.
         (2) Polygonality of a list of words depends on the choice of a free-
             basis. An example given in [19] is the word $w = abab^2ab^3$ in
             $F_2 = \langle a, b \rangle$. It was shown that while w is not polygonal, the
             automorphism $(a \mapsto ab^{-2}, b \mapsto b)$ maps w to a polygonal word
             $ab^{-1}a^2b$.

Theorem 7 ([19, 17])**.** *If U is a polygonal list of words in F_n, then
$D(U)$ contains a hyperbolic surface group.*

Proof. Let S be a closed surface obtained from a side-pairing $\sim$ on
polygonal disks $P_1, P_2 \ldots, P_m$, equipped with immersions $\partial P_i \to S^{(1)} \to$
$\mathrm{Cayley}(F_n)/F_n$ satisfying the conditions in Definition 6. Choose a point

p_i in the interior of P_i and triangulate P_i so that p_i is the common vertex, similarly to the triangulation of D_i in $Z(U)$. There is a natural extension $\phi\colon S \to Z(U)$ of the immersion $S^{(1)} \to \mathrm{Cayley}(F)/F$. In particular, ϕ respects the triangulation and is locally injective away from $p_1, \ldots, p_m$. We obtain a square complex S' from S by taking out small open disks around $p_1, \ldots, p_m$; see Figure 2 (b). Similarly to what we have done for Z', we glue two copies of S' along the corresponding boundary components. The resulting square complex S'' is a closed surface such that $\chi(S'') = 2\chi(S') = 2(\chi(S) - m) < 0$. With the square complex structure on $X(U)$ described previously, we have a locally injective cubical map $\phi''\colon S'' \to X(U)$. For a vertex $v \in S''^{(0)}$, $\mathrm{Link}(f; v)$ embeds $\mathrm{Link}_{S''}(v) \approx S^1$ onto a cycle in a link $W(U)'$ of $X(U)$. Since each cycle in $W(U)'$ is a full subcomplex, Propositions 3 and 5 imply that ϕ'' is locally an isometric embedding and so, ϕ''_* is injective. $\square$

2.4. **Tiling Conjecture and its implication.** A list U of words in F_n is said to be *diskbusting* if one cannot write $F_n = A * B$ in such a way that $A, B \neq \{1\}$ and each word in U is conjugate into A or B [7, 29, 28].

Conjecture 8 (Tiling Conjecture; see [19, 17])**.** *A minimal and diskbusting list of cyclically reduced words in F_n is polygonal when $n > 1$.*

We note that $D(U)$ is one-ended if and only if U is diskbusting [12]. By [19, 17] and Theorem 7, the double along a polygonal list contains a hyperbolic surface group. Hence, if Tiling Conjecture is true, then every one-ended double of a nonabelian free group has a hyperbolic surface subgroup, answering Question 2. Moreover, one would be able to precisely describe when doubles contain hyperbolic surface groups as follows.

Proposition 9. *Let $n > 1$. Suppose that every minimal and diskbusting list of cyclically reduced words in F_m is polygonal for all $m = 2, 3, \ldots, n$. Then for a list U of cyclically reduced words in F_n, $D(U)$ contains a hyperbolic surface group if and only if F_n cannot be written as $F_n = G_1 * G_2 * \cdots G_n$ in such a way that each G_i is infinite cyclic and each word in U is conjugate into one of $G_1, \ldots, G_n$.*

For the proof, we need the following:

Lemma 10. *A double of $\mathbb{Z}$ is virtually $\mathbb{Z} \times F_s$ for some $s \geq 0$.*

Proof. We let $m_1, \ldots, m_k$ be given positive integers and M be their least common multiple. We consider a graph of spaces X where there are two vertex spaces and k edge spaces joining the two vertex spaces

as follows. The vertex spaces are circles denoted by α and β, and each edge space E_i is a cylinder whose boundary components are attached to α^{m_i} and β^{m_i} for $i = 1, \ldots, k$. Then the double of $\mathbb{Z}$ along the words $m_1, \ldots, m_k$ is $\pi_1(X)$. There exists a degree-M cover Y of X with precisely two vertex spaces and $\sum_{i=1}^{k} m_i$ edge spaces; the vertex spaces are circles projecting onto α^M and β^M, and E_i lifts to m_i cylinders whose attaching maps are homeomorphisms, for $i = 1, \ldots, k$. Note that $\pi_1(Y) \cong \mathbb{Z} \times F_s$ where $s + 1 = \sum_{i=1}^{k} m_i$. $\square$

Proof of Proposition 9. There exists a maximum k such that $F_n = G_1 * \cdots * G_k$ for some nontrivial groups $G_1, \ldots, G_k$ and each word in U is conjugate into one of the $G_1, \ldots, G_k$. Note that $1 \le k \le n$.

For the forward implication, suppose $k < n$. Then we may assume that G_1 has rank $m > 1$. Let U_1 be the list of all the words in U conjugate into G_1. Then suitably chosen conjugates of the words in U_1 form a diskbusting list U_1' in the rank-m free group G_1. We note that $D(U_1') \subseteq D(U_1') \subseteq D(U_1' \cup (U \setminus U_1)) \cong D(U)$; here, the second inclusion can be seen by Propositions 3 and 5. From the hypothesis, a free basis $\mathcal{B}$ of G_1 can be chosen so that U_1' is polygonal as a list of words written in $\mathcal{B}$. By Theorem 7, DU_1') contains a hyperbolic surface group; hence, so does $D(U)$.

For the backward implication, assume $k = n$ and we claim that $D(U)$ does not contain a hyperbolic surface group. Since we are only interested in the isomorphism type of $D(U)$, we may assume that each word in U is contained in one of $G_1, \ldots, G_n$, by taking conjugation if necessary. By choosing the basis $\mathcal{A}_n$ of F_n from the bases of $G_1, \ldots, G_n$, one may write $\mathcal{A}_n = \{a_1, \ldots, a_n\}$ and $G_i = \langle a_i \rangle$ for $i = 1, \ldots, n$. One sees that up to homotopy equivalence, $X(U)$ is obtained from graphs of spaces of the form considered in Lemma 10 by adding closed intervals and circles. So, $D(U) \cong \pi_1(X(U))$ can be written as a free product such that each free factor has a finite-index subgroup isomorphic to $\mathbb{Z} \times F_s$ for some $s \ge 0$. In particular, $D(U)$ does not contain a hyperbolic surface group. $\square$

Remark. Tiling Conjecture would actually imply that the fundamental group of every one-ended graph of virtually free groups with virtually cyclic edge group either is virtually $\mathbb{Z} \times F_m$ for some $m > 0$ or contains a hyperbolic surface group [18]. Moreover, since minimal diskbusting words are *generic* [24, 5], Tiling Conjecture (Conjecture 8) would imply that polygonal words are generic.

3. Combinatorial Formulation of Tiling Conjecture

Throughout this section, we let U be a list of cyclically reduced words in F_n for some $n > 1$.

3.1. **Terminology on graphs.** We allow graphs to have parallel edges or loops; a *loop* is an edge with only one endpoint. For a graph G, we write $V(G)$ and $E(G)$ to denote the vertex set and the edge set of G, respectively. The *degree* $\deg_G(v)$ of a vertex v is the number of edges incident with v, assuming that loops are counted twice. A graph is k-*regular* if every vertex has degree k, and it is *regular* if it is k-regular for some k. A *cycle* is a (finite) 2-regular connected graph. For a set X of vertices, we write $\delta_G(X)$ to denote the set of edges having endpoints in both X and $V(G) \setminus X$. In particular, $\delta_G(v)$ is the set of non-loop edges incident with v. For two distinct vertices x and y of a graph G, the *local edge-connectivity* $\lambda_G(x, y)$ is the maximum number of pairwise edge-disjoint paths from x to y in G. We omit the subscript G in $\deg_G$, δ_G, and λ_G if the underlying graph G is clear from the context. Menger's theorem [23] states that $\lambda(x, y) = \min\{|\delta(X)| \colon x \in X, y \notin X\}$.

3.2. **Whitehead graph and the associated connecting map.** The following characterization of a minimal set of words is given in [2, Section 8]: a set A of cyclically reduced words in F_n is not minimal if and only if for some i, there exists a set C of edges in the Whitehead graph $W(A)$ such that $|C| < \deg(a_i)$ and $W(A) \setminus C$ has no path from a_i to a_i^{-1}. By Menger's theorem [23], it follows that $A \subseteq F_n$ is minimal if and only if
$$\lambda(a_i, a_i^{-1}) = \deg(a_i) \text{ for each } i.$$
Also, a minimal set $A \subseteq F_n$ is diskbusting if and only if $W(A)$ is connected [30, 29, 28]. These results on sets of words immediately generalize to lists of words as follows.

Proposition 11 ([30, 2, 29, 28]). *A list U of cyclically reduced words in F_n is minimal and diskbusting if and only if $W(U)$ is connected and $\lambda(v, v^{-1}) = \deg(v)$ for each vertex v of $W(U)$.*

There is a canonical fixed point free involution μ on $\mathcal{A}_n \cup \mathcal{A}_n^{-1}$ such that $\mu(a) = a^{-1}$ for all $a \in A_n \cup \mathcal{A}_n^{-1}$. For each vertex v of $W(U)$, the *connecting map σ_v associated with $W(U)$ at v* is a bijection from $\delta(v)$ to $\delta(\mu(v))$ defined as follows. For an edge e given by $x_i x_{i+1}$ in a word $w = x_1 x_2 \ldots x_l$ in U, $\sigma_{x_{i+1}^{-1}}$ maps the edge e joining x_i and x_{i+1}^{-1} to the edge f joining x_{i+1} and x_{i+2}^{-1} created by the consequently following length-2 cyclic subword $x_{i+1} x_{i+2}$ of w. We assume that $x_{l+1} = x_1$ and $x_{l+2} = x_2$. We note that if $\sigma_{y^{-1}} \circ \sigma_{x^{-1}}(e)$ is well-defined for an edge
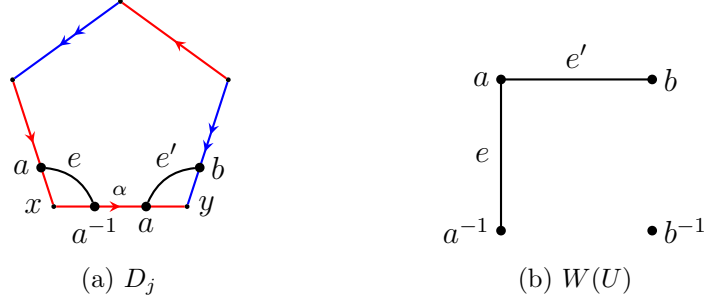
(a) D_j (b) $W(U)$

FIGURE 3. Each corner of a cell D_j in $Z(U)$ corresponds an edge in $W(U)$. Here, $F_2 = \langle a, b \rangle$ and $U = \{b^{-1}aba^2\}$. In these two figures, we note that $\sigma_{a^{-1}}(e) = e'$ and $\sigma_a(e') = e$.

e and vertices $x \neq y^{-1}$, then there exists a word w in U such that xy is a length-2 cyclic subword of w or w^{-1}. The proof of the following observation is now elementary.

Lemma 12. *Let U be a list of cyclically reduced words in F_n. In $W(U)$, consider an edge f_0 and vertices $x_1, x_2, \ldots, x_l$ where $l > 0$, such that $x_{i+1} \neq x_i^{-1}$ for $i = 1, \ldots, l$. Suppose that*

$$\sigma_{x_l^{-1}} \circ \sigma_{x_{l-1}^{-1}} \circ \cdots \circ \sigma_{x_1^{-1}}(f_0)$$

is well-defined and equal to f_0. Then $x_1 x_2 \cdots x_l$ is a nontrivial power of a cyclic conjugation of a word in U. $\qquad\square$

Connecting maps can be described in $Z(U)$. The link of a vertex p in a polygonal disk P is called the *corner* of P at p. Suppose an edge e is incident with a_i^{-1} in $W(U)$, where e corresponds to the corner of a vertex x in some D_j attached to $Z(U)$. Since we are assuming that every word in U is cyclically reduced, there exists a unique a_i-edge α outgoing from x. Choose the other endpoint y of α, and let $e' \in E(W(U))$ correspond to the corner of D_j at y; see Figure 3. Then we observe that $\sigma_{a_i^{-1}}(e) = e'$ and $\sigma_{a_i}(e') = e$.

3.3. **Graph-theoretic formulation of Tiling Conjecture.** The polygonality was described in terms of Whitehead graphs [17, Propositions 17 and 21]. But this description required infinitely many graphs to be examined. In the following lemma, we obtain a simpler formulation of polygonality requiring only one finite graph to be examined.

Lemma 13. *Let $n > 1$. A list U of cyclically reduced words in F_n is polygonal if and only if $W(U)$ has a nonempty list of cycles such*

that one of the cycles has length at least three and for each pair of edges e and f incident with a vertex v, the number of cycles in the list containing both e and f is equal to the number of cycles in the list containing both $\sigma_v(e)$ and $\sigma_v(f)$. Here, σ_v denotes the connecting map associated with $W(U)$ at v.

We prove the necessity part by similar arguments to [17, Propositions 17 and 21]. The sufficiency part is what we mainly need for this paper.

Proof. We denote by μ the involution on the vertices of $W(U)$ defined by $\mu(a_i^{\pm 1}) = a_i^{\mp 1}$.

To prove the necessity, assume U is polygonal; we can find a U-polygonal surface $S = \coprod_{1 \le i \le m} P_i / \sim$ as in Definition 6. In particular, each edge in $S^{(1)}$ is oriented and labeled by $\mathcal{A}_n$. Put $S^{(0)} = \{v_1, \ldots, v_t\}$. Fix p_i in the interior of each P_i. In Section 2.2, we have seen that there exists a map $\phi \colon S \to Z(U)$ such that ϕ is locally injective away from $p_1, \ldots, p_m$. Since S is a closed surface and ϕ is locally injective at v_i, the image of each $\text{Link}_S(v_i)$ by ϕ is a cycle, say C_i, in $W(U)$.

Choose a vertex $v \in W(U)$ and two edges e, f incident with v. Without loss of generality, we may assume that $v = a^{-1}$ for some generator $a \in \mathcal{A}_n$ and $C_1, \ldots, C_{t'}$ is the list of the cycles among $C_1, \ldots, C_t$ which contain both e and f. Then for each $i = 1, \ldots, t'$, there exists a unique a-edge e_i outgoing from v_i. Let $v_{i'}$ be the endpoint of e_i other than v_i. There exist exactly two polygonal disks Q_i and R_i sharing e_i in S, so that $\text{Link}(\phi; v_i)$ sends the corner of Q_i at v_i to e, and that of R_i at v_i to f. By the definition of a connecting map, $\text{Link}(\phi; v_{i'})$ maps the corners of Q_i and R_i at $v_{i'}$ to $\sigma_{a^{-1}}(e)$ and $\sigma_{a^{-1}}(f)$, respectively; see Figure 4, which is similar to [17, Figure 7]. The correspondence $e \cup f \to \sigma_{a^{-1}}(e) \cup \sigma_{a^{-1}}(f)$ defines an involution on the list of length-2 subpaths of $C_1, \ldots, C_t$. The conclusion follows.

For the sufficiency, consider a list of cycles $C_1, \ldots, C_t$ in $W(U)$ satisfying the given condition. For each C_i, let V_i be a polygonal disk such that ∂V_i is a cycle of the same length as C_i. We will regard ∂V_i as the dual cycle of C_i, in the sense that each edge of ∂V_i corresponds to a vertex of C_i and incident edges correspond to adjacent vertices. Choose a linear order $\prec$ on $\{(v, e) : e \in \delta(v)\}$ for each $v \in V(W(U))$ such that $(v, e) \prec (v, e')$ if and only if $(\mu(v), \sigma_v(e)) \prec (\mu(v), \sigma_v(e'))$. An edge g of ∂V_i will be labeled by $(a, \{e, f\})$ if the vertex v of $W(U)$ corresponding to g is labeled by a or a^{-1} for some $a \in \mathcal{A}_n$, and e and f are the two edges of C_i incident with v; see Figure 5 (a) and (b). Considered as a side of V_i, g will be given with a transverse orientation, which is incoming into V_i if $v \in \mathcal{A}_n$ and outgoing if $v \in \mathcal{A}_n^{-1}$. If
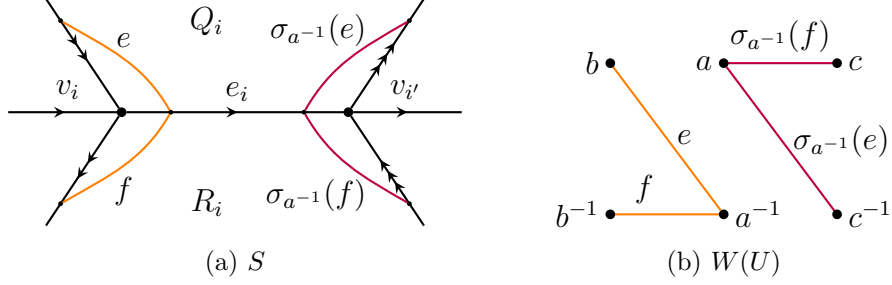
(a) S

(b) $W(U)$

FIGURE 4. Consecutive corners in S and their images by a connecting map. $F_3 = \langle a, b, c \rangle$, and single, double and triple arrows denote the labels a, b and c, respectively.

w_e and w_f denote the vertices of g corresponding to e and f respectively, and $(v, e) \prec (v, f)$, then we shall orient g from w_f to w_e. Define a side-paring $\sim_0$ on $V_1, \ldots, V_t$ such that $\sim_0$ respects the orientations, and moreover, an incoming side labeled by $(a, \{e, f\})$ is paired with an outgoing side labeled by $(a, \{\sigma_a(e), \sigma_a(f)\})$ for each $a \in \mathcal{A}_n$ and $e, f \in \delta(a)$ where e and f are consecutive edges of some cycle C_i; the existence of such a side-pairing is guaranteed by the given condition. Consider the closed surface $S_0 = \coprod_i V_i/\sim_0$. Denote by η and ζ the numbers of the edges and the faces in S_0, respectively. Each edge in S_0 is shared by two faces, and each face has at least two edges; moreover, at least one face has more than two edges by the given condition. So, $2\zeta < \sum_i (\text{the number of sides in } V_i) = 2\eta$.

By the duality between C_i and V_i, each corner of V_i corresponds to an edge in C_i. Then the link of a vertex q of S_0 corresponds to the union of edges in $W(U)$ written as the following sequence

$$f_0, f_1 = \sigma_{x_1^{-1}}(f_0), f_2 = \sigma_{x_2^{-1}}(f_1), \ldots, f_l = \sigma_{x_l^{-1}}(f_{l-1})$$

so that $f_0 = f_l = \sigma_{x_l^{-1}} \circ \sigma_{x_{l-1}^{-1}} \circ \cdots \circ \sigma_{x_1^{-1}}(f_0)$ for some vertices $x_1, \ldots, x_l$ of $W(U)$; see Figure 5 (c). By Lemma 12, $x_1 \cdots x_l$ can be taken as a nontrivial power of a word in U. We will follow the boundary curve α of a small neighborhood of q with some orientation, and whenever α crosses an edge of S_0 with the first component of the label being $a \in \mathcal{A}_n$, we record a if the crossing coincides with the transverse orientation of the edge, and a^{-1} otherwise. Let $w_q \in F$ be the word obtained by this process. Then $w_q = x_1 \cdots x_l$, up to taking an inverse and cyclic conjugations.

Let S be a surface homeomorphic to S_0. We give S a 2-dimensional cell complex structure, by letting the homeomorphic image of the dual

(a) C_i

(b) V_i
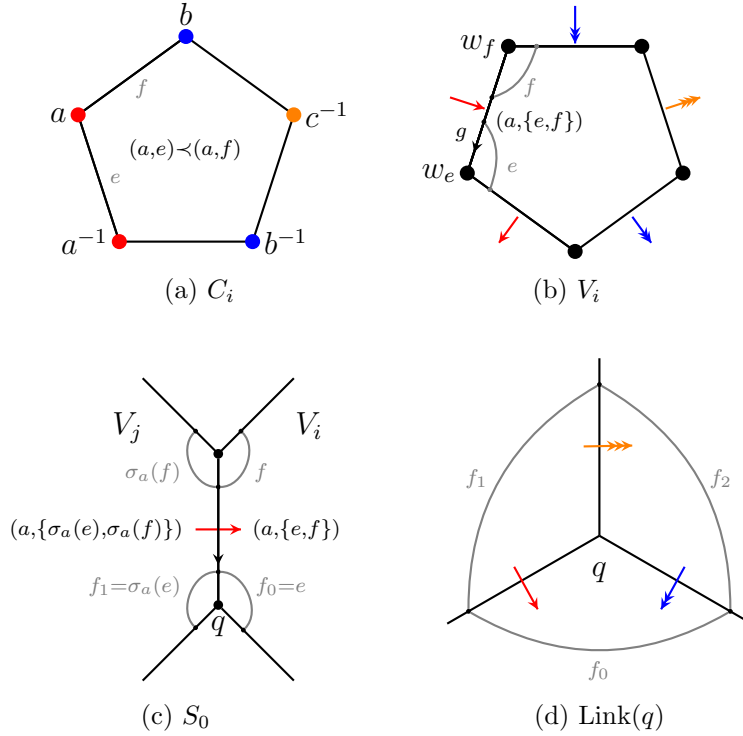

(c) S_0

(d) $\mathrm{Link}(q)$

FIGURE 5. Constructing V_i and S_0 from C_i in the proof of Lemma 13. In this example, we note from (d) that $f_1 = \sigma_a(f_0)$, $f_2 = \sigma_{c^{-1}}(f_1)$ and $f_0 = \sigma_{b^{-1}}(f_2)$.

graph of $S_0^{(1)}$ to be $S^{(1)}$. In particular, the 2-cells $P_1, \ldots, P_m$ in S are the connected regions bounded by $S^{(1)}$. The transverse orientations and the first components of the labels of the sides in $V_1, \ldots, V_t$ induce orientations and labels of the sides of $P_1, \ldots, P_m$. By duality, the boundary reading of each P_i in S is of the form w_q for some vertex q of S_0; hence, ∂P_i reads a nontrivial power of a word in U. Finally, if we let ν be the number of the vertices in S_0, then

$$\chi(S) - m = \chi(S_0) - \nu = -\eta + \zeta < 0. \qquad \square$$

Remark. There is a polynomial-time algorithm to decide whether a list of words in a free group is polygonal [17]. We note that diskbusting property can also be determined in polynomial time [30, 29, 28, 25].

A graph is *non-acyclic* if it contains at least one cycle. We now restate Tiling Conjecture combinatorially as follows.

Conjecture 14. *Let $G = (V, E)$ be a non-acyclic graph with a fixed point free involution $\mu : V \to V$ and a bijection $\sigma_v : \delta(v) \to \delta(\mu(v))$ for every vertex v such that $\lambda(v, \mu(v)) = \deg(v)$ and $\sigma_{\mu(v)} = \sigma_v^{-1}$. Then there exists a nonempty list of cycles of G such that for each pair of edges e and f incident with a vertex v, the number of cycles in the list containing both e and f is equal to the number of cycles in the list containing both $\sigma_v(e)$ and $\sigma_v(f)$. Moreover, the list can be required to contain at least one cycle of length greater than two if G has a connected component which has at least four vertices.*

Proposition 15. *Let $n' > 1$. Tiling Conjecture holds for all $n = 2, \ldots, n'$ if and only if Conjecture 14 holds for graphs on $2n'$ vertices.*

Proof. (Conjecture 14 $\Rightarrow$ Tiling Conjecture) Let $2 \leq n \leq n'$ and let U be a minimal and diskbusting list of cyclically reduced words in F_n. If Conjecture 14 holds for $2n'$, then it holds for $2n$ because we can add isolated vertices. By Proposition 11, the connected graph $W(U)$ is equipped with the fixed point free involution $\mu(v) = v^{-1}$ on $V(W(U))$ and the associated connecting map σ_v at each vertex v such that $\lambda(v, \mu(v)) = \deg(v)$ and $\sigma_{\mu(v)} = \sigma_v^{-1}$. Note that $W(U)$ is non-acyclic; because otherwise $\deg(v) = \lambda(v, \mu(v)) \leq 1$ for each vertex v and therefore $W(U)$ would be disconnected, as $W(U)$ has at least four vertices. The conclusion of Conjecture 14 along with Lemma 13 implies that U is polygonal.

(Tiling Conjecture $\Rightarrow$ Conjecture 14) We let G, μ, σ_v be as in the hypothesis of Conjecture 14 such that $|V(G)| = 2n'$. Let $n = n'$. Since for each vertex v, v and $\mu(v)$ belong to the same connected component of G, we may assume that G is connected by taking a non-acyclic component of G. If $|V(G)| = 2$, then the list of all bigons is a desired collection of cycles. So we assume G is connected and $|V(G)| \geq 4$. Label the vertices of G as $a_1, a_1^{-1}, \ldots, a_n, a_n^{-1}$ so that $a_i^{-1} = \mu(a_i)$. Then G can be regarded as the Whitehead graph of a list U of cyclically reduced words in F_n. Proposition 11 implies that U is minimal and diskbusting, as well. As we are assuming Tiling Conjecture for F_n, U is polygonal. Lemma 13 completes the proof. $\square$

In Sections 4 and 5, we will prove Conjecture 14 for regular graphs and four-vertex graphs, respectively. This amounts to proving Tiling Conjecture for k-regular lists of words and for rank-two free groups.

4. REGULAR GRAPH AND PROOF OF THEOREM 2

We will prove that Conjecture 14 holds for regular graphs. It turns out that we can prove a slightly stronger theorem.

Theorem 16. *Let $k > 1$. Let $G = (V, E)$ be a k-regular graph with a fixed point free involution $\mu : V \to V$ such that $\lambda(v, \mu(v)) = k$ for every vertex $v \in V$. Then there exists a nonempty list of cycles of G with positive integers m_1, m_2 such that every edge is in exactly m_1 cycles in the list and each adjacent pair of edges is contained in exactly m_2 cycles in the list.*

Theorem 2 is now an immediate corollary of the following.

Corollary 17. *A minimal, diskbusting, k-regular list of words in F_n is polygonal when $n > 1$.*

Proof of Corollary 17. Let U be such a list. By Proposition 11, $W(U)$ satisfies the hypotheses of Theorem 16, and moreover, $W(U)$ is connected and k-regular. Since $W(U)$ has $2n$ vertices and $n > 1$, it has two adjacent edges e and f, not parallel to each other. By Theorem 16, there must be a cycle in the list containing both e and f and that cycle must have length at least three. Lemma 13 completes the proof. $\square$

A graph H is a *subdivision* of G if H is obtained from G by replacing each edge by a path of length at least one. We remark that Conjecture 14 is also true for all subdivisions of k-regular graphs if $k > 1$, because every edge appears the same number of times in Theorem 16.

Let us start proving Theorem 16. A graph $G = (V, E)$ is called a *k-graph* if it is k-regular and $|\delta(X)| \geq k$ for every subset X of V with $|X|$ odd. In particular if $k > 0$, then every k-graph must have an even number of vertices, because otherwise $|\delta(V(G))| \geq k$.

It turns out that every k-regular graph with the properties required by Conjecture 14 is a k-graph.

Lemma 18. *Let $G = (V, E)$ be a k-regular graph with a fixed point free involution μ such that $\lambda(v, \mu(v)) = k$ for every vertex $v \in V$. Then G is a k-graph.*

Proof. Supposes $X \subseteq V$ and $|X|$ is odd. Then there must be $x \in X$ with $\mu(x) \notin X$ because μ is an involution such that $\mu(v) \neq v$ for all $v \in V$. Then there exist k edge-disjoint paths from x to $\mu(x)$ and therefore $|\delta(X)| \geq k$. $\square$

By the previous lemma, it is sufficient to consider k-graphs in order to prove Theorem 16. By using the characterization of the perfect matching polytope by Edmonds [10], Seymour [27] showed the following theorem. This is also explained in Corollary 7.4.7 of the book by Lovász and Plummer [21]. A *matching* is a set of edges in which no two are adjacent. A *perfect matching* is a matching meeting every vertex.

Theorem 19 (Seymour [27])**.** *Every k-graph is* fractionally k-edge-colorable. *In other words, every k-graph has a nonempty list of perfect matchings M_1, M_2, ..., M_ℓ such that every edge is in exactly ℓ/k of them.*

For sets A and B, we write $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

Lemma 20. *Let $k > 1$. Every k-graph has a nonempty list of cycles such that every edge appears in the same number of cycles and for each pair of adjacent edges e, f, the number of cycles in the list containing both e and f is identical.*

Proof. Let M_1, M_2, ..., M_ℓ be a nonempty list of perfect matchings of a k-graph $G = (V, E)$ such that each edge appears in ℓ/k of them. Then for distinct i, j, the set $M_i \Delta M_j$ induces a subgraph of G such that every vertex has degree 2 or 0. Thus each component of the subgraph $(V, M_i \Delta M_j)$ is a cycle. Let C_1, C_2, ..., C_m be the list of cycles appearing as a component of the subgraph of G induced by $M_i \Delta M_j$ for each pair of distinct i and j. We allow repeated cycles. This list is nonempty because $k > 1$ and so there exist i, j such that $M_i \neq M_j$.

Since each edge is contained in exactly ℓ/k of M_1, M_2, ..., M_ℓ, every edge is in exactly $\frac{\ell}{k}(\ell - \frac{\ell}{k})$ cycles in the list. For two adjacent edges e and f, since no perfect matching contains both e and f, there are $(\ell/k)^2$ cycles in C_1, C_2, ..., C_m using both e and f. $\qquad\square$

Lemmas 18 and 20 clearly imply Theorem 16. We also note that even the minimality assumption can be lifted for rank-two free groups:

Corollary 21. *Let U be a k-regular list of cyclically reduced words in F_2. Then U is diskbusting if and only if U is polygonal; in this case, $D(U)$ contains a hyperbolic surface group.*

Proof. We note that a k-regular 4-vertex graph is always a k-graph.

For the sufficiency, we recall that if U is diskbusting in F_2, then $W(U)$ is connected [29, 28]. Since a connected 4-vertex graph contains at least one pair of incident edges which are not parallel, Lemma 20 implies that $W(U)$ contains a list of cycles, not all bigons, such that each pair of incident edges appears the same number of times in the list. Lemma 13 proves the claim.

For the necessity, we note that the proof of the sufficiency part of Proposition 9 shows if U is not diskbusting in F_2, then $D(U)$ does not contain a hyperbolic surface group. $\qquad\square$

5. Graphs on four vertices

Let G be a graph with a fixed point free involution $\mu : V(G) \to V(G)$ and a bijection $\sigma_v : \delta(v) \to \delta(\mu(v))$ for each vertex v so that $\lambda(v, \mu(v)) = \deg(v)$ and $\sigma_{\mu(v)} = \sigma_v^{-1}$. For a vertex w of G, a permutation π on $\delta(w)$ is called w-good if $\{e, \sigma_w(\pi(e))\}$ is a matching of G for every edge e incident with w. Note that $\{e, f\}$ is a matching of G if and only if either $e = f$ or e, f share no vertex. In particular, if x is an edge joining w and $\mu(w)$, then $\sigma_w(\pi(x)) = x$.

A permutation π on a set X induces a permutation $\pi^{(2)}$ on 2-element subsets of X such that $\pi^{(2)}(\{x, y\}) = \{\pi(x), \pi(y)\}$ for all distinct $x, y \in X$. A w-good permutation π on $\delta(w)$ is *uniform* if $\pi^{(2)}$ has a list of orbits X_1, X_2, ..., X_t satisfying the following.

(i) If $\{x, y\} \in X_i$, then x and y do not share a vertex other than w or $\mu(w)$ in G.

(ii) There is a constant $c > 0$ such that for every edge $e \in \delta(w)$,

$$|\{(X_i, F) : 1 \leq i \leq t, \ F \in X_i \text{ and } e \in F\}| = c.$$

The following lemma shows that in order to prove Conjecture 14 for 4-vertex graphs, it is enough to find a w-good uniform permutation on the edges incident with a vertex w of minimum degree.

Lemma 22. *Let G be a connected 4-vertex graph with a fixed point free involution $\mu : V(G) \to V(G)$ such that $\lambda(v, \mu(v)) = \deg(v)$ for each vertex v. Let w be a vertex of G with the minimum degree. Let $\sigma_w : \delta(w) \to \delta(\mu(w))$ be a bijection.*

If there is a w-good uniform permutation π on $\delta(w)$, then G admits a nonempty list of cycles satisfying the following properties.

(a) *For distinct edges $e_1, e_2 \in \delta(w)$, the number of cycles in the list containing both e_1 and e_2 is equal to the number of cycles in the list containing both $\sigma_w(e_1)$ and $\sigma_w(e_2)$.*

(b) *There is a constant $c_1 > 0$ such that each edge appears in exactly c_1 cycles in the list.*

(c) *There is a constant $c_2 > 0$ such that for a vertex $v \in V(G) \setminus \{w, \mu(w)\}$ and each pair of distinct edges $e_1, e_2 \in \delta(v)$, exactly c_2 cycles in the list contain both e_1 and e_2.*

(d) *The list contains a cycle of length at least three.*

Proof. We say that a list of cycles is *good* if it satisfies (a), (b), (c), and (d). We proceed by induction on $|E(G)|$. Let u be a vertex of G other than w and $\mu(w)$. If $\deg(u) = \deg(w)$, then the conclusion follows by Theorem 16. Therefore we may assume that $\deg(u) > \deg(w)$. There should exist an edge e joining u and $\mu(u)$. Moreover $G \setminus e$ is connected

because otherwise G would not have $\deg(w)$ edge-disjoint paths from w to $\mu(w)$.

By the induction hypothesis, $G \setminus e$ has a good list of cycles C_1', C_2', ..., C_s'. Note that we use the fact that $\deg(u) > \deg(w)$ so that $G \setminus e$ has $\deg_{G \setminus e}(v)$ edge-disjoint paths from v to $\mu(v)$ for each vertex v of $G \setminus e$. Let c_1', c_2' be the constants given by (b) and (c), respectively, for the list C_1', C_2', ..., C_s' of cycles of $G \setminus e$.

Since π is w-good uniform, $\pi^{(2)}$ has a list of orbits X_1, X_2, ..., X_t satisfying (i) and (ii), where each edge in $\delta(w)$ appears c times in this list.

Suppose that $\{x, y\} \in X_i$. Then $\{\pi(x), \pi(y)\} \in X_i$. If $x, y \in \delta(\mu(w))$, then we let C_{xy} be a cycle formed by two edges $x = \sigma_w(\pi(x))$ and $y = \sigma_w(\pi(y))$. If $x, y \notin \delta(\mu(w))$, let C_{xy} be a list of two cycles, one formed by three edges e, x, y, and the other formed by three edges e, $\sigma_w(\pi(x))$, $\sigma_w(\pi(y))$. If exactly one of x and y, say y, is incident with $\mu(w)$, then let C_{xy} be the cycle formed by four edges e, x, $y = \sigma_w(\pi(y))$, $\sigma_w(\pi(x))$. Since x and y never share u or $\mu(u)$ by (i), C_{xy} always consists of one or two cycles of G.

Let $C_1, C_2, \ldots, C_p$ be the list of all cycles in C_{xy} for each member $\{x, y\}$ of X_i for all $i = 1, 2, \ldots, t$. Notice that we allow repetitions of cycles.

We claim that the list $C_1, C_2, \ldots, C_p$ satisfies (a). For each occurrence of $x, y \in \delta(w)$ in a cycle in the list, there is a corresponding i such that $\{x, y\} \in X_i$. Since X_i is an orbit, there is $\{x', y'\} \in X_i$ where $\pi(x') = x$ and $\pi(y') = y$. Then the list contains cycles in $C_{x'y'}$ for X_i. This proves the claim because $\sigma_w(x) = \sigma_w(\pi(x'))$ and $\sigma_w(y) = \sigma_w(\pi(y'))$.

By (ii) of the definition of a uniform permutation, for each edge f incident with w, there are c cycles in the list $C_1, C_2, \ldots, C_p$ containing the edge f of G. Notice that whenever an edge f in C_{xy} is in $\delta(\{w, \mu(w)\})$, C_{xy} contains e and $\sigma_w(\pi(f))$ by the construction. Therefore every edge incident with w or $\mu(w)$ appears c times in the list $C_1, C_2, \ldots, C_p$.

We now construct a good list of cycles for G as follows: We take c_2' copies of $C_1, C_2, \ldots, C_p$, c copies of $C_1', C_2', \ldots, C_s'$, and cc_2' copies of cycles formed by e and another edge $f \neq e$ joining u and $\mu(u)$. We claim that this is a good list of cycles of G. It is trivial to check (a). For distinct edges e_1, e_2 incident with u, the list contains cc_2' cycles containing both of them, verifying (c). Let a be the number of edges in $\delta(u)$ incident with w or $\mu(w)$ and let b be the number of edges joining u and $\mu(u)$. By (c) on $G \setminus e$, we have $c_1' = c_2'(a + b - 2)$. Finally to prove (b), every edge incident with w or $\mu(w)$ appears $cc_2' + cc_1' = cc_2'(a + b - 1)$

times in the list and the edge e appears $acc_2' + (b-1)cc_2' = cc_2'(a+b-1)$ times in the list. An edge $f \neq e$ joining u and $\mu(u)$ appears $cc_1' + cc_2' = cc_2'(a+b-1)$ times in the list. $\square$

5.1. **Lemma on Odd Paths and Even Cycles.** To find a w-good uniform permutation of $\delta(w)$, we need a combinatorial lemma on a set of disjoint odd paths and even cycles. The *length* of a path or a cycle is the number of its edges.

Lemma 23. *Let D be a directed graph with at least four vertices such that each component is a directed path of odd length or a directed cycle of even length. Suppose that every vertex of in-degree 0 or out-degree 0 in D is colored with red or blue, while the number of red vertices of in-degree 0 is equal to the number of red vertices of out-degree 0. We say that a graph is* good *if at most half of all the vertices are blue and at most half of all the vertices are red. We say that a directed path or a cycle is* long *if its length is at least three. A directed path or a cycle is said to be* short *if it is not long. A R-R* path *denotes a directed path starting with a red vertex and ending with a red vertex. Similarly we say* R-B *paths,* B-B *paths,* B-R *paths. A set of paths is called* monochromatic *if it has no blue vertex or no red vertex.*

If D is good, then D can be partitioned into good subgraphs, each of which is one of eight types listed below. (See Figure 6.)

 (1) A short R-R path, a short B-B path, and possibly a short cycle.
 (2) A monochromatic path and one or two short cycles.
 (3) A short cycle, a B-R path, and an R-B path.
 (4) At least two short cycles.
 (5) A long monochromatic path and monochromatic short paths, possibly none.
 (6) A B-R path, a R-B path, and monochromatic short paths, possibly none.
 (7) A long cycle and monochromatic short paths, possibly none.
 (8) A long cycle and a short cycle.

We remark that in a subgraph of type (5), we require that the long path is monochromatic and the set of short paths monochromatic, but we allow the long path to have a color unused in short paths.

Proof. We proceed by induction on $|V(D)|$. If D has a subgraph H that is a disjoint union of a short R-R path and a short B-B path, then $D \setminus V(H)$, the subgraph obtained by removing vertices of H from D, is still good. If $D = H$, then we have nothing to prove. If $|V(D) \setminus V(H)| = 2$, then D is the disjoint union of a short R-R path, a short B-B path, and a short cycle, and therefore D is a directed graph of type (1). If
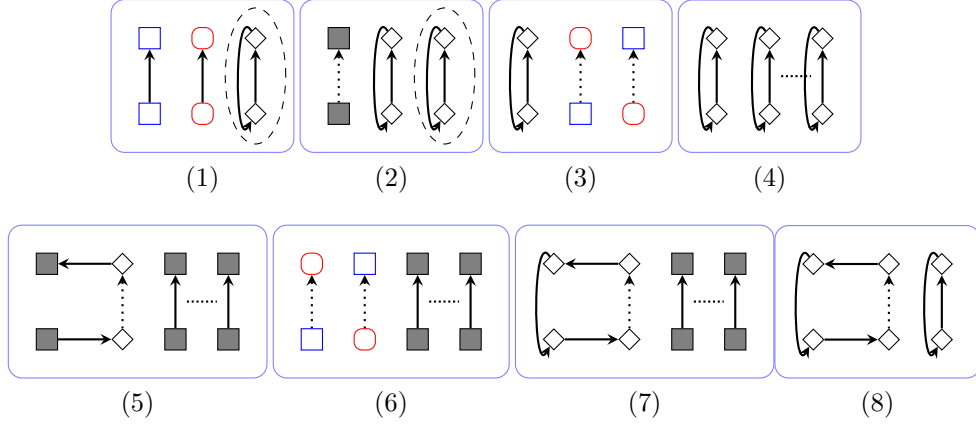
FIGURE 6. Description of eight types of good subgraphs

$|V(D) \setminus V(H)| \geq 4$, then H is a good subgraph of type (1). Then we apply the induction hypothesis to get a partition for $D \setminus V(H)$.

Therefore we may assume that D has no pair of a short B-B path and a short R-R path. By symmetry, we may assume that D has no short R-R path. Then in each component, the number of red vertices is at most half of the number of vertices. Thus, in order to check whether some disjoint union of components is good, it is enough to count blue vertices.

Suppose that D has a short cycle and a short B-B path. We are done if D is a graph of type (2). Thus we may assume that D has at least eight vertices. Let X be the set of vertices in the pair of a short cycle and a short B-B path. Then the subgraph of D induced on X is a subgraph of type (2). Because X has two blue vertices and two uncolored vertices, $D \setminus X$ is good and has at least four vertices. By the induction hypothesis, we obtain a good partition of $D \setminus X$. This together with the subgraph induced by X is a good partition of D.

We may now assume that either D has no short cycles, or D has no short B-B path.

(Case 1) Suppose that D has no short cycles. The subgraph of D consisting of all components other than short B-B paths can be partitioned into good subgraphs $P_1, P_2, \ldots, P_k$ of type (5), (6), or (7), because the number of R-B paths is equal to the number of B-R paths. We claim that short B-B paths can be assigned to those subgraphs while maintaining each P_i to be good. Suppose that P_i has $2b_i$ blue vertices and $2n_i = |V(P_i)|$. Notice that b_i and n_i are integers. Let x be the number of short B-B paths in D. Since D is good,
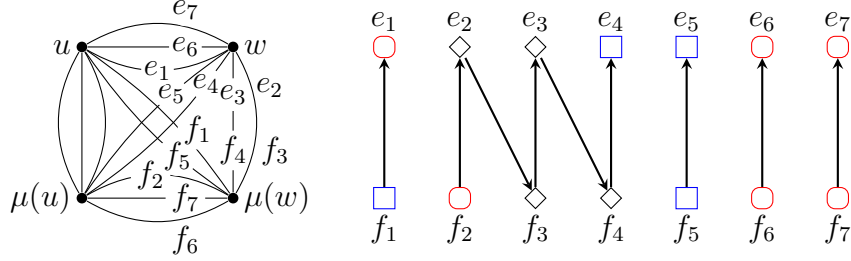
FIGURE 7. A graph and its auxiliary directed graph at w

$2(2x + \sum_{i=1}^{k} 2b_i) \leq \sum_{i=1}^{k} 2n_i + 2x$ and therefore $x \leq \sum_{i=1}^{k}(n_i - 2b_i)$. Each P_i can afford to have $n_i - 2b_i$ short B-B paths to be good. Overall all $P_1, \ldots, P_k$ can afford $\sum_{i=1}^{k}(n_i - 2b_i)$ short B-B paths; thus consuming all short B-B paths. This proves the claim.

(Case 2) Suppose D has short cycles but has no short B-B paths. If D has at least two short cycles, then we can take all short cycles as a subgraph of type (4) and the subgraph of D consisting of all components other than short cycles can be decomposed into subgraphs, each of which is type (5), (6), or (7).

Thus we may assume D has exactly one short cycle. Since D has at least four vertices, D must have a subgraph P consisting of components of D that is one of the following type: a monochromatic path, a long cycle, or a pair of a B-R path and an R-B path. Then P with the short cycle forms a good subgraph of type (2), (8), or (3), respectively. The subgraph of D induced by all the remaining components can be decomposed into subgraphs of type (5), (6), and (7). □

5.2. **Finding a Good Uniform Permutation.** Let G be a connected 4-vertex graph with a fixed point free involution $\mu : V(G) \to V(G)$ such that $\lambda(v, \mu(v)) = \deg(v)$ for each vertex v. Let w be a vertex of G with the minimum degree and let u be a vertex of G other than w and $\mu(w)$. Let $\sigma_w : \delta(w) \to \delta(\mu(w))$ be a bijection.

Let $e_1, e_2, \ldots, e_m$ be the edges incident with w and let $f_1, f_2, \ldots, f_m$ be the edges incident with $\mu(w)$ so that $f_i = \sigma_w(e_i)$. We construct an auxiliary directed graph D on the disjoint union of $\{e_1, e_2, \ldots, e_m\}$ and $\{f_1, f_2, \ldots, f_m\}$ as follows:

(i) For all $i \in \{1, 2, \ldots, m\}$, D has an edge from f_i to e_i.
(ii) If e_i and f_j denote the same edge in G, then D has an edge from e_i to f_j.

We have an example in Figure 7. It is easy to observe the following.

 • Every vertex in $\{e_1, e_2, \ldots, e_m\}$ of D has in-degree 1.

- Every vertex in $\{f_1, f_2, \ldots, f_m\}$ of D has out-degree 1.
- A vertex e_i of D has out-degree 1 if the edge e_i of G is incident with $\mu(u)$, and out-degree 0 if otherwise.
- A vertex f_i of D has in-degree 1 if the edge f_i of G is incident with u, and in-degree 0 if otherwise.

By the degree condition, D is the disjoint union of odd directed paths and even directed cycles.

Let r be the number of edges of G joining u and w and let b be the number of edges of G joining $\mu(u)$ and w. For each i, we color e_i red if it is incident with u and blue if it is incident with $\mu(u)$. Similarly for each i, we color f_i blue if it is incident with u and red if it is incident with $\mu(u)$. Clearly there are r red vertices and b blue vertices in $\{e_1, e_2, \ldots, e_m\}$.

Let r' be the number of edges of G joining $\mu(u)$ and $\mu(w)$ and let b' be the number of edges of G joining u and $\mu(w)$. We claim that $r' = r$ and $b' = b$. Of course, there are r red vertices and b blue vertices in $\{f_1, f_2, \ldots, f_m\}$. Since $\deg w = \deg \mu(w)$ and $\deg u = \deg \mu(u)$, we have $r + b' = b + r'$ and $r + b = r' + b'$. We deduce that $r = r'$ and $b = b'$.

We also assume that G has $\deg(u)$ edge-disjoint paths from u to $\mu(u)$. Therefore $|\delta(\{u, w\})| \geq |\delta(u)|$ and $|\delta(\{u, \mu(w)\})| \geq |\delta(u)|$. This implies that $b + b + (m - r - b) \geq b + r$ and $r + r + (m - r - b) \geq b + r$. Thus

$$2r \leq m \text{ and } 2b \leq m.$$

From now on, our goal is to describe a w-good permutation π on $\delta(w)$ from a directed graph D with a few extra edges.

Lemma 24. *Let D' be a directed graph obtained by adding one edge from each vertex of out-degree 0 to a vertex of in-degree 0 with the same color so that every vertex has in-degree 1 and out-degree 1 in D'. Let π be a permutation on $\delta(w) = \{e_1, e_2, \ldots, e_m\}$ so that $\pi(e_i) = e_j$ if and only if D' has a directed walk from e_i to e_j of length two. Then π is w-good.*

Let us call such a directed graph D' a *completion* of D. A completion of D' always exists, because the number of red vertices of in-degree 0 is equal to the number of red vertices of out-degree 0. Clearly there are $r! \, b!$ completions of D.

Proof. It is enough to show that if D' has an edge e from e_i to f_j, then $\{e_i, f_j\}$ is a matching of G. If $e \in E(D)$, then $e_i = f_j$ and therefore $\{e_i, f_j\} = \{e_i\}$ is a matching of G. If $e \notin E(D)$, then e_i and f_j should have the same color and therefore e_i and f_j do not share any vertex. □

Out of $r!\,b!$ completions of D', we wish to find a completion D' of D so that the w-good permutation induced by D' is uniform.

Lemma 25. *If D is a directed graph of type (1), (2), ..., (8) described in Lemma 23, then D has a completion D' so that the induced w-good permutation is uniform.*

Proof. We claim that for each type of a directed graph D, there is a completion D' of D such that its induced w-good permutation π on $\delta(w)$ is uniform. Recall that a w-good permutation π is uniform if $\pi^{(2)}$ has a list of orbits X_1, X_2, ..., X_t satisfying the following conditions:

(i) If $\{x, y\} \in X_i$, then x and y do not share a vertex other than w or $\mu(w)$ in G.

(ii) There is a constant $c > 0$ such that for every edge $e \in \delta(w)$,

$$|\{(X_i, F) : 1 \leq i \leq t, \ F \in X_i \text{ and } e \in F\}| = c.$$

Case 1: Suppose that D is of type (1) or (4) with k components. Then There is a unique completion D' of D. It is easy to verify that the list of all orbits of $\pi^{(2)}$ satisfies the conditions (i) and (ii) where $c = k - 1$.

Case 2: Suppose that D is of type (2). Then D consists of a monochromatic path P and one or two short cycles. A completion D' of D is unique, as it is obtained by adding an edge from the terminal vertex of P to the initial vertex of P. Let π be the permutation of $\delta(w)$ induced by D'. Let $x_1, x_2, \ldots, x_m$ be the edges in $\delta(w)$ that are in P such that $\pi(x_i) = x_{i+1}$ for all $i = 1, 2, \ldots, m$ where $x_{m+1} = x_1$. Let $y_1 \in \delta(w)$ be the vertex in the first short cycle such that $\pi(y_1) = y_1$. If D has two cycles, then let $y_2 \in \delta(w)$ be the vertex in the second short cycle such that $\pi(y_2) = y_2$.

Then $O_j = \{\{x_i, y_j\} : 1 \leq i \leq m\}$ is an orbit of $\pi^{(2)}$ satisfying (i). If $m > 1$, then $O_P = \{\{x_i, x_{i+1}\} : 1 \leq i \leq m\}$ is an orbit of $\pi^{(2)}$ satisfying (i) in which each x_i appears twice if $m > 2$ and each x_i appears once if $m = 2$.

If D has only one cycle, then each x_i appears once and y_1 appears m times in O_1. So if $m = 1$, then O_1 satisfies (i) and (ii). If $m = 2$, then O_1 and O_P form a list of orbits of $\pi^{(2)}$ satisfying (i) and (ii). If $m > 2$, then a list of two copies of O_1 and $(m - 1)$ copies of O_P satisfies (i) and (ii).

If D has two short cycles, then in O_1 and O_2, each x_i appears twice and each y_j appears m times. Notice that $\{\{y_1, y_2\}\}$ is an orbit of $\pi^{(2)}$. If $m = 1$, then a list of O_1, O_2, and $\{\{y_1, y_2\}\}$ satisfies (i) and (ii). If $m = 2$, then a list of O_1 and O_2 satisfies (i) and (ii). If $m > 3$, then

a list of two copies of O_1, two copies of O_2, and $(m-2)$ copies of O_P satisfies (i) and (ii).

Case 3: If D is of type (3), then D has a unique completion D'. Let π be the permutation of $\delta(w)$ induced by D'. Let $y \in \delta(w)$ be a vertex of D in the short cycle such that $\pi(y) = y$. Let $x_1, x_2, \ldots, x_m \in \delta(w)$ be the vertices on the long cycle in D' such that $\pi(x_i) = x_{i+1}$ for all $i = 1, 2, \ldots, m$ where $x_{m+1} = x_1$. Since D has two paths, $m > 1$. Then $O_P = \{\{x_i, x_{i+1}\} : i = 1, 2, \ldots, m\}$ and $O_C = \{\{y, x_i\} : i = 1, 2, \ldots, m\}$ are orbits of $\pi^{(2)}$. In O_P, each x_i appears twice if $m > 2$ and once if $m = 2$. In O_C each x_i appears once and y appears m times. Now it is routine to create a list of orbits satisfying (i) and (ii) by taking copies of O_C and copies of O_P.

Case 4: Suppose that D is of type (5) having both red and blue vertices or D is of type (7) or (8). Let D' be a completion of D obtained by making each path of D to be a cycle of D'. Let $x_1, x_2, \ldots, x_m \in \delta(w)$ be vertices in the long cycle of D' so that $\pi(x_i) = x_{i+1}$ for all $i = 1, 2, \ldots, m$ where $x_{m+1} = x_1$. Let $y_1, y_2, \ldots, y_k \in \delta(w)$ be vertices in short cycles of D' such that $\pi(y_i) = y_i$. Since D is good, $k \le m$. Let $O_j = \{\{x_i, y_j\} : i = 1, 2, \ldots, m\}$ for $j = 1, 2, \ldots, k$ and $O_P = \{\{x_i, x_{i+1}\} : i = 1, 2, \ldots, m\}$ where $x_{m+1} = x_1$. In the list of O_1, O_2, $\ldots$, O_k, each x_i appears k times and each y_j appears m times. In O_P, each x_i appears twice if $m > 2$ and once if $m = 2$. To satisfy (i) and (ii), we can take a list of two copies of each O_j for $j = 1, 2, \ldots, k$ and copies of O_P.

Case 5: Suppose that D is a directed graph of type (5) not having both red and blue, or D is a directed graph of type (6). Then D has a completion D' consisting of a single cycle. Let π be the permutation of $\delta(w)$ induced by D'. Let $x_1, x_2, \ldots, x_m \in \delta(w)$ be vertices in D such that $\pi(x_i) = x_{i+1}$ for all $i = 1, 2, \ldots, m$. We $O_P = \{\{x_i, x_{i+\lfloor m/2 \rfloor}\} : i = 1, 2, \ldots, m\}$ where $x_{j+m} = x_j$ for all $j = 1, \cdots, \lfloor m/2 \rfloor$. Then in O_P, each x_i appears twice if m is odd and once if m is even. Moreover, since all the vertices of the same color appear consecutively in D' and the number of vertices of the same color is at most half of m, O_P never contains a pair $\{x_i, x_j\}$ of vertices of the same color, red or blue. Therefore O_P satisfies (i) and (ii). This completes the proof. $\square$

Lemma 26. *There exists a completion D' of D so that the w-good permutation induced by D' is uniform.*

Proof. By Lemma 23, D can be partitioned into good subgraphs D_1, D_2, $\ldots$, D_t of type (1), (2), $\ldots$, (8). Lemma 25 shows that each D_i admits a completion that induces a w-good uniform permutation π_i

with a list L_i of orbits of $\pi_i^{(2)}$ satisfying (i) and (ii). Let us assume that each vertex of D_i appears $c_i > 0$ times in L_i. Let $c = \text{lcm}(c_1, c_2, \ldots, c_t)$. Then let L be the list of orbits obtained by taking c/c_i copies of L_i for each $i = 1, 2, \ldots, t$. Then L satisfies (i) and (ii). This proves the lemma. $\qquad \square$

Now we are ready to prove Conjecture 14 for 4-vertex graphs:

Theorem 27. *Let G be a connected 4-vertex graph with a fixed point free involution $\mu : V(G) \to V(G)$ and a bijection $\sigma_v : \delta(v) \to \delta(\mu(v))$ for each vertex v such that $\lambda(v, \mu(v)) = \deg(v)$ and $\sigma_{\mu(v)} = \sigma_v^{-1}$. Then G has a nonempty list of cycles satisfying the following.*

(a) *For each pair of edges e and f incident with a vertex v, the number of cycles in the list containing both e and f is equal to the number of cycles in the list containing both $\sigma_v(e)$ and $\sigma_v(f)$.*
(b) *Each edge of G appears in the same number of cycles in the list.*
(c) *The list contains a cycle of length at least three.*

Proof. Let w be a vertex of minimum degree. By Lemma 26, G has a w-good uniform permutation π on $\delta(w)$. By Lemma 22, G has a nonempty list of cycles satisfying (a), (b), and (c). $\qquad \square$

We remark that Conjecture 14 is true for subdivisions of connected 4-vertex graphs because of (b). By Proposition 15 and Theorem 27, we verify Tiling Conjecture for rank-two free groups:

Corollary 28. *A minimal and diskbusting list of cyclically reduced words in F_2 is polygonal.*

Now Theorem 1 is an immediate consequence of the following.

Corollary 29. *For a list U of words in F_2, the following are equivalent.*

(1) *The list U is diskbusting.*
(2) *$D(U)$ contains a hyperbolic surface group.*
(3) *$D(U)$ is one-ended.*

Proof. (1)$\Leftrightarrow$(3) is well-known and stated in [12], for example. (1)$\Rightarrow$(2) follows from Corollary 28 and Theorem 7. By putting $n = 2$ in the Proposition 9, we have (2)$\Rightarrow$(1). $\qquad \square$

6. Final Remarks

Minimality assumption in Tiling Conjecture. A graph G is 2-*connected* if $|V(G)| > 2$, G is connected, and $G \setminus x$ is connected for every vertex x. It is well-known that a list U of cyclically reduced words in F_n is diskbusting if and only if $W(\phi(U))$ is 2-connected for
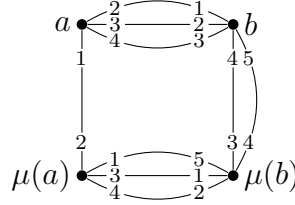
FIGURE 8. Example 30.

some $\phi \in \operatorname{Aut}(F_n)$ [29, 28]. However, the minimality assumption in Tiling Conjecture cannot be weakened to the 2-connectedness of the Whitehead graph; this is equivalent to saying that $\lambda(v, \mu(v)) = \deg(v)$ in Conjecture 14 cannot be relaxed to 2-connectedness. Daniel Král' [20] kindly provided us Example 30 showing why this relaxation is not possible.

Example 30. Let G be a 4-vertex graph shown in Figure 8. For a vertex v and edges $e \in \delta(v)$ and $f \in \delta(\mu(v))$, we let $\sigma_v(e) = f$ if and only if the number written on e near v coincides with the number written on f near $\mu(v)$. Actually, G is the Whitehead graph of $a(ab^{-1})^3 b^{-2}$ with the associated connecting maps σ_v. While G is 2-connected, one can verify that G does not have a list of cycles satisfying the conclusion of Conjecture 14. Note that $\lambda(a, \mu(a)) = 3 < 4 = \deg(a)$.

Control over positive degrees. The following lemma states that Conjecture 14 can be strengthened to require each edge to appear the same number of times.

Lemma 31. *Suppose Conjecture 14 is true. If G is connected and has at least four vertices, then the list of cycles in the conclusion of Conjecture 14 can be chosen so that each edge appears the same number of times.*

Proof. Let G be a given graph. We claim that G is 2-connected. Suppose not and let x be a vertex such that $G \setminus x$ is disconnected. Let C be a component of $G \setminus x$ containing $\mu(x)$ and D be a component of $G \setminus x$ other than C. Since G is connected, x has an edge incident with a vertex in D and therefore G can not have $\deg(x)$ edge-disjoint paths from x to $\mu(x)$, a contradiction. This proves the claim.

Let $e_1, e_2, \ldots, e_m$ be the list of edges of G. Let G' be a graph obtained from G by replacing each edge with a path of length m. Let v_{ij} be the j-th internal vertex of the path of G' representing e_i where $j = 1, 2, \ldots, m-1$. We extend μ of G to obtain μ' of G' so that $\mu'(v_{i,j}) = v_{j,i-1}$ and $\mu'(v_{j,i-1}) = v_{i,j}$ for all $1 \le j < i \le m$.

Since G is 2-connected, for each pair of edges e and f of G, there is at least one cycle containing both e and f. Thus in G', there are two edge-disjoint paths from $v_{i,j}$ to $v_{j,i-1}$ for all $1 \leq j < i \leq m$. So we can apply Conjecture 14 to G' and deduce that each edge of G is used the same number of times because the number of cycles passing $v_{i,j}$ is equal to the number of cycles passing $v_{j,i-1}$ for all $1 \leq j < i \leq m$. $\quad\square$

Suppose U is a polygonal list of cyclically reduced words $u_1, \ldots, u_r$ in F_n. There exists a closed U-polygonal surface S obtained by a side-pairing on polygonal disks $P_1, \ldots, P_m$ equipped with an immersion $S^{(1)} \to \mathrm{Cayley}(F_n)/F_n$ as in Definition 6. We shall orient each ∂P_i so that each $\partial P_i \to S^{(1)} \to \mathrm{Cayley}(F_n)/F_n$ reads a positive power of a word in U. Partition $\mathcal{P} = \{P_1, \ldots, P_m\}$ into $\mathcal{P}_1, \ldots, \mathcal{P}_r$ so that each $P_i \in \mathcal{P}_j$ reads a power of u_j. If the polygonal disks in $\mathcal{P}_j$ read $u_j^{c_1}, u_j^{c_2}, \ldots, u_j^{c_k}$, we say $c_1 + c_2 + \cdots + c_k$ is the *positive degree of u_j* with respect to the partition $\mathcal{P} = \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_r$.

Proposition 32. *Let U be a minimal and diskbusting list of cyclically reduced words $u_1, \ldots, u_r$ in F_n for some $n > 1$. We assume that either Tiling Conjecture is true, or $n = 2$. Then there exists a U-polygonal surface S such that the positive degree of every word in U is the same with respect to a suitable partition of the polygonal disks in S.*

Proof. Suppose that $W(U)$ has a list of cycles satisfying the conclusion of Conjecture 14 and each edge appears the same number of times, say s, in the list. We consider a U-polygonal surface S as in the proof of Lemma 13. We define $\mathcal{P}_j$ to be the set of polygonal disks on S so that in the construction of S, the corners of the polygonal disks in $\mathcal{P}_j$ correspond to the edges in $W(U)$ that are coming from u_j. Then every word in U has the positive degree s with respect to this natural choice of a partition of the polygonal disks. Hence, the proof follows from Part (b) of Theorem 27 and Lemma 31. $\quad\square$

Non-virtually geometric words. Let H_n denote a 3-dimensional handlebody of genus n. A word w in F_n can be realized as an embedded curve $\gamma \subseteq H_n$. A word w is said to be *virtually geometric* if there exists a finite cover $p \colon H' \to H_n$ such that $p^{-1}(\gamma)$ is homotopic to a 1-submanifold on the boundary of H' [12]. Using Dehn's lemma, Gordon and Wilton [12] proved that if $w \in F_n$ is diskbusting and virtually geometric, then $D(\{w\})$ contains a surface group; this also follows from the fact that a minimal diskbusting geometric word is polygonal [17]. On the other hand, Manning provided examples of minimal diskbusting, non-virtually geometric words as follows.

Theorem 33 (Manning [22]). *If the Whitehead graph of a word w in F_n is non-planar, k-regular and k-edge-connected for some $k \geq 3$, then w is not virtually geometric.*

Here, a graph G is said to be *k-edge-connected* if $|\delta(X)| \geq k$ for all $\emptyset \neq X \subsetneq V(G)$. So, if $W(U)$ is k-regular and k-edge-connected for a list U of words in F_n, then U is minimal and diskbusting. Hence even for the words provided by Manning, Theorem 2 finds hyperbolic surface groups in the corresponding doubles:

Corollary 34. *If the Whitehead graph of a list U of words in F_n is k-regular and k-edge-connected for some $k \geq 3$, then U is polygonal. In particular, $D(U)$ contains a hyperbolic surface group.*

Existence of separable surface subgroups. A subgroup H of a group G is said to be *separable* if H coincides with the intersection of all the finite-index subgroups of G containing H. If every finitely generated subgroup of G is separable, we say G is *subgroup separable*. The *Virtual Haken Conjecture* for a closed hyperbolic 3-manifold M asserts that there exists a π_1-injective, homeomorphically embedded, closed hyperbolic surface in some finite cover of M [26]; this is a main motivation for Question 1. If $\pi_1(M)$ contains a *separable* hyperbolic surface subgroup, then it is known that a closed hyperbolic surface π_1-injectively embeds into a finite cover of M [26]. So, it is natural to augment Question 1 as follows.

Question 3. *Does every one-ended word-hyperbolic group contain a separable hyperbolic surface group?*

Since $X(U)$ has a non-positively curved square complex structure and also decomposes a graph of free groups with cyclic edge groups, $D(U)$ is subgroup separable by [31].

References

[1] Gilbert Baumslag, Benjamin Fine, Charles F. Miller, III, and Douglas Troeger, *Virtual properties of cyclically pinched one-relator groups*, Internat. J. Algebra Comput. **19** (2009), no. 2, 213–227. MR 2512551 (2010c:20034)

[2] John Berge, *Heegaard documentation. Preprint (c.1990)*, `http://www.math.uic.edu/~t3m`.

[3] Martin R. Bridson, *Geodesics and curvature in metric simplicial complexes*, Group theory from a geometrical viewpoint (Trieste, 1990), World Sci. Publ., River Edge, NJ, 1991, pp. 373–463. MR 1170372 (94c:57040)

[4] Martin R. Bridson and André Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999. MR 1744486 (2000k:53038)

[5] José Burillo and Enric Ventura, *Counting primitive elements in free groups*, Geom. Dedicata **93** (2002), 143–162. MR 1934695 (2003i:20066)

[6] Danny Calegari, *scl*, MSJ Memoirs, vol. 20, Mathematical Society of Japan, Tokyo, 2009. MR 2527432

[7] Richard D. Canary, *Ends of hyperbolic 3-manifolds*, J. Amer. Math. Soc. **6** (1993), no. 1, 1–35. MR 1166330 (93e:57019)

[8] Ruth Charney, *The Tits conjecture for locally reducible Artin groups*, Internat. J. Algebra Comput. **10** (2000), no. 6, 783–797.

[9] John Crisp and Bert Wiest, *Embeddings of graph braid and surface groups in right-angled Artin groups and braid groups*, Algebr. Geom. Topol. **4** (2004), 439–472.

[10] Jack Edmonds, *Maximum matching and a polyhedron with $0, 1$-vertices*, J. Res. Nat. Bur. Standards Sect. B **69B** (1965), 125–130. MR 0183532 (32 #1012)

[11] Cameron McA. Gordon, Darren D. Long, and Alan W. Reid, *Surface subgroups of Coxeter and Artin groups*, J. Pure Appl. Algebra **189** (2004), no. 1–3, 135–148. MR 2038569 (2004k:20077)

[12] Cameron McA. Gordon and Henry Wilton, *On surface subgroups of doubles of free groups*, J. London Math. Soc. (2) **82** (2010), no. 1, 17–31.

[13] Mikhail Gromov, *Hyperbolic groups*, Essays in group theory, Math. Sci. Res. Inst. Publ., vol. 8, Springer, New York, 1987, pp. 75–263. MR 919829 (89e:20070)

[14] _____, *Asymptotic invariants of infinite groups*, Geometric group theory, Vol. 2 (Sussex, 1991), London Math. Soc. Lecture Note Ser., vol. 182, Cambridge Univ. Press, Cambridge, 1993, pp. 1–295. MR 1253544 (95m:20041)

[15] Frédéric Haglund and Daniel T. Wise, *Special cube complexes*, Geom. Funct. Anal. **17** (2008), no. 5, 1551–1620. MR 2377497 (2009a:20061)

[16] Jeremy Kahn and Vladimir Markovic, *Immersing almost geodesic surfaces in a closed hyperbolic three manifold*, Ann. of Math. (2) (2011), to appear.

[17] Sang-hyun Kim, *Geometricity and polygonality in free groups*, Internat. J. Algebra Comput. **21** (2011), no. 1-2, 235–256. MR 2787460

[18] Sang-hyun Kim and Henry Wilton, *priviate communication*.

[19] _____, *Polygonal words in free groups*, to appear at Q. J. of Math., 2010.

[20] Daniel Král', *Personal communication*, 2010.

[21] László Lovász and Michael D. Plummer, *Matching theory*, North-Holland Mathematics Studies, vol. 121, North-Holland Publishing Co., Amsterdam, 1986, Annals of Discrete Mathematics, 29. MR 859549 (88b:90087)

[22] Jason Fox Manning, *Virtually geometric words and Whitehead's algorithm*, Math. Res. Lett. **17** (2010), no. 5, 917–925. MR 2727618

[23] Karl Menger, *Zur allgemeinen Kurventheorie*, Fund. Math. **10** (1927), 96–115.

[24] Alexei G. Myasnikov and Vladimir Shpilrain, *Automorphic orbits in free groups*, J. Algebra **269** (2003), no. 1, 18–27. MR 2015300 (2004j:20051)

[25] Abdó Roig, Enric Ventura, and Pascal Weil, *On the complexity of the Whitehead minimization problem*, Internat. J. Algebra Comput. **17** (2007), no. 8, 1611–1634. MR 2378055 (2008m:20044)

[26] Peter Scott, *Subgroups of surface groups are almost geometric*, J. London Math. Soc. (2) **17** (1978), no. 3, 555–565. MR 0494062 (58 #12996)

[27] Paul Seymour, *On multicolourings of cubic graphs, and conjectures of Fulkerson and Tutte*, Proc. London Math. Soc. (3) **38** (1979), no. 3, 423–460. MR 532981 (81j:05061)

[28] John R. Stallings, *Whitehead graphs on handlebodies*, Geometric group theory down under (Canberra, 1996), de Gruyter, Berlin, 1999, pp. 317–330. MR 1714852 (2001i:57028)

[29] Richard Stong, *Diskbusting elements of the free group*, Math. Res. Lett. **4** (1997), no. 2-3, 201–210. MR 1453054 (98h:20049)

[30] John H. C. Whitehead, *On certain sets of elements in a free group*, Proc. London Math. Soc. **s2–41** (1936), no. 1, 48–56.

[31] Daniel T. Wise, *Subgroup separability of graphs of free groups with cyclic edge groups*, Q. J. Math. **51** (2000), no. 1, 107–129. MR 1760573 (2001e:20026)

DEPARTMENT OF MATHEMATICAL SCIENCES, KAIST, 291 DAEHAK-RO, YUSEONG-GU, DAEJEON 305-701, REPUBLIC OF KOREA

E-mail address: shkim@kaist.edu

E-mail address: sangil@kaist.edu