RANK-WIDTH AND WELL-QUASI-ORDERING OF SKEW-SYMMETRIC OR SYMMETRIC MATRICES

SANG-IL OUM

ABSTRACT. We prove that every infinite sequence of skew-symmetric or symmetric matrices M_1, M_2, ... over a fixed finite field must have a pair M_i, M_j $(i < j)$ such that M_i is isomorphic to a principal submatrix of the Schur complement of a nonsingular principal submatrix in M_j, if those matrices have bounded rank-width. This generalizes three theorems on well-quasi-ordering of graphs or matroids admitting good tree-like decompositions; (1) Robertson and Seymour's theorem for graphs of bounded tree-width, (2) Geelen, Gerards, and Whittle's theorem for matroids representable over a fixed finite field having bounded branch-width, and (3) Oum's theorem for graphs of bounded rank-width with respect to pivot-minors.

1. INTRODUCTION

For a $V_1 \times V_1$ matrix A_1 and a $V_2 \times V_2$ matrix A_2, an *isomorphism* f from A_1 to A_2 is a bijective function that maps V_1 to V_2 such that the (i, j) entry of A_1 is equal to the $(f(i), f(j))$ entry of A_2 for all $i, j \in V_1$. Two square matrices A_1, A_2 are *isomorphic* if there is an isomorphism from A_1 to A_2. Note that an isomorphism allows permuting rows and columns simultaneously. For a $V \times V$ matrix A and a subset X of its ground set V, we write $A[X]$ to denote the principal submatrix of A induced by X. Similarly, we write $A[X, Y]$ to denote the $X \times Y$ submatrix of A. Suppose that a $V \times V$ matrix M has the following form:

$$M = \begin{array}{c c} & \begin{array}{cc} Y & V \setminus Y \end{array} \\ \begin{array}{c} Y \\ V \setminus Y \end{array} & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array}.$$

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If $A = M[Y]$ is nonsingular, then we define the *Schur complement* (M/A) *of* A *in* M to be

$$(M/A) = D - CA^{-1}B.$$

(If $Y = \emptyset$, then A is nonsingular and $(M/A) = M$.) Notice that if M is skew-symmetric or symmetric, then (M/A) is skew-symmetric or symmetric, respectively.

We prove that skew-symmetric or symmetric matrices over a fixed finite field are *well-quasi-ordered* under the relation defined in terms of taking a principal submatrix and a Schur complement, if they have bounded *rank-width*. Rank-width of a skew-symmetric or symmetric matrix will be defined precisely in Section 2. Roughly speaking, it is a measure to describe how easy it is to decompose the matrix into a tree-like structure so that the connecting matrices have small rank. Rank-width of matrices generalizes rank-width of simple graphs introduced by Oum and Seymour [12], and branch-width of graphs and matroids by Robertson and Seymour [15]. Here is our main theorem.

Theorem 7.1. *Let* $\mathbb{F}$ *be a finite field and let* k *be a constant. Every infinite sequence* M_1, M_2, ... *of skew-symmetric or symmetric matrices over* $\mathbb{F}$ *of rank-width at most* k *has a pair* $i < j$ *such that* M_i *is isomorphic to a principal submatrix of* (M_j/A) *for some nonsingular principal submatrix* A *of* M_j.

It may look like a purely linear algebraic result. However, it implies the following well-quasi-ordering theorems on graphs and matroids admitting 'good tree-like decompositions.'

- (Robertson and Seymour [15]) Every infinite sequence G_1, G_2, ... of graphs of bounded tree-width has a pair $i < j$ such that G_i is isomorphic to a minor of G_j.
- (Geelen, Gerards, and Whittle [8]) Every infinite sequence M_1, M_2, ... of matroids representable over a fixed finite field having bounded branch-width has a pair $i < j$ such that M_i is isomorphic to a minor of M_j.
- (Oum [11]) Every infinite sequence G_1, G_2, ... of simple graphs of bounded rank-width has a pair $i < j$ such that G_i is isomorphic to a pivot-minnor of G_j.

We ask, as an open problem, whether the requirement on rank-width is necessary in Theorem 7.1. It is likely that our theorem for matrices of bounded rank-width is a step towards this problem, as Roberson and Seymour also started with graphs of bounded tree-width. If we have a positive answer, then this would imply Robertson and Seymour's graph minor theorem [16] as well as an open problem on the well-quasi-ordering of matroids representable over a fixed finite field [10].

A big portion of this paper is devoted to introduce Lagrangian chain-groups and prove their relations to skew-symmetric or symmetric matrices. One can regard Sections 3 and 4 as an almost separate paper introducing Lagrangian chain-groups, their matrix representations, and their relations to delta-matroids. In particular, Lagrangian chain-groups provide an alternative definition of representable delta-matroids. The situation is comparable to Tutte chain-groups,[1] introduced by Tutte [20]. Tutte [21] showed that a matroid is representable over a field $\mathbb{F}$ if and only if it is representable by a Tutte chain-group over $\mathbb{F}$. We prove an analogue of his theorem; *a delta-matroid is representable over a field* $\mathbb{F}$ *if and only if it is representable by a Lagrangian chain-group over* $\mathbb{F}$*.* We believe that the notion of Lagrangian chain-groups will be useful to extend the matroid theory to representable delta-matroids.

To prove well-quasi-ordering, we work on Lagrangian chain-groups instead of skew-symmetric or symmetric matrices for the convenience. The main proof of the well-quasi-ordering of Lagrangian chain-groups is in Sections 5 and 6. Section 5 proves a theorem generalizing Tutte's linking theorem for matroids, which in turn generalizes Menger's theorem. The proof idea in Section 6 is similar to the proof of Geelen, Gerards, and Whittle's theorem [8] for representable matroids.

The last two sections discuss how the result on Lagrangian chain-groups imply our main theorem and its other corollaries. Section 7 formulates the result of Section 6 in terms of skew-symmetric or symmetric matrices with respect to the Schur complement and explain its implications for representable delta-matroids and simple graphs of bounded rank-width. Section 8 explains why our theorem implies the theorem for representable matroids by Geelen, Gerards, and Whittle [8] via Tutte chain-groups.

2. Preliminaries

2.1. **Matrices.** For two sets X and Y, we write $X\Delta Y = (X \setminus Y) \cup (Y \setminus X)$. A $V \times V$ matrix A is called *symmetric* if $A = A^t$, *skew-symmetric* if $A = -A^t$ and all of its diagonal entries are zero. We require each diagonal entry of a skew-symmetric matrix to be zero, even if the underlying field has characteristic 2.

[1]We call Tutte's chain-groups as *Tutte chain-groups* to distinguish from chain-groups defined in Section 3.

Suppose that a $V \times V$ matrix M has the following form:

$$M = \begin{array}{c} \\ Y \\ V\setminus Y \end{array} \begin{array}{c} \begin{array}{cc} Y & V\setminus Y \end{array} \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array}.$$

If $A = M[Y]$ is nonsingular, then we define a matrix $M * Y$ by

$$M * Y = \begin{array}{c} \\ Y \\ V\setminus Y \end{array} \begin{array}{c} \begin{array}{cc} Y & V\setminus Y \end{array} \\ \begin{pmatrix} A^{-1} & A^{-1}B \\ -CA^{-1} & (M/A) \end{pmatrix} \end{array}.$$

This operation is called a *pivot*. In the literature, it has been called a *principal pivoting*, a *principal pivot transformation*, and other various names; we refer to the survey by Tsatsomeros [18].

Notice that if M is skew-symmetric, then so is $M * Y$. If M is symmetric, then so is $(I_Y)(M * Y)$, where I_Y is a diagonal matrix such that the diagonal entry indexed by an element in Y is -1 and all other diagonal entries are 1.

The following theorem implies that $(M * Y)[X]$ is nonsingular if and only if $M[X\Delta Y]$ is nonsingular.

Theorem 2.1 (Tucker [19]). *Let $M[Y]$ be a nonsingular principal submatrix of a $V \times V$ matrix M. Then for all $X \subseteq V$,*

$$\det(M * Y)[X] = \det M[Y\Delta X]/\det M[Y].$$

Proof. See Bouchet's proof in Geelen's thesis paper [7, Theorem 2.7]. □

2.2. **Rank-width.** A tree is called *subcubic* if every vertex has at most three incident edges. We define *rank-width* of a skew-symmetric or symmetric $V \times V$ matrix A over a field $\mathbb{F}$ by rank-decompositions as follows. A *rank-decomposition* of A is a pair $(T, \mathcal{L})$ of a subcubic tree T and a bijection $\mathcal{L} : V \to \{t : t \text{ is a leaf of } T\}$. For each edge $e = uv$ of the tree T, the connected components of $T \setminus e$ form a partition (X_e, Y_e) of the leaves of T and we call $\operatorname{rank} A[\mathcal{L}^{-1}(X_e), \mathcal{L}^{-1}(Y_e)]$ the *width* of e. The *width* of a rank-decomposition $(T, \mathcal{L})$ is the maximum width of all edges of T. The *rank-width* $\operatorname{rwd}(A)$ of a skew-symmetric or symmetric $V \times V$ matrix A over $\mathbb{F}$ is the minimum width of all its rank-decompositions. (If $|V| \leq 1$, then we define that $\operatorname{rwd}(A) = 0$.)

2.3. **Delta-matroids.** Delta-matroids were introduced by Bouchet [2]. A *delta-matroid* is a pair $(V, \mathcal{F})$ of a finite set V and a *nonempty* collection $\mathcal{F}$ of subsets of V such that the following *symmetric exchange* *axiom* holds.

(SEA) If $F, F' \in \mathcal{F}$ and $x \in F\Delta F'$,

$$\text{then there exists } y \in F\Delta F' \text{ such that } F\Delta\{x,y\} \in \mathcal{F}.$$

A member of $\mathcal{F}$ is called *feasible*. A delta-matroid is *even*, if cardinalities of all feasible sets have the same parity.

Let $\mathcal{M} = (V, \mathcal{F})$ be a delta-matroid. For a subset X of V, it is easy to see that $\mathcal{M}\Delta X = (V, \mathcal{F}\Delta X)$ is also a delta-matroid, where $\mathcal{F}\Delta X = \{F\Delta X : F \in \mathcal{F}\}$; this operation is referred to as *twisting*. Also, $\mathcal{M}\setminus X = (V\setminus X, \mathcal{F}\setminus X)$ defined by $\mathcal{F}\setminus X = \{F \subseteq V\setminus X : F \in \mathcal{F}\}$ is a delta-matroid if $\mathcal{F} \setminus X$ is nonempty; we refer to this operation as *deletion*. Two delta-matroids $\mathcal{M}_1 = (V, \mathcal{F}_1)$, $\mathcal{M}_2 = (V, \mathcal{F}_2)$ are called *equivalent* if there exists $X \subseteq V$ such that $\mathcal{M}_1 = \mathcal{M}_2\Delta X$. A delta-matroid that comes from $\mathcal{M}$ by twisting and/or deletion is called a *minor* of $\mathcal{M}$.

2.4. **Representable delta-matroids.** For a $V \times V$ skew-symmetric or symmetric matrix A over a field $\mathbb{F}$, let

$$\mathcal{F}(A) = \{X \subseteq V : A[X] \text{ is nonsingular}\}$$

and $\mathcal{M}(A) = (V, \mathcal{F}(A))$. Bouchet [4] showed that $\mathcal{M}(A)$ forms a delta-matroid. We call a delta-matroid *representable* over a field $\mathbb{F}$ or $\mathbb{F}$-*representable* if it is equivalent to $\mathcal{M}(A)$ for some skew-symmetric or symmetric matrix A over $\mathbb{F}$. We also say that $\mathcal{M}$ is represented by A if $\mathcal{M}$ is equivalent to $\mathcal{M}(A)$.

Twisting (by feasible sets) and deletions are both natural operations for representable delta-matroids. For $X \subseteq V$, $\mathcal{M}(A) \setminus X = \mathcal{M}(A[V \setminus X])$, and for a feasible set X, $\mathcal{M}(A)\Delta X = \mathcal{M}(A * X)$ by Theorem 2.1. Therefore minors of a $\mathbb{F}$-representable delta-matroid are $\mathbb{F}$-representable [5].

2.5. **Well-quasi-order.** In general, we say that a binary relation $\leq$ on a set X is a *quasi-order* if it is reflexive and transitive. For a quasi-order $\leq$, we say "$\leq$ is a *well-quasi-ordering*" or "X is *well-quasi-ordered* by $\leq$" if for every infinite sequence $a_1, a_2, \ldots$ of elements of X, there exist $i < j$ such that $a_i \leq a_j$. For more detail, see Diestel [6, Chapter 12].

3. Lagrangian chain-groups

3.1. **Definitions.** If W is a vector space with a bilinear form $\langle\,,\,\rangle$ and W' is a subspace of W satisfying

$$\langle x, y\rangle = 0 \text{ for all } x, y \in W',$$

then W' is called *totally isotropic*. A vector $v \in W$ is called *isotropic* if $\langle v, v\rangle = 0$. A well-known theorem in linear algebra states that if a bilinear form $\langle\,,\,\rangle$ is non-degenerate in W and W' is a totally isotropic subspace of W, then $\dim(W) = \dim(W') + \dim(W'^{\perp}) \geq 2\dim(W')$ because $W' \subseteq W'^{\perp}$.

Let V be a finite set and $\mathbb{F}$ be a field. Let $K = \mathbb{F}^2$ be a two-dimensional vector space over $\mathbb{F}$. Let $b^+\left(\binom{a}{b}, \binom{c}{d}\right) = ad + bc$ and $b^-(\binom{a}{b}, \binom{c}{d}) = ad - bc$ be bilinear forms on K. We assume that K is equipped with a bilinear form $\langle\,,\,\rangle_K$ that is either b^+ or b^-. Clearly b^+ is symmetric and b^- is skew-symmetric.

A *chain* on V to K is a mapping $f : V \to K$. If $x \in V$, the element $f(x)$ of K is called the *coefficient* of x in f. If V is nonnull, there is a *zero chain* on V whose coefficients are 0. When V is null, we say that there is just one chain on V to K and we call it a zero chain.

The *sum* $f + g$ of two chains f, g is the chain on V satisfying $(f + g)(x) = f(x)+g(x)$ for all $x \in V$. If f is a chain on V to K and $\lambda \in \mathbb{F}$, the *product* λf is a chain on V such that $(\lambda f)(x) = \lambda f(x)$ for all $x \in V$. It is easy to see that the set of all chains on V to K, denoted by K^V, is a vector space. We give a bilinear form $\langle\,,\,\rangle$ to K^V as following:

$$\langle f, g\rangle = \sum_{x\in V} \langle f(x), g(x)\rangle_K .$$

If $\langle f, g\rangle = 0$, we say that the chains f and g are *orthogonal*. For a subspace L of K^V, we write $L^{\perp}$ for the set of all chains orthogonal to every chain in L.

A *chain-group* on V to K is a subspace of K^V. A chain-group is called *isotropic* if it is a totally isotropic subspace. It is called *Lagrangian* if it is isotropic and has dimension $|V|$. We say a chain-group N is over a field $\mathbb{F}$ if K is obtained from $\mathbb{F}$ as described above.

A *simple isomorphism* from a chain-group N on V to K to another chain-group N' on V' to K is defined as a bijective function $\mu : V \to V'$ satisfying that $N = \{f \circ \mu : f \in N'\}$ where $f \circ \mu$ is a chain on V to K such that $(f \circ \mu)(x) = f(\mu(x))$ for all $x \in V$. We require both N and N' have the same type of bilinear forms on K, that is either skew-symmetric or symmetric. A chain-group N on V to K is *simply isomorphic* to another chain-group N' on V' to K if there is a simple isomorphism from N to N'.

Remark. Bouchet's definition [4] of isotropic chain-groups is slightly more general than ours, since he allows $\left\langle\binom{a}{b}, \binom{c}{d}\right\rangle_K = -ad \pm bc$. His notation, however, is different; he uses $\mathbb{F}^{V'}$ instead of K^V where V' is a union of V and its disjoint copy $V^{\sim}$. Since $K = \mathbb{F}^2$, two definitions are equivalent. Our notation has advantages which we will see in the next subsection. Bouchet's notation also has its own virtues because, in Bouchet's sense, isotropic chain-groups are Tutte chain-groups. Strictly speaking, our isotropic chain-groups are not Tutte chain-groups, because we define chains differently. We are mainly interested in Lagrangian chain-groups because they are closely related to representable delta-matroids. We note that the notion of Lagrangian chain-groups is motivated by Tutte's chain-groups and Bouchet's isotropic systems [3].

3.2. **Minors.** Consider a subset T of V. If f is a chain on V to K, we define its *restriction* $f \cdot T$ to T as the chain on T such that $(f \cdot T)(x) = f(x)$ for all $x \in T$. For a chain-group N on V,

$$N \cdot T = \{f \cdot T : f \in N\}$$

is a chain-group on T to K. We note that $N \cdot T$ is not necessarily isotropic, even if N is isotropic. We write

$$N \times T = \{f \cdot T : f \in N,\ f(x) = 0 \text{ for all } x \in V \setminus T\}.$$

For a chain-group N on V, we define

$$N \setminus\!\!\setminus T = \{f \cdot (V \setminus T) : f \in N, \left\langle f(x), \tbinom{1}{0}\right\rangle_K = 0 \text{ for all } x \in T\}.$$

We call this the *deletion*. Similarly we define

$$N /\!\!/ T = \{f \cdot (V \setminus T) : f \in N, \left\langle f(x), \tbinom{0}{1}\right\rangle_K = 0 \text{ for all } x \in T\}.$$

We call this the *contraction*. We refer to a chain-group of the form $N /\!\!/ X \setminus\!\!\setminus Y$ on $V \setminus (X \cup Y)$ as a *minor* of N.

Proposition 3.1. *A minor of a minor of a chain-group N on V to K is a minor of N.*

Proof. We can deduce this from the following easy facts.

$$\begin{aligned} N /\!\!/ X /\!\!/ Y &= N /\!\!/ (X \cup Y), \\ N /\!\!/ X \setminus\!\!\setminus Y &= N \setminus\!\!\setminus Y /\!\!/ X, \\ N \setminus\!\!\setminus X \setminus\!\!\setminus Y &= N \setminus\!\!\setminus (X \cup Y). \end{aligned}$$ □

Lemma 3.2. *Let $x, y \in K$. If $x \in K$ is isotropic, $x \neq 0$, and $\langle x, y\rangle_K = 0$, then $y = cx$ for some $c \in \mathbb{F}$.*

Proof. Since $\langle\, , \rangle_K$ is nondegenerate, there exists a vector $x' \in K$ such that $\langle x, x'\rangle_K \neq 0$. Hence $\{x, x'\}$ is a basis of K. Let $y = cx + dx'$ for some $c, d \in \mathbb{F}$. Since $\langle x, cx + dx'\rangle_K = d\,\langle x, x'\rangle_K = 0$, we deduce $d = 0$. □

Proposition 3.3. *A minor of an isotropic chain-group on V to K is isotropic.*

Proof. By Lemma 3.2, if $\left\langle x, \binom{1}{0} \right\rangle_K = \left\langle y, \binom{1}{0} \right\rangle_K = 0$, then $\langle x, y \rangle_K = 0$ and similarly if $\left\langle x, \binom{0}{1} \right\rangle_K = \left\langle y, \binom{0}{1} \right\rangle_K = 0$, then $\langle x, y \rangle_K = 0$. This easily implies the lemma. □

We will prove that every minor of a Lagrangian chain-group is Lagrangian in the next section.

3.3. **Algebraic duality.** For an element v of a finite set V, if N is a chain-group on V to K and B is a basis of N, then we may assume that the coefficient at v of every chain in B is zero except at most two chains in B because $\dim(K) = 2$. So, it is clear that dimensions of $N \times (V \setminus \{v\})$, $N \cdot (V \setminus \{v\})$, $N \setminus\!\!\setminus \{v\}$, and $N /\!\!/ \{v\}$ are at least $\dim(N) - 2$. In this subsection, we discuss conditions for those chain-groups to have dimension $\dim(N) - 2$, $\dim(N) - 1$, or $\dim(N)$. Note that we do not assume that N is isotropic.

Theorem 3.4. *If N is a chain-group on V to K and $X \subseteq V$, then*

$$(N \cdot X)^\perp = N^\perp \times X.$$

Proof. (Tutte [25, Theorem VIII.7.]) Let $f \in (N \cdot X)^\perp$. There exists a chain f_1 on V to K such that $f_1 \cdot X = f$ and $f_1(v) = 0$ for all $v \in V \setminus X$. Since $\langle f_1, g \rangle = \langle f, g \cdot X \rangle = 0$ for all $g \in N$, we have $f \in N^\perp \times X$.

Conversely, if $f \in N^\perp \times X$, it is the restriction to X of a chain f_1 of $N^\perp$ specified as above. Hence $\langle f, g \cdot X \rangle = \langle f_1, g \rangle = 0$ for all $g \in N$. Therefore $f \in (N \cdot X)^\perp$. □

Lemma 3.5. *Let N be a chain-group on V to K. If $X \cup Y = V$ and $X \cap Y = \emptyset$, then*

$$\dim(N \cdot X) + \dim(N \times Y) = \dim(N).$$

Proof. Let $\varphi : N \to N \cdot X$ be a linear transformation defined by $\varphi(f) = f \cdot X$. The kernel $\ker(\varphi)$ of this transformation is the set of all chains f in N having $f \cdot X = 0$. Thus, $\dim(\ker(\varphi)) = \dim(N \times Y)$. Since φ is surjective, we deduce that $\dim(N \cdot X) = \dim(N) - \dim(N \times Y)$. □

For $v \in V$, let v^*, v_* be chains on V to K such that

$$v^*(v) = \binom{1}{0}, \quad v_*(v) = \binom{0}{1},$$
$$v^*(w) = v_*(w) = 0 \quad \text{for all } w \in V \setminus \{v\}.$$

Proposition 3.6. *Let N be a chain-group on V to K and $v \in V$. Then*

$$\dim(N \setminus\!\!\setminus \{v\}) = \begin{cases} \dim N & \text{if } v^* \notin N, v^* \in N^\perp, \\ \dim N - 2 & \text{if } v^* \in N, v^* \notin N^\perp, \\ \dim N - 1 & \text{otherwise,} \end{cases}$$

$$\dim(N /\!\!/ \{v\}) = \begin{cases} \dim N & \text{if } v_* \notin N, v_* \in N^\perp, \\ \dim N - 2 & \text{if } v_* \in N, v_* \notin N^\perp, \\ \dim N - 1 & \text{otherwise.} \end{cases}$$

Proof. By symmetry, it is enough to show for $\dim(N \setminus\!\!\setminus \{v\})$. Let $N' = \{f \in N : \langle f(v), \binom{1}{0} \rangle_K = 0\}$. By definition, $N \setminus\!\!\setminus \{v\} = N' \cdot (V \setminus \{v\})$.

Observe that $N' = N$ if and only if $v^* \in N^\perp$. If $N' \neq N$, then there is a chain g in N such that $\langle g(v), \binom{1}{0} \rangle_K \neq 0$. Then, for every chain $f \in N$, there exists $c \in \mathbb{F}$ such that $f - cg \in N'$. Therefore $\dim(N') = \dim N - 1$ if $v^* \notin N^\perp$ and $\dim(N') = \dim N$ if $v^* \in N^\perp$.

By Lemma 3.5, $\dim(N' \cdot (V \setminus \{v\})) = \dim N' - \dim(N' \times \{v\})$. Clearly, $\dim(N' \times \{v\}) = 0$ if $v^* \notin N$ and $\dim(N' \times \{v\}) = 1$ if $v^* \in N$. This concludes the proof. □

Corollary 3.7. *If N is an isotropic chain-group on V to K and M is a minor of N on V', then*

$$|V'| - \dim M \leq |V| - \dim N.$$

Proof. We proceed by induction on $|V \setminus V'|$. Since N is isotropic, every minor of N is isotropic by Proposition 3.3. Since $v^* \notin N \setminus N^\perp$ and $v_* \notin N \setminus N^\perp$, $\dim(N) - \dim(N \setminus\!\!\setminus \{v\}) \in \{0, 1\}$ and $\dim(N) - \dim(N /\!\!/ \{v\}) \in \{0, 1\}$. So $|V \setminus \{v\}| - \dim(N \setminus\!\!\setminus \{v\}) \leq |V| - \dim N$ and $|V \setminus \{v\}| - \dim(N /\!\!/ \{v\}) \leq |V| - \dim N$. Since M is a minor of either $N \setminus\!\!\setminus \{v\}$ or $N /\!\!/ \{v\}$, $|V'| - \dim M \leq |V| - \dim N$ by the induction hypothesis. □

Proposition 3.8. *A minor of a Lagrangian chain-group is Lagrangian.*

Proof. Let N be a Lagrangian chain-group on V to K and N' be its minor on V' to K. By Proposition 3.3, N' is isotropic and therefore $\dim(N') \leq |V'|$. Thus it is enough to show that $\dim(N') \geq |V'|$. Since $\dim(N) = |V|$, it follows that $\dim(N') \geq |V'|$ by Corollary 3.7. □

Theorem 3.9. *If N is a chain-group on V to K and $X \subseteq V$, then*

$$(N \setminus\!\!\setminus X)^\perp = N^\perp \setminus\!\!\setminus X \text{ and } (N /\!\!/ X)^\perp = N^\perp /\!\!/ X.$$

Proof. By symmetry, it is enough to show that $(N \setminus\!\!\setminus X)^\perp = N^\perp \setminus\!\!\setminus X$. By induction, we may assume $|X| = 1$. Let $v \in X$.

Let f be a chain in $N^\perp \setminus\!\!\setminus X$. There is a chain $f_1 \in N^\perp$ such that $f_1 \cdot (V \setminus X) = f$ and $\left\langle f_1(v), \binom{1}{0} \right\rangle_K = 0$. Let $g \in N$ be a chain such that $\left\langle g(v), \binom{1}{0} \right\rangle_K = 0$. Then $\langle f_1(v), g(v) \rangle_K = 0$ by Lemma 3.2. Therefore $\langle f, g \cdot (V \setminus X) \rangle = \langle f_1, g \rangle = 0$ and so $f \in (N \setminus\!\!\setminus X)^\perp$. We conclude that $N^\perp \setminus\!\!\setminus X \subseteq (N \setminus\!\!\setminus X)^\perp$.

We now claim that $\dim(N^\perp \setminus\!\!\setminus X) = \dim(N \setminus\!\!\setminus X)^\perp$. We apply Proposition 3.6 to deduce that

$$\dim(N \setminus\!\!\setminus X) - \dim(N) = \begin{cases} 0 & \text{if } v^* \notin N, v^* \in N^\perp, \\ -2 & \text{if } v^* \in N, v^* \notin N^\perp, \\ -1 & \text{otherwise,} \end{cases}$$

$$\dim(N^\perp \setminus\!\!\setminus X) - \dim(N^\perp) = \begin{cases} 0 & \text{if } v^* \notin N^\perp, v^* \in N, \\ -2 & \text{if } v^* \in N^\perp, v^* \notin N, \\ -1 & \text{otherwise.} \end{cases}$$

By summing these equations, we obtain the following:

$$\dim(N \setminus\!\!\setminus X) - \dim(N) + \dim(N^\perp \setminus\!\!\setminus X) - \dim(N^\perp) = -2.$$

Since $\dim(N) + \dim(N^\perp) = 2|V|$ and $\dim(N \setminus\!\!\setminus X) + \dim(N \setminus\!\!\setminus X)^\perp = 2(|V| - 1)$, we deduce that $\dim(N^\perp \setminus\!\!\setminus X) = \dim(N \setminus\!\!\setminus X)^\perp$.

Since $N^\perp \setminus\!\!\setminus X \subseteq (N \setminus\!\!\setminus X)^\perp$ and $\dim(N^\perp \setminus\!\!\setminus X) = \dim(N \setminus\!\!\setminus X)^\perp$, we conclude that $N^\perp \setminus\!\!\setminus X = (N \setminus\!\!\setminus X)^\perp$. □

3.4. **Connectivity.** We define the connectivity of a chain-group. Later it will be shown that this definition is related to the connectivity function of matroids (Lemma 8.5) and rank functions of matrices (Theorem 4.13).

Let N be a chain-group on V to K. If U is a subset of V, then we write

$$\lambda_N(U) = \frac{\dim N - \dim(N \times (V \setminus U)) - \dim(N \times U)}{2}.$$

This function λ_N is called the *connectivity function* of a chain-group N. By Lemma 3.5, we can rewrite λ_N as follows:

$$\lambda_N(U) = \frac{\dim(N \cdot U) - \dim(N \times U)}{2}.$$

From Theorem 3.4, it is easy to derive that $\lambda_{N^\perp}(U) = \lambda_N(U)$.

In general $\lambda_N(X)$ need not be an integer. But if N is Lagrangian, then $\lambda_N(X)$ is always an integer by the following lemma.

Lemma 3.10. *If N is a Lagrangian chain-group on V to K, then*

$$\lambda_N(X) = |X| - \dim(N \times X)$$

for all $X \subseteq V$.

Proof. From the definition of $\lambda_N(X)$,

$$\begin{aligned}2\lambda_N(X) &= \dim(N \cdot X) - \dim(N \times X)\\ &= 2|X| - \dim(N \cdot X)^{\perp} - \dim(N \times X)\\ &= 2|X| - \dim(N^{\perp} \times X) - \dim(N \times X),\end{aligned}$$

and since $N = N^{\perp}$, we have

$$= 2(|X| - \dim(N \times X)). \qquad \square$$

By definition, it is easy to see that $\lambda_N(U) = \lambda_N(V \setminus U)$. Thus λ_N is symmetric. We prove that λ_N is submodular.

Lemma 3.11. *Let* N *be a chain-group on* V *to* K *and* X, Y *be two subsets of* V. *Then,*

$$\dim(N \times (X \cup Y)) + \dim(N \times (X \cap Y)) \geq \dim(N \times X) + \dim(N \times Y).$$

Proof. For $T \subseteq V$, let $N_T = \{f \in N : f(v) = 0 \text{ for all } v \notin T\}$. Let $N_X + N_Y = \{f + g : f \in N_X, g \in N_Y\}$. We know that $\dim(N_X + N_Y) + \dim(N_X \cap N_Y) = \dim N_X + \dim N_Y$ from a standard theorem in the linear algebra. Since $N_X \cap N_Y = N_{X\cap Y}$ and $N_X + N_Y \subseteq N_{X\cup Y}$, we deduce that

$$\dim N_{X\cup Y} + \dim N_{X\cap Y} \geq \dim N_X + \dim N_Y.$$

Since $\dim N_T = \dim(N \times T)$, we are done. $\square$

Theorem 3.12 (Submodular inequality)**.** *Let* N *be a chain-group on* V *to* K. *Then* λ_N *is submodular; in other words,*

$$\lambda_N(X) + \lambda_N(Y) \geq \lambda_N(X \cup Y) + \lambda_N(X \cap Y)$$

for all $X, Y \subseteq V$.

Proof. We use Lemma 3.11. Let $S = V \setminus X$ and $T = V \setminus Y$.

$$\begin{aligned}&2\lambda_N(X) + 2\lambda_N(Y)\\ &= 2\dim(N)\\ &\quad - (\dim(N \times X) + \dim(N \times S) + \dim(N \times Y) + \dim(N \times T))\\ &\geq 2\dim(N) - \dim(N \times (X \cup Y)) - \dim(N \times (X \cap Y))\\ &\quad - \dim(N \times (S \cap Y)) - \dim(N \times (S \cup Y))\\ &= 2\lambda_N(X \cup Y) + 2\lambda_N(X \cap Y). \qquad \square\end{aligned}$$

What happens to the connectivity functions if we take minors of a chain-group? As in the matroid theory, the connectivity does not increase.

Theorem 3.13. *Let N, M be chain-groups on V, V' respectively. If M is a minor of a chain-group N, then $\lambda_M(T) \leq \lambda_N(T \cup U)$ for all $T \subseteq V'$ and all $U \subseteq V \setminus V'$.*

Proof. By induction on $|V \setminus V'|$, it is enough to prove this when $|V \setminus V'| = 1$. Let $v \in V \setminus V'$. By symmetry we may assume that $M = N \setminus\!\!\setminus \{v\}$.

We claim that $\lambda_M(T) \leq \lambda_N(T)$. From the definition, we deduce

$$\begin{aligned} 2\lambda_M(T) - 2\lambda_N(T) = \dim(N \setminus\!\!\setminus \{v\} \cdot T) - \dim(N \setminus\!\!\setminus \{v\} \times T) \\ - \dim(N \cdot T) + \dim(N \times T). \end{aligned}$$

Clearly $N \setminus\!\!\setminus \{v\} \cdot T \subseteq N \cdot T$ and $N \times T \subseteq N \setminus\!\!\setminus \{v\} \times T$. Thus $\lambda_M(T) \leq \lambda_N(T)$.

Since λ_N and λ_M are symmetric, $\lambda_M(T) = \lambda_M(V' \setminus T) \leq \lambda_N(V' \setminus T) = \lambda_N(T \cup \{v\})$. □

3.5. **Branch-width.** A *branch-decomposition* of a chain-group N on V to K is a pair $(T, \mathcal{L})$ of a subcubic tree T and a bijection $\mathcal{L} : V \to \{t : t \text{ is a leaf of } T\}$. For each edge $e = uv$ of the tree T, the connected components of $T \setminus e$ form a partition (X_e, Y_e) of the leaves of T and we call $\lambda_N(\mathcal{L}^{-1}(X_e))$ the *width* of e. The *width* of a branch-decomposition $(T, \mathcal{L})$ is the maximum width of all edges of T. The *branch-width* $\mathrm{bw}(N)$ of a chain-group N is the minimum width of all its branch-decompositions. (If $|V| \leq 1$, then we define that $\mathrm{bw}(N) = 0$.)

4. Matrix Representations of Lagrangian Chain-groups

4.1. **Matrix Representations.** We say that two chains f and g on V to K are *supplementary* if, for all $x \in V$,

(i) $\langle f(x), f(x) \rangle_K = \langle g(x), g(x) \rangle_K = 0$ and
(ii) $\langle f(x), g(x) \rangle_K = 1$.

Given a skew-symmetric or symmetric matrix A, we may construct a Lagrangian chain-group as follows.

Proposition 4.1. *Let $M = (m_{ij} : i, j \in V)$ be a skew-symmetric or symmetric $V \times V$ matrix over a field $\mathbb{F}$. Let a, b be supplementary chains on V to $K = \mathbb{F}^2$ where $\langle\, ,\, \rangle_K$ is skew-symmetric if M is symmetric and symmetric if M is skew-symmetric.*

For $i \in V$, let f_i be a chain on V to K such that for all $j \in V$,

$$f_i(j) = \begin{cases} m_{ij} a(j) + b(j) & \text{if } j = i, \\ m_{ij} a(j) & \text{if } j \neq i. \end{cases}$$

Then the subspace N of K^V spanned by chains $\{f_i : i \in V\}$ is a Lagrangian chain-group on V to K.

If M is a skew-symmetric or symmetric matrix and a, b are supplementary chains on V to K, then we call (M, a, b) a *(general) matrix representation* of a Lagrangian chain-group N. Furthermore if $a(v), b(v) \in \{\pm\binom{1}{0}, \pm\binom{0}{1}\}$ for each $v \in V$, then (M, a, b) is called a *special matrix representation* of N.

Proof. For all $i \in V$,

$$\langle f_i, f_i \rangle = \sum_{j \in V} \langle f_i(j), f_i(j) \rangle_K = m_{ii}(\langle a(i), b(i) \rangle_K + \langle b(i), a(i) \rangle_K) = 0,$$

because either $m_{ii} = 0$ (if M is skew-symmetric) or $\langle\, , \rangle_K$ is skew-symmetric.

Now let i and j be two distinct elements of V. Then,

$$\begin{aligned} \langle f_i, f_j \rangle &= \langle f_i(i), f_j(i) \rangle_K + \langle f_i(j), f_j(j) \rangle_K \\ &= m_{ji} \langle b(i), a(i) \rangle_K + m_{ij} \langle a(j), b(j) \rangle_K \\ &= 0, \end{aligned}$$

because either $m_{ij} = -m_{ji}$ and $\langle b(i), a(i) \rangle_K = \langle a(j), b(j) \rangle_K$ or $m_{ij} = m_{ji}$ and $\langle b(i), a(i) \rangle_K = -\langle a(j), b(j) \rangle_K$.

It is easy to see that $\{f_i : i \in V\}$ is linearly independent and therefore $\dim(N) = |V|$. This proves that N is a Lagrangian chain-group. $\square$

4.2. **Eulerian chains.** A chain a on V to K is called a *(general) eulerian* chain of an isotropic chain-group N if

(i) $a(x) \neq 0$, $\langle a(x), a(x) \rangle_K = 0$ for all $x \in V$ and
(ii) there is no non-zero chain $f \in N$ such that $\langle f(x), a(x) \rangle_K = 0$ for all $x \in V$.

A general eulerian chain a is a *special eulerian* chain if for all $v \in V$, $a(v) \in \{\pm\binom{1}{0}, \pm\binom{0}{1}\}$. It is easy to observe that if (M, a, b) is a general (special) matrix representation of a Lagrangian chain-group N, then a is a general (special) eulerian chain of N. We will prove that every general eulerian chain of a Lagrangian chain-group induces a matrix representation. Before proving that, we first show that every Lagrangian chain-group has a special eulerian chain.

Proposition 4.2. *Every isotropic chain-group has a special eulerian chain.*

Proof. Let N be an isotropic chain-group on V to $K = \mathbb{F}^2$. We proceed by induction on $|V|$. We may assume that $\dim(N) > 0$. Let $v \in V$.

If $|V| = 1$, then $\dim(N) = 1$. Then either v^* or v_* is a special eulerian chain.

Now let us assume that $|V| > 1$. Let $W = V \setminus \{v\}$. Both $N \mathbin{\backslash\!\!\backslash} \{v\}$ and $N \mathbin{/\!\!/} \{v\}$ are isotropic chain-groups on W to K. By the induction hypothesis, both $N \mathbin{\backslash\!\!\backslash} \{v\}$ and $N \mathbin{/\!\!/} \{v\}$ have special eulerian chains a_1', a_2', respectively, on W to K such that $a_i'(x) \in \{\binom{1}{0}, \binom{0}{1}\}$ for all $x \in W$.

Let a_1, a_2 be chains on V to K such that $a_1(v) = \binom{1}{0}$, $a_2(v) = \binom{0}{1}$, and $a_i \cdot W = a_i'$ for $i = 1, 2$. We claim that either a_1 or a_2 is a special eulerian chain of N. Suppose not. For each $i = 1, 2$, there is a nonzero chain $f_i \in N$ such that $\langle f_i(x), a_i(x)\rangle_K = 0$ for all $x \in V$. By construction $f_1 \cdot W \in N \mathbin{\backslash\!\!\backslash} \{v\}$ and $f_2 \cdot W \in N \mathbin{/\!\!/} \{v\}$. Since a_1', a_2' are special eulerian chains of $N \mathbin{\backslash\!\!\backslash} \{v\}$ and $N \mathbin{/\!\!/} \{v\}$, respectively, we have $f_1 \cdot W = f_2 \cdot W = 0$.

Since $f_i \neq 0$, by Lemma 3.2, $f_1 = c_1 v^*$ and $f_2 = c_2 v_*$ for some nonzero $c_1, c_2 \in \mathbb{F}$. Then $\langle f_1, f_2\rangle = \langle f_1(v), f_2(v)\rangle_K = c_1 c_2 \neq 0$, contradictory to the assumption that N is isotropic. □

Proposition 4.3. *Let N be a Lagrangian chain-group on V to K and let a be a general eulerian chain of N and let b be a chain supplementary to a.*

1. *For every $v \in V$, there exists a unique chain $f_v \in N$ satisfying the following two conditions.*
   (i) $\langle a(v), f_v(v)\rangle_K = 1$,
   (ii) $\langle a(w), f_v(w)\rangle_K = 0$ *for all $w \in V \setminus \{v\}$.*
   Moreover, $\{f_v : v \in V\}$ is a basis of N. This basis is called the fundamental basis *of N with respect to a.*
2. *If $\langle\, ,\rangle_K$ is symmetric and either the characteristic of $\mathbb{F}$ is not 2 or $f_v(v) = b(v)$ for all $v \in V$, then $M = (\langle f_i(j), b(j)\rangle_K : i, j \in V)$ is a skew-symmetric matrix such that (M, a, b) is a general matrix representation of N.*
3. *If $\langle\, ,\rangle_K$ is skew-symmetric, $M = (\langle f_i(j), b(j)\rangle_K : i, j \in V)$ is a symmetric matrix such that (M, a, b) is a general matrix representation of N.*

Proof. Existence in (1): For each $x \in V$, let g_x be a chain on V to K such that $g_x(x) = a(x)$ and $g_x(y) = 0$ for all $y \in V \setminus \{x\}$. Let W be a chain-group spanned by $\{g_x : x \in V\}$. It is clear that $\dim(W) = |V|$. Let $N + W = \{f + g : f \in N, g \in W\}$. Since a is eulerian, $N \cap W = \{0\}$ and therefore $\dim(N + W) = \dim(N) + \dim(W) = 2|V|$, because N is Lagrangian. We conclude that $N + W = K^V$. Let h_v be a chain on V to K such that $\langle a(v), h_v(v)\rangle_K = 1$ and $h_v(w) = 0$ for all $w \in V \setminus \{v\}$. We express $h_v = f_v + g$ for some $f_v \in N$ and $g \in W$. Then $\langle a(v), f_v(v)\rangle_K = \langle a(v), h_v(v)\rangle_K - \langle a(v), g(v)\rangle_K = 1$

and $\langle a(w), f_v(w)\rangle_K = \langle a(w), h_v(w)\rangle_K - \langle a(w), g(w)\rangle_K = 0$ for all $w \in V \setminus \{v\}$.

Uniqueness in (1): Suppose that there are two chains f_v and f'_v in N satisfying two conditions (i), (ii) in (1). Then $\langle a(v), f_v(v) - f'_v(v)\rangle_K = 0$. By Lemma 3.2, there exists $c \in \mathbb{F}$ such that $f_v(v) - f'_v(v) = ca(v)$. Let $f = f_v - f'_v \in N$. Then $\langle a(w), f(w)\rangle_K = 0$ for all $w \in V$. Since a is eulerian, $f = 0$ and therefore $f_v = f'_v$.

Being a basis in (1): We claim that $\{f_v : v \in V\}$ is linearly independent. Suppose that $\sum_{w\in V} c_w f_w = 0$ for some $c_w \in \mathbb{F}$. Then $c_v = \sum_{w\in V} c_w \langle a(v), f_w(v)\rangle_K = 0$ for all $v \in V$.

Constructing a matrix for (2) and (3): Let $i, j \in V$. By (ii) and Lemma 3.2, there exists $m_{ij} \in \mathbb{F}$ such that $f_i(j) = m_{ij}a(j)$ if $i \neq j$ and $f_i(i) - b(i) = m_{ii}a(i)$. Then, $\langle f_i(j), b(j)\rangle_K = m_{ij}$ for all $i, j \in V$. Therefore $M = (m_{ij} : i, j \in V)$.

Since N is isotropic,

$$\langle f_i, f_j\rangle = \sum_{v\in V} \langle f_i(v), f_j(v)\rangle_K = 0$$

and we deduce that $\langle f_i(i), f_j(i)\rangle_K + \langle f_i(j), f_j(j)\rangle_K = 0$ if $i \neq j$ and $\langle f_i(i), f_i(i)\rangle_K = 0$. This implies that

$$m_{ji} \langle b(i), a(i)\rangle_K + m_{ij} \langle a(j), b(j)\rangle_K = 0 \text{ for all } i, j \in V.$$

If $\langle\, ,\, \rangle_K$ is skew-symmetric, then $\langle b(i), a(i)\rangle_K = -1$ and therefore $m_{ji} = m_{ij}$.

If $\langle\, ,\, \rangle_K$ is symmetric, then $\langle b(i), a(i)\rangle_K = 1$ and so $m_{ji} = -m_{ij}$. This also imply that $m_{ii} = 0$ if the characteristic of $\mathbb{F}$ is not 2. If the characteristic of $\mathbb{F}$ is 2, then we assumed that $f_i(i) = b(i)$ and therefore $m_{ii} = 0$. Note that $\langle f_i(i), f_i(i)\rangle_K = 0$ and therefore the chain b with $b(i) = f_i(i)$ for all $i \in V$ is supplementary to a.

It is easy to observe that (M, a, b) is a general matrix representation of N because a, b are supplementary and $f_i(j) = m_{ij}a(j) + b(j)$ if $i = j \in V$ and $f_i(j) = m_{ij}a(j)$ if $i \neq j$. □

Proposition 4.4. *Let (M, a, b) be a special matrix representation of a Lagrangian chain-group N on V to $K = \mathbb{F}^2$. Suppose that a' is a chain such that $a'(v) \in \{\pm\binom{1}{0}, \pm\binom{0}{1}\}$ for all $v \in V$. Then a' is special eulerian if and only if $M[Y]$ is nonsingular for $Y = \{x \in V : a'(x) \neq \pm a(x)\}$.*

Proof. Let $M = (m_{ij} : i, j \in V)$. Let $f_i \in N$ be a chain such that $f_i(j) = m_{ij}a(j)$ if $j \neq i$ and $f_i(i) = m_{ii}a(i) + b(i)$.

We first prove that if $M[Y]$ is nonsingular, then f is special eulerian. Suppose that there is a chain $f \in N$ such that $\langle f(x), a'(x)\rangle_K = 0$ for all $x \in V$. We may express f as a linear combination $\sum_{i\in V} c_i f_i$

with some $c_i \in \mathbb{F}$. If $j \notin Y$, then $a'(j) = \pm a(j)$ and $\langle f(j), a(j)\rangle_K = c_j \langle b(j), a(j)\rangle_K = 0$ and therefore $c_j = 0$ for all $j \notin Y$.

If $j \in Y$, then $a'(j) = \pm b(j)$ and so

$$\langle f(j), b(j)\rangle_K = \sum_{i\in Y} c_i m_{ij} \langle a(j), b(j)\rangle_K = \sum_{i\in Y} c_i m_{ij} = 0.$$

Since $M[Y]$ is invertible, the only solution $\{c_i : i \in Y\}$ satisfying the above linear equation is zero. So $c_i = 0$ for all $i \in V$ and therefore $f = 0$, meaning that a' is special eulerian.

Conversely suppose that $M[Y]$ is singular. Then there is a linear combination of rows in $M[Y]$ whose sum is zero. Thus there is a non-zero linear combination $\sum_{i\in Y} c_i f_i$ such that

$$\left\langle \sum_{i\in Y} c_i f_i(x), b(x) \right\rangle_K = 0 \text{ for all } x \in Y.$$

Clearly $\left\langle \sum_{i\in Y} c_i f_i(x), a(x)\right\rangle_K = 0$ for all $x \notin Y$. Since at least one c_i is non-zero, $\sum_{i\in Y} c_i f_i$ is non-zero. Therefore a' can not be special eulerian. □

For a subset Y of V, let I_Y be a $V \times V$ *indicator diagonal matrix* such that each diagonal entry corresponding to Y is -1 and all other diagonal entries are 1.

Proposition 4.5. *Suppose that (M, a, b) is a special matrix representation of a Lagrangian chain-group N on V to $K = \mathbb{F}^2$. Let $Y \subseteq V$. Assume that $M[Y]$ is nonsingular.*

(1) If $\langle\, ,\, \rangle_K$ is symmetric, then $(M * Y, a', b')$ is another special matrix representation of N where $M * Y$ is skew-symmetric and

$$a'(v) = \begin{cases} a(v) & \text{if } v \notin Y, \\ b(v) & \text{otherwise,} \end{cases} \qquad b'(v) = \begin{cases} b(v) & \text{if } v \notin Y, \\ a(v) & \text{otherwise.} \end{cases}$$

(2) If $\langle\, ,\, \rangle_K$ is skew-symmetric, then $(I_Y(M * Y), a', b')$ is another special matrix representation of N where $I_Y(M*Y)$ is symmetric and

$$a'(v) = \begin{cases} a(v) & \text{if } v \notin Y, \\ b(v) & \text{otherwise,} \end{cases} \qquad b'(v) = \begin{cases} b(v) & \text{if } v \notin Y, \\ -a(v) & \text{otherwise.} \end{cases}$$

Proof. Let $M = (m_{ij} : i, j \in V)$. For each $i \in V$, let $f_i \in N$ be a chain such that $f_i(j) = m_{ij} a(j)$ if $j \neq i$ and $f_i(i) = m_{ij} a(j) + b(j)$ if $j = i$. Since (M, a, b) is a special matrix representation of N, $\{f_i : i \in V\}$ is a fundamental basis of N.

Proposition 4.4 implies that a' is eulerian. According to Proposition 4.3, we should be able to construct a special matrix representation with respect to the eulerian chain a'. To do so, we first construct the fundamental basis $\{g_v : v \in V\}$ of N with respect to a'.

Suppose that for each $x \in V$, $g_x = \sum_{i\in V} c_{xi} f_i$ for some $c_{xi} \in \mathbb{F}$. By definition, $\langle a'(x), g_x(x)\rangle_K = 1$ and $\langle a'(j), g_x(j)\rangle_K = 0$ for all $j \neq x$. Then

$$\langle a'(j), g_x(j)\rangle_K = \begin{cases} \sum_{i\in V} c_{xi} m_{ij} \langle b(j), a(j)\rangle_K\,, & \text{if } j \in Y, \\ c_{xj}. & \text{if } j \notin Y. \end{cases}$$

Suppose that $x \in Y$. If $j \in Y$, then

$$\sum_{i\in Y} c_{xi} m_{ij} \langle b(j), a(j)\rangle_K = \begin{cases} 1 & \text{if } x = j, \\ 0 & \text{if } x \neq j. \end{cases}$$

Let $(m'_{ij} : i,j \in Y) = (M[Y])^{-1}$. Then c_{xi} is given by the row of x in $(M[Y])^{-1}$; in other words, if $x, i \in Y$, then $c_{xi} = m'_{xi}$ if $\langle\, ,\, \rangle_K$ is symmetric and $c_{xi} = -m'_{xi}$ otherwise. If $x \in Y$ and $i \notin Y$, then $c_{xi} = 0$.

If $x \notin Y$, then clearly $c_{xx} = 1$ and $c_{xi} = 0$ for all $i \in V \setminus (Y \cup \{x\})$. If $j \in Y$, then $\sum_{i\in Y} c_{xi} m_{ij} \langle b(j), a(j)\rangle_K + c_{xx} m_{xj} \langle b(j), a(j)\rangle_K = 0$ and therefore $\sum_{i\in Y} c_{xi} m_{ij} = -m_{xj}$. For each k in Y, we have $c_{xk} = \sum_{i\in Y} c_{xi} \sum_{j\in Y} m_{ij} m'_{jk} = \sum_{j\in Y} m'_{jk} \sum_{i\in Y} c_{xi} m_{ij} = -\sum_{j\in Y} m'_{jk} m_{xj}$ and therefore for $x \notin Y$ and $i \in Y$, $c_{xi} = -\sum_{j\in Y} m_{xj} m'_{ji}$

We determined the fundamental basis $\{g_x : x \in V\}$ with respect to a'. We now wish to compute the matrix according to Proposition 4.3. Let us compute $\langle g_x(y), b'(y)\rangle_K$.

If $x, y \in Y$, then

$$\begin{aligned} &\left\langle \sum_{i\in Y} c_{xi} f_i(y), b'(y) \right\rangle_K \\ &\quad = c_{xy} \langle b(y), b'(y)\rangle_K = c_{xy} = \begin{cases} m'_{xy} & \text{if } \langle\, ,\, \rangle_K \text{ is symmetric,} \\ -m'_{xy} & \text{if } \langle\, ,\, \rangle_K \text{ is skew-symmetric.} \end{cases} \end{aligned}$$

If $x \in Y$ and $y \notin Y$, then

$$\begin{aligned} \left\langle \sum_{i\in Y} c_{xi} f_i(y), b'(y) \right\rangle_K &= \sum_{i\in Y} c_{xi} m_{iy} \langle a(y), b(y)\rangle_K \\ &= \begin{cases} \sum_{i\in Y} m'_{xi} m_{iy}. & \text{if } \langle\, ,\, \rangle_K \text{ is symmetric,} \\ -\sum_{i\in Y} m'_{xi} m_{iy}. & \text{if } \langle\, ,\, \rangle_K \text{ is skew-symmetric.} \end{cases} \end{aligned}$$

If $x \notin Y$ and $y \in Y$, then

$$\left\langle \sum_{i \in Y} c_{xi} f_i(y) + f_x(y), b'(y) \right\rangle_K = c_{xy} = -\sum_{j \in Y} m_{xj} m'_{jy}.$$

If $x \notin Y$ and $y \notin Y$, then

$$\left\langle \sum_{i \in Y} c_{xi} f_i(y) + f_x(y), b'(y) \right\rangle_K = -\sum_{i,j \in Y} m_{xj} m'_{ji} m_{iy} + m_{xy}$$

If $\langle\, , \rangle_K$ is symmetric and the characteristic of $\mathbb{F}$ is 2, then we need to ensure that M has no non-zero diagonal entries by verifying the additional assumption in (2) of Proposition 4.3 asking that $b'(x) = g_x(x)$ for all $x \in V$. It is enough to show that

$$\langle g_x(x), b'(x) \rangle_K = 0 \text{ for all } x \in V,$$

because, if so, then $\langle a'(x), b'(x) \rangle_K = 1 = \langle a'(x), g_x(x) \rangle_K$ implies that $g_x(x) = b'(x)$. Since $M[Y]$ is skew-symmetric, so is its inverse and therefore $m'_{xx} = 0$ for all $x \in Y$. Furthermore, for each $i, j \in Y$ and $x \in V \setminus Y$, we have $m_{xj} m'_{ji} m_{ix} = -m_{xi} m'_{ij} m_{jx}$ because M and $(M[Y])^{-1}$ are skew-symmetric and therefore $\sum_{i,j \in Y} m_{xj} m'_{ji} m_{ix} = 0$. Thus $g_x(x) = b'(x)$ for all $x \in V$ if $\langle\, , \rangle_K$ is symmetric and the characteristic of $\mathbb{F}$ is 2.

We conclude that the matrix $(\langle g_i(j), b'(j) \rangle_K : i, j \in V)$ is indeed $M * Y$ if $\langle\, , \rangle_K$ is symmetric or $(I_Y)(M * Y)$ if $\langle\, , \rangle_K$ is skew-symmetric. This concludes the proof. $\square$

A matrix M is called a *fundamental matrix* of a Lagrangian chain-group N if (M, a, b) is a special matrix representation of N for some chains a and b. We aim to characterize when two matrices M and M' are fundamental matrices of the same Lagrangian chain-group.

Theorem 4.6. *Let M and M' be $V \times V$ skew-symmetric or symmetric matrices over $\mathbb{F}$. The following are equivalent.*

(i) *There is a Lagrangian chain-group N such that both (M, a, b) and (M', a', b') are special matrix representations of N for some chains a, a', b, b'.*

(ii) *There is $Y \subseteq V$ such that $M[Y]$ is nonsingular and*

$$M' = \begin{cases} D(M * Y)D & \text{if } \langle\, , \rangle_K \text{ is symmetric,} \\ DI_Y(M * Y)D & \text{if } \langle\, , \rangle_K \text{ is skew-symmetric} \end{cases}$$

for some diagonal matrix D whose diagonal entries are ± 1.

Proof. To prove (i) from (ii), we use Proposition 4.5. Let $a(v) = \binom{1}{0}$ and $b(v) = \binom{0}{1}$ for all $v \in V$. Let N be the Lagrangian chain-group with the special matrix representation (M, a, b). Let $M_0 = M * Y$ if $\langle\, , \rangle_K$ is symmetric and $M_0 = I_Y(M*Y)$ if $\langle\, , \rangle_K$ is skew-symmetric. By Proposition 4.5, there are chains a_0, b_0 so that (M_0, a_0, b_0) is a special matrix representation of N. Let Z be a subset of V such that $I_Z = D$. For each $v \in V$, let

$$a'(v) = \begin{cases} -a_0(v) & \text{if } v \in Z, \\ a_0(v) & \text{if } v \notin Z, \end{cases} \qquad b'(v) = \begin{cases} -b_0(v) & \text{if } v \in Z, \\ b_0(v) & \text{if } v \notin Z. \end{cases}$$

Then a', b' are supplementary and (M', a', b') is a special matrix representation of N because $M' = DM_0D$.

Now let us assume (i) and prove (ii). Let $Y = \{x \in V : a'(x) \neq \pm a(x)\}$. Since a' is a special eulerian chain of N, $M[Y]$ is nonsingular by Proposition 4.4. By replacing M with $M * Y$ if $\langle\, , \rangle_K$ is symmetric, or $I_Y(M * Y)$ if $\langle\, , \rangle_K$ is skew-symmetric, we may assume that $Y = \emptyset$. Thus $a'(x) = \pm a(x)$ and $b'(x) = \pm b(x)$ for all $x \in V$. Let $Z = \{x \in V : a'(x) = -a(x)\}$ and $D = I_Z$. Since $\langle a'(x), b'(x)\rangle_K = 1$, $b'(x) = -b(x)$ if and only if $x \in Z$. Then (DMD, a', b') is a special matrix representation of N, because the fundamental basis generated by (DMD, a', b') spans the same subspace N spanned by the fundamental basis generated by (M, a, b). We now have two special matrix representations (M', a', b') and (DMD, a', b'). By Proposition 4.3, $M' = DMD$ because of the uniqueness of the fundamental basis with respect to a'. This concludes the proof. □

Negating a row or a column of a matrix is to multiply -1 to each of its entries. Obviously a matrix obtained by negating some rows and columns of a $V \times V$ matrix M is of the form $I_X M I_Y$ for some $X, Y \subseteq V$. We now prove that the order of applying pivots and negations can be reversed.

Lemma 4.7. *Let M be a $V \times V$ matrix and let Y be a subset of V such that $M[Y]$ is nonsingular. Let M' be a matrix obtained from M by negativing some rows and columns. Then $M' * Y$ can be obtained from $M * Y$ by negating some rows and columns. (See Figure 4.2.)*

Proof. More generally we write M and M' as follows:

$$M = \begin{array}{c} \\ Y \\ V\setminus Y \end{array} \!\!\begin{array}{c} \begin{array}{cc} Y & V\setminus Y \end{array} \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array}, \quad M' = \begin{array}{c} \\ Y \\ V\setminus Y \end{array} \!\!\begin{array}{c} \begin{array}{cc} Y & V\setminus Y \end{array} \\ \begin{pmatrix} JAK & JBL \\ UCK & UDL \end{pmatrix} \end{array},$$

$$\begin{tikzcd} M \arrow[r, "\text{pivot}"] \arrow[d, "\text{negating some rows and columns}"'] & M * Y \arrow[d, "\text{negating some rows and columns}"] \\ M' \arrow[r, "\text{pivot}"] & M' * Y \end{tikzcd}$$

FIGURE 1. Commuting pivots and negations

for some nonsingular diagonal matrices J, K, L, U. Then

$$\begin{aligned} M * Y &= \begin{pmatrix} A^{-1} & A^{-1}B \\ -CA^{-1} & D - CA^{-1}B \end{pmatrix}, \\ M' * Y &= \begin{pmatrix} K^{-1}A^{-1}J^{-1} & K^{-1}A^{-1}J^{-1}JBL \\ -UCKK^{-1}A^{-1}J^{-1} & UDL - UCKK^{-1}A^{-1}J^{-1}JBL \end{pmatrix} \\ &= \begin{pmatrix} K^{-1}(A^{-1})J^{-1} & K^{-1}(A^{-1}B)L \\ U(-CA^{-1})J^{-1} & U(D - CA^{-1}B)L \end{pmatrix}. \end{aligned}$$

This lemma follows because we can set J, K, L, U to be diagonal matrices with ± 1 on the diagonal entries and then $M' * Y$ can be obtained from $M * Y$ by negating some rows and columns. □

4.3. **Minors.** Suppose that (M, a, b) is a special matrix representation of a Lagrangian chain-group N. We will find special matrix representations of minors of N.

Lemma 4.8. *Let (M, a, b) be a special matrix representation of a Lagrangian chain-group N on V to $K = \mathbb{F}^2$. Let $v \in V$ and $T = V \setminus \{v\}$. Suppose that $a(v) = \pm\binom{1}{0}$.*

1. *The triple $(M[T], a \cdot T, b \cdot T)$ is a special matrix representation of $N \setminus\!\!\setminus \{v\}$.*
2. *There is $Y \subseteq V$ such that $M[Y]$ is nonsingular and $(M'[T], a' \cdot T, b' \cdot T)$ is a special matrix representation of $N /\!\!/ \{v\}$, where*
$$M' = \begin{cases} M * Y & \text{if } \langle\, ,\, \rangle_K \text{ is symmetric,} \\ (I_Y)(M * Y) & \text{if } \langle\, ,\, \rangle_K \text{ is skew-symmetric,} \end{cases}$$
and a' and b' are given by Proposition 4.5.

Proof. Let $M = (m_{ij} : i, j \in V)$ and for each $i \in V$, let $f_i \in N$ be a chain as it is defined in Proposition 4.1.

(1): We know that $f_i \cdot T \in N \setminus\!\!\setminus \{v\}$ for all $i \neq v$. Since a is eulerian, $v^* \notin N$ and therefore $\{f_i \cdot T : i \in T\}$ is linearly independent. Then $\{f_i \cdot T : i \in T\}$ is a basis of $N \setminus\!\!\setminus \{v\}$, because $\dim(N \setminus\!\!\setminus \{v\}) = |T| =$

$|V|-1$. Now it is easy to verify that $(M[T], a\cdot T, b\cdot T)$ is a special matrix representation of $N \setminus\!\!\setminus \{v\}$.

(2): If $m_{iv} = m_{vi} = 0$ for all $i \in V$, then we may simply replace $a(v)$ with $\pm\binom{0}{1}$ and $b(v)$ with $\pm\binom{1}{0}$ without changing the Lagrangian chain-group N. In this case, we simply apply (1) to deduce that $Y = \emptyset$ works.

Otherwise, there exists $Y \subseteq V$ such that $v \in Y$ and $M[Y]$ is non-singular because M is skew-symmetric or symmetric. We apply $M * Y$ to get (M', a', b') as an alternative special matrix representation of N by Proposition 4.5. Then $a'(v) = \pm\binom{0}{1}$ and then we apply (1) to (M', a', b'). □

Theorem 4.9. *For $i = 1, 2$, let M_i be a fundamental matrix of a Lagrangian chain-group N_i on V_i to $K = \mathbb{F}^2$. If N_1 is simply isomorphic to a minor of N_2, then M_1 is isomorphic to a principal submatrix of a matrix obtained from M_2 by taking a pivot and negating some rows and columns.*

Proof. Since K is shared by N_1 and N_2, M_1 and M_2 are skew-symmetric if $\langle\, ,\rangle_K$ is symmetric and symmetric if $\langle\, ,\rangle_K$ is skew-symmetric.

We may assume that N_1 is a minor of N_2 and $V_1 \subseteq V_2$. Then by Lemmas 4.7 and 4.8, N_1 has a fundamental matrix M' that is a principal submatrix of a matrix obtained from M by taking a pivot and negativing some rows if necessary. Then both M' and M_1 are fundamental matrices of N_1. By Theorem 4.6, there is a method to get M_1 from M' by applying a pivot and negating some rows and columns if necessary. □

4.4. **Representable Delta-matroids.** Theorem 2.1 implies the following proposition.

Proposition 4.10. *Let A, B be skew-symmetric or symmetric matrices over a field $\mathbb{F}$. If A is a principal submatrix of a matrix obtained from B by taking a pivot and negating some rows and columns, then the delta-matroid $\mathcal{M}(A)$ is a minor of $\mathcal{M}(B)$.*

Bouchet [4] showed that there is a natural way to construct a delta-matroid from an isotropic chain-group.

Theorem 4.11 (Bouchet [4])**.** *Let N be an isotropic chain-groups N on V to K. Let a and b be supplementary chains on V to K. Let*

$$\begin{aligned}\mathcal{F} = \{X \subseteq V :& \textit{there is no non-zero chain } f \in N \\ & \textit{such that } \langle f(x), a(x)\rangle_K = 0 \textit{ for all } x \in V \setminus X \\ & \textit{and } \langle f(x), b(x)\rangle_K = 0 \textit{ for all } x \in X.\}\end{aligned}$$

Then, $\mathcal{M} = (V, \mathcal{F})$ is a delta-matroid.

The triple (N, a, b) given as above is called the *chain-group representation* of the delta-matroid $\mathcal{M}$. In addition, if $a(v), b(v) \in \{\pm\binom{1}{0}, \pm\binom{0}{1}\}$, then (N, a, b) is called the *special chain-group representation* of $\mathcal{M}$.

We remind you that a delta-matroid $\mathcal{M}$ is representable over a field $\mathbb{F}$ if $\mathcal{M} = \mathcal{M}(A)\Delta Y$ for some skew-symmetric or symmetric $V \times V$ matrix A over $\mathbb{F}$ and a subset Y of V where $\mathcal{M}(A) = (V, \mathcal{F})$ where $\mathcal{F} = \{Y : A[Y] \text{ is nonsingular}\}$.

Suppose that N is a Lagrangian chain-group represented by a special matrix representation (M, a, b). Then (N, a, b) induces a delta-matroid $\mathcal{M}$ by the above theorem. Proposition 4.4 characterizes all the special eulerian chains in terms of the singularity of $M[Y]$ and special eulerian chains coincide with the feasible sets of $\mathcal{M}$ given by Theorem 4.11. In other words, Y is feasible in $\mathcal{M}$ if and only if a chain a' is special eulerian in N when $a(v) = a'(v)$ if $v \in Y$ and $a'(v) = b(v)$ if $v \notin Y$.

Then twisting operations $\mathcal{M}\Delta Y$ on delta-matroids can be simulated by swapping supplementary chains $a(x)$ and $b(x)$ for $x \in Y$ in the chain-group representation as it is in Proposition 4.5. Thus we can alternatively define representable delta-matroids as follows.

Theorem 4.12. *A delta-matroid on V is representable over a field $\mathbb{F}$ if and only if it admits a special chain-group representation (N, a, b) for a Lagrangian chain-group N on V to $K = \mathbb{F}^2$ and special supplementary chains a, b on V to K where $\langle\, ,\, \rangle_K$ is either skew-symmetric or symmetric.*

4.5. **Connectivity.** When the rank-width of matrices is defined, the function $\operatorname{rank} M[X, V \setminus X]$ is used to describe how complex the connection between X and $V \setminus X$ is. In this subsection, we express $\operatorname{rank} M[X, V \setminus X]$ in terms of a Lagrangian chain-group represented by M.

Theorem 4.13. *Let M be a skew-symmetric or symmetric $V \times V$ matrix over a field $\mathbb{F}$. Let N be a Lagrangian chain-group on V to $K = \mathbb{F}^2$ such that (M, a, b) is a matrix representation of N with supplementary chains a and b on V to K. Then,*

$$\operatorname{rank} M[X, V \setminus X] = \lambda_N(X) = |X| - \dim(N \times X).$$

Proof. Let $M = (m_{ij} : i, j \in V)$. As we described in Proposition 4.1, we let $f_i(j) = m_{ij}a(j)$ if $j \in V \setminus \{i\}$ and $f_i(i) = m_{ii} + b(i)$. We know that $\{f_i : i \in V\}$ is a fundamental basis of N. Let $A = M[X, V \setminus X]$. We have $\operatorname{rank} A = \operatorname{rank} A^t = |X| - \operatorname{nullity}(A^t)$, where the *nullity* of A^t is $\dim(\{x \in \mathbb{F}^X : A^t x = 0\})$, that is eqaul to $\dim(\{x \in \mathbb{F}^X : x^t A = 0\})$.

Let $\varphi : \mathbb{F}^V \to N$ be a linear transformation with $\varphi(p) = \sum_{v\in V} p(v) f_v$. Then, φ is an isomorphism and therefore we have the following:

$$\begin{aligned}
\dim(N \times X) &= \dim(\{y \in N : y(j) = 0 \text{ for all } j \in V \setminus X\}) \\
&= \dim(\varphi^{-1}(\{y \in N : y(j) = 0 \text{ for all } j \in V \setminus X\})) \\
&= \dim(\{x \in \mathbb{F}^V : \sum_{i\in V} x(i) f_i(j) = 0 \text{ for all } j \in V \setminus X\}) \\
&= \dim(\{x \in \mathbb{F}^X : \sum_{i\in X} x(i) m_{ij} = 0 \text{ for all } j \in V \setminus X\}) \\
&= \dim(\{x \in \mathbb{F}^X : x^t A = 0\}) \\
&= \text{nullity}(A^t).
\end{aligned}$$

We deduce that $\operatorname{rank} A = |X| - \dim(N \times X)$. □

The above theorem gives the following corollaries.

Corollary 4.14. *Let $\mathbb{F}$ be a field and let N be a Lagrangian chain-group on V to $K = \mathbb{F}^2$. If M_1 and M_2 are two fundamental matrices of N, then* $\operatorname{rank} M_1[X, V \setminus X] = \operatorname{rank} M_2[X, V \setminus X]$ *for all $X \subseteq V$.*

Corollary 4.15. *Let M be a skew-symmetric or symmetric $V \times V$ matrix over a field $\mathbb{F}$. Let N be a Lagrangian chain-group on V to $K = \mathbb{F}^2$ such that (N, a, b) is a matrix representation of N. Then the rank-width of M is equal to the branch-width of N.*

5. Generalization of Tutte's linking theorem

We prove an analogue of Tutte's linking theorem [23] for Lagrangian chain-groups. Tutte's linking theorem is a generalization of Menger's theorem of graphs to matroids. Robertson and Seymour [14] uses Menger's theorem extensively for proving well-quasi-ordering of graphs of bounded tree-width. When generalizing this result to matroids, Geelen, Gerards, and Whittle [8] used Tutte's linking theorem for matroids. To further generalize this to Lagrangian chain-groups, we will need a generalization of Tutte's linking theorem for Lagrangian chain-groups.

A crucial step for proving this is to ensure that the connectivity function behaves nicely on one of two minors $N \setminus\!\!\setminus \{v\}$ and $N /\!\!/ \{v\}$ of a Lagrangian chain-group N. The following inequality was observed by Bixby [1] for matroids.

Proposition 5.1. *Let $v \in V$. Let N be a chain-group on V to $K = \mathbb{F}^2$ and let $X, Y \subseteq V \setminus \{v\}$. Then,*

$$\lambda_{N\setminus\!\!\setminus\{v\}}(X) + \lambda_{N/\!\!/\{v\}}(Y) \geq \lambda_N(X \cap Y) + \lambda_N(X \cup Y \cup \{v\}) - 1.$$

We first prove the following lemma for the above proposition.

Lemma 5.2. *Let $v \in V$. Let N be a chain-group on V to $K = \mathbb{F}^2$ and let $X, Y \subseteq V \setminus \{v\}$. Then,*

$$\dim(N \times (X \cap Y)) + \dim(N \times (X \cup Y \cup \{v\})) \geq \dim((N \setminus\!\!\setminus \{v\}) \times X) + \dim((N /\!\!/ \{v\}) \times Y).$$

Moreover, the equality does not hold if $v^* \in N$ or $v_* \in N$.

Proof. We may assume that $V = X \cup Y \cup \{v\}$. Let

$$N_1 = \left\{ f \in N : \left\langle f(v), \binom{1}{0} \right\rangle_K = 0, f(x) = 0 \text{ for all } x \in V \setminus X \setminus \{v\} \right\},$$
$$N_2 = \left\{ f \in N : \left\langle f(v), \binom{0}{1} \right\rangle_K = 0, f(x) = 0 \text{ for all } x \in V \setminus Y \setminus \{v\} \right\}.$$

We use the fact that $\dim(N_1+N_2)+\dim(N_1\cap N_2) = \dim(N_1)+\dim(N_2)$. It is easy to see that if $f \in N_1 \cap N_2$, then $f(v) = 0$ and therefore $(N_1\cap N_2)\cdot(X\cap Y) = N\times(X\cap Y)$ and $\dim(N_1\cap N_2) = \dim(N\times(X\cap Y))$. Moreover, $N_1 + N_2 \subseteq N$ and therefore $\dim(N) \geq \dim(N_1 + N_2)$. It is clear that $\dim(N\setminus\!\!\setminus\{v\}\times X) \leq \dim N_1$ and $\dim(N/\!\!/\{v\}\times X) \leq \dim N_2$. Therefore we conclude that $\dim(N \times (X \cap Y)) + \dim N \geq \dim(N \setminus\!\!\setminus \{v\} \times X) + \dim(N /\!\!/ \{v\} \times Y)$.

If $v^* \in N$, then $\dim(N \setminus\!\!\setminus \{v\} \times X) < \dim N_1$ and therefore the equality does not hold. Similarly if $v_* \in N$, then the equality does not hold as well. $\square$

Proof of Proposition 5.1. Since N and $N^\perp$ have the same connectivity function λ and $N^\perp \setminus\!\!\setminus \{v\} = (N \setminus\!\!\setminus \{v\})^\perp$, $N^\perp /\!\!/ \{v\} = (N /\!\!/ \{v\})^\perp$, (Lemma 3.9), we may assume that $\dim N - \dim(N \setminus\!\!\setminus \{v\}) \in \{0,1\}$ (Proposition 3.6) by replacing N by $N^\perp$ if necessary. Let $X' = V \setminus X \setminus \{v\}$ and $Y' = V \setminus Y \setminus \{v\}$. We recall that

$$\begin{aligned}
&2\lambda_N(X \cap Y)\\
&\qquad = \dim N - \dim(N \times (X \cap Y)) - \dim(N \times (X' \cup Y' \cup \{v\})),\\
&2\lambda_N(X \cup Y \cup \{v\})\\
&\qquad = \dim N - \dim(N \times (X \cup Y \cup \{v\})) - \dim(N \times (X' \cap Y')),\\
&2\lambda_{N\setminus\!\!\setminus\{v\}}(X)\\
&\qquad = \dim(N \setminus\!\!\setminus \{v\}) - \dim(N \setminus\!\!\setminus \{v\} \times X) - \dim(N \setminus\!\!\setminus \{v\} \times X'),\\
&2\lambda_{N/\!\!/\{v\}}(Y)\\
&\qquad = \dim(N /\!\!/ \{v\}) - \dim(N /\!\!/ \{v\} \times Y) - \dim(N /\!\!/ \{v\} \times Y').
\end{aligned}$$

It is easy to deduce this lemma from Lemma 5.2 if

$$2\dim N - \dim(N \setminus\!\!\setminus \{v\}) - \dim(N /\!\!/ \{v\}) \leq 2. \tag{1}$$

Therefore we may assume that (1) is false. Since we have assumed that $\dim N - \dim(N \setminus\!\!\setminus \{v\}) \in \{0,1\}$, we conclude that $\dim N - \dim(N /\!\!/ \{v\}) \geq 2$. By Proposition 3.6, we have $v_* \in N$. Then the equality in the inequality of Lemma 5.2 does not hold. So, we conclude that $\dim(N \times (X \cap Y)) + \dim(N \times (X \cup Y \cup \{v\})) \geq \dim(N \setminus\!\!\setminus \{v\} \times X) + \dim(N /\!\!/ \{v\} \times Y) + 1$ and the same inequality for X' and Y'. Then, $\lambda_{N\setminus\!\!\setminus\{v\}}(X) + \lambda_{N/\!\!/\{v\}}(Y) \geq \lambda_N(X \cap Y) + \lambda_N(X \cup Y \cup \{v\}) - 3/2 + 1$. □

We are now ready to prove an analogue of Tutte's linking theorem for Lagrangian chain-groups.

Theorem 5.3. *Let V be a finite set and X, Y be disjoint subsets of V. Let N be a Lagrangian chain-group on V to K. The following two conditions are equivalent:*

(i) *$\lambda_N(Z) \geq k$ for all sets Z such that $X \subseteq Z \subseteq V \setminus Y$,*
(ii) *there is a minor M of N on $X \cup Y$ such that $\lambda_M(X) \geq k$.*

In other words,

$$\begin{aligned}&\min\{\lambda_N(Z) : X \subseteq Z \subseteq V \setminus Y\}\\ &\qquad = \max\{\lambda_{N\setminus\!\!\setminus U/\!\!/W}(X) : U \cup W = V \setminus (X \cup Y), U \cap W = \emptyset\}.\end{aligned}$$

Proof. By Theorem 3.13, (ii) implies (i). Now let us assume (i) and show (ii). We proceed by induction on $|V \setminus (X \cup Y)|$. If $V = X \cup Y$, then it is trivial. So we may assume that $|V \setminus (X \cup Y)| \geq 1$. Since $\lambda_N(X)$ are integers for all $X \subseteq V$ by Lemma 3.10, we may assume that k is an integer.

Let $v \in V \setminus (X \cup Y)$. Suppose that (ii) is false. Then there is no minor M of $N \setminus\!\!\setminus \{v\}$ or $N /\!\!/ \{v\}$ on $X \cup Y$ having $\lambda_M(X) \geq k$. By the induction hypothesis, we conclude that there are sets X_1 and X_2 such that $X \subseteq X_1 \subseteq V \setminus Y \setminus \{v\}$, $X \subseteq X_2 \subseteq V \setminus Y \setminus \{v\}$, $\lambda_{N\setminus\!\!\setminus\{v\}}(X_1) < k$, and $\lambda_{N/\!\!/\{v\}}(X_2) < k$. By Lemma 3.10, $\lambda_{N\setminus\!\!\setminus\{v\}}(X_1)$ and $\lambda_{N/\!\!/\{v\}}(X_2)$ are integers. Therefore $\lambda_{N\setminus\!\!\setminus\{v\}}(X_1) \leq k-1$ and $\lambda_{N/\!\!/\{v\}}(X_2) \leq k-1$. By Proposition 5.1,

$$\lambda_{N\setminus\!\!\setminus\{v\}}(X_1) + \lambda_{N/\!\!/\{v\}}(X_2) \geq \lambda_N(X_1 \cap X_2) + \lambda_N(X_1 \cup X_2 \cup \{v\}) - 1.$$

This is a contradiction because $\lambda_N(X_1 \cap X_2) \geq k$ and $\lambda_N(X_1 \cup X_2 \cup \{v\}) \geq k$. □

Corollary 5.4. *Let N be a Lagrangian chain-group on V to K and let $X \subseteq Y \subseteq V$. If $\lambda_N(Z) \geq \lambda_N(X)$ for all Z satisfying $X \subseteq Z \subseteq Y$, then there exist disjoint subsets C and D of $Y \setminus X$ such that $C \cup D = Y \setminus X$ and $N \times X = N \times Y /\!\!/ C \setminus\!\!\setminus D$.*

Proof. For all C and D if $C \cup D = Y \setminus X$ and $C \cap D = \emptyset$, then $N \times X \subseteq N \times Y \sslash C \bbslash D$. So it is enough to show that there exists a partition (C, D) of $Y \setminus X$ such that

$$\dim(N \times X) \geq \dim(N \times Y \sslash C \bbslash D).$$

By Theorem 5.3, there is a minor $M = N \sslash C \bbslash D$ of N on $X \cup (V \setminus Y)$ such that $\lambda_M(X) \geq \lambda_N(X)$. It follows that $|X| - \dim(N \sslash C \bbslash D \times X) \geq |X| - \dim(N \times X)$. Now we use the fact that $N \sslash C \bbslash D \times X = N \times Y \sslash C \bbslash D$. □

6. Well-quasi-ordering of Lagrangian chain-groups

In this section, we prove that Lagrangian chain-groups of bounded branch-width are well-quasi-ordered under taking a minor. Here we state its simplified form.

Theorem 6.1 (Simplified)**.** *Let $\mathbb{F}$ be a finite field and let k be a constant. Every infinite sequence $N_1, N_2, \ldots$ of Lagrangian chain-groups over $\mathbb{F}$ having branch-width at most k has a pair $i < j$ such that N_i is simply isomorphic to a minor of N_j.*

This simplified version is enough to obtain results in Sections 7 and 8. One may first read corollaries in later sections and return to this section.

6.1. Boundaried chain-groups.

For an isotropic chain-group N on V to $K = \mathbb{F}^2$, we write $N^\perp/N$ for a vector space over $\mathbb{F}$ containing vectors of the form $a + N$ where $a \in N^\perp$ such that

(i) $a + N = b + N$ if and only if $a - b \in N$,
(ii) $(a + N) + (b + N) = (a + b) + N$,
(iii) $c(a + N) = ca + N$ for $c \in \mathbb{F}$.

An *ordered basis* of a vector space is a sequence of vectors in the vector space such that the vectors in the sequence form a basis of the vector space. An ordered basis of $N^\perp/N$ is called a *boundary* of N. An isotropic chain-group N on V to K with a boundary B is called a *boundaried chain-group* on V to K, denoted by (V, N, B).

By the theorem in the linear algebra, we know that

$$|B| = \dim(N^\perp) - \dim(N) = 2(|V| - \dim N).$$

We define contractions and deletions of boundaries B of an isotropic chain-group N on V to K. Let $B = \{b_1 + N, b_2 + N, \ldots, b_m + N\}$ be a boundary of N. For a subset X of V, if $|V \setminus X| - \dim(N \bbslash X) = |V| - \dim N$, then we define $B \bbslash X$ as a sequence

$$\{b'_1 \cdot (V \setminus X) + N \bbslash X, b'_2 \cdot (V \setminus X) + N \bbslash X, \ldots, b'_m \cdot (V \setminus X) + N \bbslash X\}$$

where $b_i + N = b'_i + N$ and $\left\langle b'_i(v), \binom{1}{0}\right\rangle_K = 0$ for all $v \in X$. Similarly if $|V \setminus X| - \dim(N /\!\!/ X) = |V| - \dim N$, then we define $B /\!\!/ X$ as a sequence

$$\{b'_1 \cdot (V \setminus X) + N /\!\!/ X, b'_2 \cdot (V \setminus X) + N /\!\!/ X, \ldots, b'_m \cdot (V \setminus X) + N /\!\!/ X\}$$

where $b_i + N = b'_i + N$ and $\left\langle b'_i(v), \binom{0}{1}\right\rangle_K = 0$ for all $v \in X$. We prove that $B \setminus\!\!\setminus X$ and $B /\!\!/ X$ are well-defined.

Lemma 6.2. *Let N be an isotropic chain-group on V to K. Let X be a subset of V. If $\dim N - \dim(N \setminus\!\!\setminus X) = |X|$ and $f \in N^\perp$, then there exists a chain $g \in N^\perp$ such that $f - g \in N$ and $\left\langle g(x), \binom{1}{0}\right\rangle_K = 0$ for all $x \in X$.*

Proof. We proceed by induction on $|X|$. If $X = \emptyset$, then it is trivial. Let us assume that X is nonempty. Notice that $N \subseteq N^\perp$ because N is isotropic. We may assume that there is $v \in X$ such that $\left\langle f(v), \binom{1}{0}\right\rangle_K \neq 0$, because otherwise we can take $g = f$.

Then $v^* \notin N$. Since $|V \setminus X| - \dim(N \setminus\!\!\setminus X) = |V| - \dim N$, we have $|V| - 1 - \dim(N \setminus\!\!\setminus \{v\}) = |V| - \dim N$ (Corollary 3.7) and therefore $v^* \notin N^\perp$ by Proposition 3.6.

Thus there exists a chain $h \in N$ such that $\langle h, v^*\rangle = \left\langle h(v), \binom{1}{0}\right\rangle_K \neq 0$. By multiplying a nonzero constant to h, we may assume that

$$\left\langle f(v) - h(v), \binom{1}{0}\right\rangle_K = 0.$$

Let $f' = f - h \in N^\perp$. Then $\left\langle f'(v), \binom{1}{0}\right\rangle_K = 0$ and therefore $f' \cdot (V \setminus \{v\}) \in N^\perp \setminus\!\!\setminus \{v\} = (N \setminus\!\!\setminus \{v\})^\perp$. By using the induction hypothesis based on the fact that $\dim(N \setminus\!\!\setminus \{v\}) - \dim(N \setminus\!\!\setminus X) = |X| - 1$, we deduce that there exists a chain $g' \in (N \setminus\!\!\setminus \{v\})^\perp$ such that $f' \cdot (V \setminus \{v\}) - g' \in N \setminus\!\!\setminus \{v\}$ and $\left\langle g'(x), \binom{1}{0}\right\rangle_K = 0$ for all $x \in X \setminus \{v\}$. Let g be a chain in $N^\perp$ such that $g \cdot (V \setminus \{v\}) = g'$ and $\left\langle g(v), \binom{1}{0}\right\rangle_K = 0$.

We know that $\left\langle f'(v) - g(v), \binom{1}{0}\right\rangle_K = 0$. Since $(f' - g) \cdot (V \setminus \{v\}) \in N \setminus\!\!\setminus \{v\}$ and $v^* \notin N$, we deduce that $f' - g \in N$. Thus $f - g = f' - g + h \in N$. Moreover for all $x \in X$, $\left\langle g(x), \binom{1}{0}\right\rangle_K = 0$. ☐

Lemma 6.3. *Let N be an isotropic chain-group on V to K. Let X be a subset of V. Let f be a chain in $N^\perp$ such that $\left\langle f(x), \binom{1}{0}\right\rangle_K = 0$ if $x \in X$ and $f(x) = 0$ if $x \in V \setminus X$. If $\dim N - \dim(N \setminus\!\!\setminus X) = |X|$, then $f \in N$.*

Proof. We proceed by induction on $|X|$. We may assume that X is nonempty. Let $v \in X$. By Corollary 3.7, $\dim(N \setminus\!\!\setminus \{v\}) = \dim N - 1$ and $\dim(N \setminus\!\!\setminus \{v\}) - \dim(N \setminus\!\!\setminus X) = |X| - 1$. Proposition 3.6 implies that either $v^* \in N$ or $v^* \notin N^\perp$.

By Theorem 3.9, $f \cdot (V \setminus \{v\}) \in (N \mathbin{\backslash\!\!\backslash} \{v\})^\perp$. By the induction hypothesis, $f \cdot (V \setminus \{v\}) \in N \mathbin{\backslash\!\!\backslash} \{v\}$. There is a chain $f' \in N$ such that $f'(x) = f(x)$ for all $x \in V \setminus \{v\}$ and $\left\langle f'(v), \binom{1}{0} \right\rangle_K = 0$. Then $f - f' = cv^*$ for some $c \in \mathbb{F}$ by Lemma 3.2. Because N is isotropic, $f - f' \in N^\perp$.

If $v^* \in N$, then $f = f' + cv^* \in N$. If $v^* \notin N^\perp$, then $c = 0$ and therefore $f \in N$. □

Proposition 6.4. *Let N be an isotropic chain-group on V to K with a boundary B. Let X be a subset of V. If $|V \setminus X| - \dim(N \mathbin{\backslash\!\!\backslash} X) = |V| - \dim N$, then $B \mathbin{\backslash\!\!\backslash} X$ is well-defined and it is a boundary of $N \mathbin{\backslash\!\!\backslash} X$. Similarly if $|V \setminus X| - \dim(N \mathbin{/\!\!/} X) = |V| - \dim N$, then $B \mathbin{/\!\!/} X$ is well-defined and it is a boundary of $N \mathbin{/\!\!/} X$.*

Proof. By symmetry it is enough to show for $B \mathbin{\backslash\!\!\backslash} X$. Let $B = \{b_1 + N, b_2 + N, \ldots, b_m + N\}$.

By Lemma 6.2, there exists a chain $b_i' \in N^\perp$ such that $b_i + N = b_i' + N$ and $\left\langle b_i'(x), \binom{1}{0} \right\rangle_K = 0$ for all $x \in X$.

Suppose that there are chains c_i and d_i in $N^\perp$ such that $b_i + N = c_i + N = d_i + N$ and $\left\langle c_i(x), \binom{1}{0} \right\rangle_K = \left\langle d_i(x), \binom{1}{0} \right\rangle_K = 0$ for all $x \in X$. Since $c_i - d_i \in N$ and $\left\langle c_i(x) - d_i(x), \binom{1}{0} \right\rangle_K = 0$ for all $x \in X$, we deduce that $(c_i - d_i) \cdot (V \setminus X) \in N \mathbin{\backslash\!\!\backslash} X$ and therefore

$$c_i \cdot (V \setminus X) + N \mathbin{\backslash\!\!\backslash} X = d_i \cdot (V \setminus X) + N \mathbin{\backslash\!\!\backslash} X.$$

Hence $B \mathbin{\backslash\!\!\backslash} X$ is well-defined.

Now we claim that $B \mathbin{\backslash\!\!\backslash} X$ is a boundary of $N \mathbin{\backslash\!\!\backslash} X$. Since $\dim((N \mathbin{\backslash\!\!\backslash} X)^\perp / (N \mathbin{\backslash\!\!\backslash} X)) = 2|V \setminus X| - 2\dim(N \mathbin{\backslash\!\!\backslash} X) = 2|V| - 2\dim N = \dim N^\perp / N = |B| = |B \mathbin{\backslash\!\!\backslash} X|$, it is enough to show that $B \mathbin{\backslash\!\!\backslash} X$ is linearly independent in $(N \mathbin{\backslash\!\!\backslash} X)^\perp / N \mathbin{\backslash\!\!\backslash} X$. We may assume that $\left\langle b_i(x), \binom{1}{0} \right\rangle_K = 0$ for all $x \in X$. Let $f_i = b_i \cdot (V \setminus X) \in N^\perp \mathbin{\backslash\!\!\backslash} X$. We claim that $\{f_i + N \mathbin{\backslash\!\!\backslash} X : i = 1, 2, \ldots, m\}$ is linearly independent. Suppose that $\sum_{i=1}^m a_i (f_i + N \mathbin{\backslash\!\!\backslash} X) = 0$ for some constants $a_i \in \mathbb{F}$. This means $\sum_{i=1}^m a_i f_i \in N \mathbin{\backslash\!\!\backslash} X$. Let f be a chain in N such that $f \cdot (V \setminus X) = \sum_{i=1}^m a_i f_i$ and $\left\langle f(x), \binom{1}{0} \right\rangle_K = 0$ for all $x \in X$. Let $b = \sum_{i=1}^m a_i b_i$. Clearly $b \in N^\perp$.

We consider the chain $b - f$. Since N is isotropic, $f \in N^\perp$ and so $b - f \in N^\perp$. Moreover $(b - f) \cdot (V \setminus X) = 0$ and $\left\langle b(x) - f(x), \binom{1}{0} \right\rangle_K = 0$ for all $x \in X$. By Lemma 6.3, we deduce that $b - f \in N$ and therefore $b = (b - f) + f \in N$. Since B is a basis of $N^\perp / N$, $a_i = 0$ for all i. We conclude that $B \mathbin{\backslash\!\!\backslash} X$ is linearly independent. □

A boundaried chain-group (V', N', B') is a *minor* of another boundaried chain-group (V, N, B) if

$$|V'| - \dim N' = |V| - \dim N$$

and there exist disjoint subsets X and Y of V such that $V' = V \setminus (X \cup Y)$, $N' = N \setminus\!\setminus X /\!/ Y$, and $B' = B \setminus\!\setminus X /\!/ Y$.

Proposition 6.5. *A minor of a minor of a boundaried chain-group is a minor of the boundaried chain-group.*

Proof. Let (V_0, N_0, B_0), (V_1, N_1, B_1), (V_2, N_2, B_2) be boundaried chain-groups. Suppose that for $i \in \{0, 1\}$, $(V_{i+1}, N_{i+1}, B_{i+1})$ is a minor of (V_i, N_i, B_i) as follows:

$$N_{i+1} = N_i \setminus\!\setminus X_i /\!/ Y_i, \quad B_{i+1} = B_i \setminus\!\setminus X_i /\!/ Y_i.$$

It is easy to deduce that $|V_0| - \dim N_0 = |V_2| - \dim N_2$ and $N_2 = N_0 \setminus\!\setminus (X_0 \cup X_1) /\!/ (Y_0 \cup Y_1)$.

We claim that $B_2 = B_0 \setminus\!\setminus (X_0 \cup X_1) /\!/ (Y_0 \cup Y_1)$. By Corollary 3.7, we deduce that $|V_0 \setminus (X_0 \cup X_1)| - \dim N_0 \setminus\!\setminus (X_0 \cup X_1) = |V_0| - \dim N_0 = |V_2| - \dim N_2$ and so it is possible to delete $X_0 \cup X_1$ from V_0 and then contract $Y_0 \cup Y_1$. From the definition, it is easy to show that $B \setminus\!\setminus (X_0 \cup X_1) /\!/ (Y_0 \cup Y_1) = B_2$. □

6.2. **Sums of boundaried chain-groups.** Two boundaried chain-groups over the same field are *disjoint* if their ground sets are disjoint. In this subsection, we define *sums* of disjoint boundaried chain-groups and their *connection types*.

A boundaried chain-group (V, N, B) over a field $\mathbb{F}$ is a *sum* of disjoint boundaried chain-groups (V_1, N_1, B_1) and (V_2, N_2, B_2) over $\mathbb{F}$ if

$$N_1 = N \times V_1, \; N_2 = N \times V_2, \text{ and } V = V_1 \cup V_2.$$

For a chain f on V_1 to K and a chain g on V_2 to K, we denote $f \oplus g$ for a chain on $V_1 \cup V_2$ to K such that $(f \oplus g) \cdot V_1 = f$ and $(f \oplus g) \cdot V_2 = g$. The *connection type* of the sum is a sequence $(C_0, C_1, \ldots, C_{|B|})$ of sets of sequences in $\mathbb{F}^{|B_1|} \times \mathbb{F}^{|B_2|}$ such that, for $B = \{b_1 + N, b_2 + N, \ldots, b_{|B|} + N\}$, $B_1 = \{b^1_1 + N_1, b^1_2 + N_1, \ldots, b^1_{|B_1|} + N_1\}$, and $B_2 = \{b^2_1 + N_2, b^2_2 + N_2, \ldots, b^2_{|B_2|} + N_2\}$,

$$C_0 = \left\{ (x, y) \in \mathbb{F}^{|B_1|} \times \mathbb{F}^{|B_2|} : \left(\sum_{i=1}^{|B_1|} x_i b^1_i \right) \oplus \left(\sum_{j=1}^{|B_2|} y_j b^2_j \right) \in N \right\},$$

and for $s \in \{1, 2, \dots, |B|\}$,

$$C_s = \left\{ (x, y) \in \mathbb{F}^{|B_1|} \times \mathbb{F}^{|B_2|} : \left(\sum_{i=1}^{|B_1|} x_i b_i^1 \right) \oplus \left(\sum_{j=1}^{|B_2|} y_j b_j^2 \right) - b_s \in N \right\}.$$

Proposition 6.6. *The connection type is well-defined.*

Proof. It is enough to show that the choices of b_i, b_i^1, and b_i^2 do not affect C_s for $s \in \{0, 1, 2, \dots, |B|\}$. Suppose that $b_i + N = d_i + N$, $b_i^1 + N_1 = d_i^1 + N_1$, and $b_i^2 + N_2 = d_i^2 + N_2$. Then for every $(x, y) \in \mathbb{F}^{|B_1|} \times \mathbb{F}^{|B_2|}$,

$$\sum_{i=1}^{|B_1|} x_i (b_i^1 - d_i^1) \oplus \sum_{j=1}^{|B_2|} y_j (b_j^2 - d_j^2) \in N$$

because $(b_i^1 - d_i^1) \oplus 0 \in N$ and $0 \oplus (b_j^2 - d_j^2) \in N$. Moreover if $s \neq 0$, then $b_s - d_s \in N$. Hence C_s is well-defined. $\square$

Proposition 6.7. *The connection type uniquely determines the sum of two disjoint boundaried chain-groups.*

Proof. Suppose that both (V, N, B) and (V, N', B') are sums of disjoint boundaried chain-groups (V_1, N_1, B_1), (V_2, N_2, B_2) over a field $\mathbb{F}$ with the same connection type $(C_0, C_1, \dots, C_{|B|})$.

We first claim that $N = N'$. By symmetry, it is enough to show that $N \subseteq N'$. Let $a \in N$. Since $a \in N^\perp$ and $(N \times V_1)^\perp = N^\perp \cdot V_1$ by Theorem 3.4, we deduce that $a \cdot V_1 \in (N \times V_1)^\perp$ and similarly $a \cdot V_2 \in (N \times V_2)^\perp$. Therefore there exists $(x, y) \in \mathbb{F}^{|B_1|} \times \mathbb{F}^{|B_2|}$ such that

$$f = \sum_{i=1}^{|B_1|} x_i b_i^1 - a \cdot V_1 \in N_1 \quad \text{and} \quad g = \sum_{j=1}^{|B_2|} y_j b_j^2 - a \cdot V_2 \in N_2.$$

Since $f \oplus 0 \in N$ and $0 \oplus g \in N$, we have $f \oplus g \in N$. We deduce that $\sum_{i=1}^{|B_1|} x_i b_i^1 \oplus \sum_{j=1}^{|B_2|} y_j b_j^2 = a + (f \oplus g) \in N$. Therefore $(x, y) \in C_0$. So, $a + (f \oplus g) \in N'$ as well. Since $f \oplus 0, 0 \oplus g \in N'$, we have $a \in N'$. We conclude that $N \subseteq N'$.

Now we show that $B = B'$. Let $b_s + N$ be the s-th element of B where $b_s \in N^\perp$. Let $b_s' + N$ be the s-th element of B' with $b_s' \in N^\perp$. Since $b_s \cdot V_1 \in (N \times V_1)^\perp$ and $b_s \cdot V_2 \in (N \times V_2)^\perp$, there is $(x, y) \in \mathbb{F}^{|B_1|} \times \mathbb{F}^{|B_2|}$ such that

$$f = \sum_{i=1}^{|B_1|} x_i b_i^1 - b_s \cdot V_1 \in N_1 \quad \text{and} \quad g = \sum_{j=1}^{|B_2|} y_j b_j^2 - b_s \cdot V_2 \in N_2.$$

Since $f \oplus 0, 0 \oplus g \in N$, we have $f \oplus g \in N$. Therefore $\sum_{i=1}^{|B_1|} x_i b_i^1 \oplus \sum_{j=1}^{|B_2|} y_j b_j^2 - b_s \in N$. This implies that $(x, y) \in C_s$ and therefore $\sum_{i=1}^{|B_1|} x_i b_i^1 \oplus \sum_{j=1}^{|B_2|} y_j b_j^2 - b'_s \in N' = N$. Thus, $b_s + N = b'_s + N$. $\square$

In the next proposition, we prove that minors of a sum of disjoint boundaried chain-groups are sums of minors of the boundaried chain-groups with the same connection type.

Proposition 6.8. *Suppose that a boundaried chain-group (V, N, B) is a sum of disjoint boundaried chain-groups (V_1, N_1, B_1), (V_2, N_2, B_2) over a field $\mathbb{F}$. Let $(C_0, C_1, \ldots, C_{|B|})$ be the connection type of the sum. If*

$$|V_1 \setminus (X \cup Y)| - \dim(N_1 \setminus\!\!\setminus X /\!\!/ Y) = |V_1| - \dim N_1$$

and

$$|V_2 \setminus (Z \cup W)| - \dim(N_2 \setminus\!\!\setminus Z /\!\!/ W) = |V_2| - \dim N_2,$$

then $(V \setminus (X \cup Y \cup Z \cup W), N \setminus\!\!\setminus (X \cup Z) /\!\!/ (Y \cup W), B \setminus\!\!\setminus (X \cup Z) /\!\!/ (Y \cup W))$ is a well-defined minor of (V, N, B). Moreover it is a sum of $(V_1 \setminus (X \cup Y), N_1 \setminus\!\!\setminus X /\!\!/ Y, B_1 \setminus\!\!\setminus X /\!\!/ Y)$ and $(V_2 \setminus (Z \cup W), N_2 \setminus\!\!\setminus Z /\!\!/ W, B_2 \setminus\!\!\setminus Z /\!\!/ W)$ with the connection type $(C_0, C_1, \ldots, C_{|B|})$.

Proof. We proceed by induction on $|X \cup Y \cup Z \cup W|$. If $X \cup Y \cup Z \cup W = \emptyset$, then it is trivial.

Suppose that $|X \cup Y \cup Z \cup W| = 1$. By symmetry, we may assume that $Y = Z = W = \emptyset$. Let $v \in X$. Since $|V_1 \setminus \{v\}| - \dim(N_1 \setminus\!\!\setminus \{v\}) = |V_1| - \dim N_1$, either $v^* \in N_1$ or $v^* \notin N_1^\perp$ by Proposition 3.6. Since $N_1 = N \times V_1$, we deduce that either $v^* \in N$ or $v^* \notin N^\perp$. Thus, $|V \setminus \{v\}| - \dim(N \setminus\!\!\setminus \{v\}) = |V| - \dim N$ and so $(V \setminus \{v\}, N \setminus\!\!\setminus \{v\}, B \setminus\!\!\setminus \{v\})$ is a minor of (V, N, B).

To show that $(V \setminus \{v\}, N \setminus\!\!\setminus \{v\}, B \setminus\!\!\setminus \{v\})$ is a sum of $(V_1 \setminus \{v\}, N_1 \setminus\!\!\setminus \{v\}, B \setminus\!\!\setminus \{v\})$ and (V_2, N_2, B_2), it is enough to show that

$$N \times V_1 \setminus\!\!\setminus \{v\} = N \setminus\!\!\setminus \{v\} \times (V_1 \setminus \{v\}), \tag{2}$$

$$N \times V_2 = N \setminus\!\!\setminus \{v\} \times V_2. \tag{3}$$

It is easy to see (2) and $N \times V_2 \subseteq N \setminus\!\!\setminus \{v\} \times V_2$. We claim that $N \setminus\!\!\setminus \{v\} \times V_2 \subseteq N \times V_2$. Suppose that f is a chain in $N \setminus\!\!\setminus \{v\} \times V_2$. There exists a chain f' in N such that $f' \cdot V_2 = f$, $\left\langle f'(v), \binom{1}{0} \right\rangle_K = 0$, and $f'(x) = 0$ for all $x \in V \setminus (V_2 \cup \{v\}) = V_1 \setminus \{v\}$.

If $f'(v) \neq 0$, then $f' \cdot V_1 = cv^*$ for a nonzero $c \in \mathbb{F}$ by Lemma 3.2. Since $N_1^\perp = N^\perp \cdot V_1$ (Theorem 3.4), we deduce $v^* = c^{-1} f' \cdot V_1 \in N_1^\perp$. Therefore $v^* \in N_1$ and so $v^* \in N$. We may assume that $f'(v) = 0$ by adding a multiple of v^* to f'. This implies that $f \in N \times V_2$. We conclude (3).

Let $(C'_0, C'_1, \ldots, C'_{|B|})$ be the connection type of the sum of $(V_1 \setminus \{v\}, N_1 \setminus\!\!\setminus \{v\}, B_1 \setminus\!\!\setminus \{v\})$ and (V_2, N_2, B_2). Let $B = \{b_1 + N, b_2 + N, \ldots, b_{|B|} + N\}$, $B_1 = \{b^1_1 + N_1, b^1_2 + N_1, \ldots, b^1_{|B_1|} + N_1\}$, and $B_2 = \{b^2_1 + N_2, b^2_2 + N_2, \ldots, b^2_{|B_2|} + N_2\}$. We may assume that $\left\langle b_i(v), \binom{1}{0} \right\rangle_K = 0$ and $\left\langle b^1_i(v), \binom{1}{0} \right\rangle_K = 0$ by Lemma 6.2.

We claim that $C_s = C'_s$ for all $s \in \{0, 1, \ldots, |B|\}$. Let g be a chain in $N^\perp$ such that $g = 0$ if $s = 0$ or $g = b_s$ otherwise. If $(x, y) \in C_s$, then

$$\left(\sum_{i=1}^{|B_1|} x_i b^1_i \oplus \sum_{j=1}^{|B_2|} y_j b^2_j \right) - g \in N. \tag{4}$$

Since $\left\langle b^1_i(v), \binom{1}{0} \right\rangle_K = 0$ and $\left\langle g(v), \binom{1}{0} \right\rangle_K = 0$, we conclude that

$$\left(\sum_{i=1}^{|B_1|} x_i b^1_i \cdot (V_1 \setminus \{v\}) \oplus \sum_{j=1}^{|B_2|} y_j b^2_j \right) - g \cdot (V \setminus \{v\}) \in N \setminus\!\!\setminus \{v\}, \tag{5}$$

and therefore $(x, y) \in C'_s$.

Conversely suppose that $(x, y) \in C'_s$. Then (5) is true. By Lemma 6.3, we deduce (4). Therefore $(x, y) \in C_s$.

To complete the inductive proof, we now assume that $|X \cup Y \cup Z \cup W| > 1$. If X is nonempty, let $v \in X$. Let $X' = X \setminus \{v\}$. Then, by Corollary 3.7 we have $|V_1 \setminus \{v\}| - \dim N_1 \setminus\!\!\setminus \{v\} = |V_1| - \dim N_1$. So $(V_1 \setminus \{v\}, N \setminus\!\!\setminus \{v\}, B \setminus\!\!\setminus \{v\})$ is the sum of $(V_1 \setminus \{v\}, N_1 \setminus\!\!\setminus \{v\}, B_1 \setminus\!\!\setminus \{v\})$ and (V_2, N_2, B_2) with the connection type $(C_0, C_1, \ldots, C_{|B|})$. We deduce our claim by applying the induction hypothesis to $(V_1 \setminus \{v\}, N_1 \setminus\!\!\setminus \{v\}, B_1 \setminus\!\!\setminus \{v\})$ and (V_2, N_2, B_2). Similarly if one of Y or Z or W is nonempty, we deduce our claim. $\square$

6.3. **Linked branch-decompositions.** Suppose $(T, \mathcal{L})$ is a branch-decomposition of a Lagrangian chain-group N on V to $K = \mathbb{F}^2$. For two edges f and g of T, let F be the set of elements in V corresponding to the leaves in the component of $T \setminus f$ not containing g and let G be the set of elements in V corresponding to the leaves in the component of $T \setminus g$ not containing f. Let P be the unique path from e to f in T. We say that f and g are *linked* if the minimum width of the edges on P is equal to $\min_{F \subseteq X \subseteq V \setminus G} \lambda_N(X)$. We say that a branch-decomposition $(T, \mathcal{L})$ is *linked* if every pair of edges in T is linked.

The following lemma is shown by Geelen, Gerards, and Whittle [8, 9]. We state it in terms of Lagrangian chain-groups, because the connectivity function of chain-groups are symmetric submodular (Theorem 3.12).

Lemma 6.9 (Geelen et al. [8, 9, Theorem (2.1)]). *A chain-group of branch-width n has a linked branch-decomposition of width n.*

Having a linked branch-decomposition will be very useful for proving well-quasi-ordering because it allows Tutte's linking theorem to be used. It was the first step to prove well-quasi-ordering of matroids of bounded branch-width by Geelen et al. [8]. An analogous theorem by Thomas [17] was used to prove well-quasi-ordering of graphs of bounded tree-width in [14].

6.4. **Lemma on cubic trees.** We use "lemma on trees," proved by Robertson and Seymour [14]. It has been used by Robertson and Seymour to prove that a set of graphs of bounded tree-width is well-quasi-ordered by the graph minor relation. It has been also used by Geelen et al. [8] to prove that a set of matroids representable over a fixed finite field and having bounded branch-width is well-quasi-ordered by the matroid minor relation. We need a special case of "lemma on trees," in which a given forest is cubic, which was also useful for branch-decompositions of matroids in [8].

The following definitions are in [8]. A *rooted tree* is a finite directed tree where all but one of the vertices have indegree 1. A *rooted forest* is a collection of countably many vertex disjoint rooted trees. Its vertices with indegree 0 are called *roots* and those with outdegree 0 are called *leaves*. Edges leaving a root are *root edges* and those entering a leaf are *leaf edges*.

An *n-edge labeling* of a graph F is a map from the set of edges of F to the set $\{0, 1, \ldots, n\}$. Let λ be an n-edge labeling of a rooted forest F and let e and f be edges in F. We say that *e is λ-linked to f* if F contains a directed path P starting with e and ending with f such that $\lambda(g) \geq \lambda(e) = \lambda(f)$ for every edge g on P.

A *binary forest* is a rooted orientation of a cubic forest with a distinction between left and right outgoing edges. More precisely, we call a triple (F, l, r) a *binary forest* if F is a rooted forest where roots have outdegree 1 and l and r are functions defined on non-leaf edges of F, such that the head of each non-leaf edge e of F has exactly two outgoing edges, namely $l(e)$ and $r(e)$.

Lemma 6.10 (Geelen et al. [8, (3.2)]). *Let (F, l, r) be an infinite binary forest with an n-edge labeling λ. Moreover, let $\leq$ be a quasi-order on the set of edges of F with no infinite strictly descending sequences, such that $e \leq f$ whenever f is λ-linked to e. If the set of leaf edges of F is well-quasi-ordered by $\leq$ but the set of root edges of F is not, then F contains an infinite sequence $(e_0, e_1, \ldots)$ of non-leaf edges such that*

(i) $\{e_0, e_1, \ldots\}$ *is an antichain with respect to* $\leq$,
(ii) $l(e_0) \leq l(e_1) \leq l(e_2) \leq \cdots$,
(iii) $r(e_0) \leq r(e_1) \leq r(e_2) \leq \cdots$.

6.5. **Main theorem.** We are now ready to prove our main theorem. To make it more useful, we label each element of the ground set by a well-quasi-ordered set Q with an ordering $\preceq$ and enforce the minor relation to follow the ordering $\preceq$. More precisely, for a chain-group N on V to K, a *Q-labeling* is a mapping from V to Q. A *Q-labeled chain-group* is a chain-group equipped with a Q-labeling. A Q-labeled chain-group N' on V' to K with a Q-labeling μ' is a *Q-minor* of a Q-labeled chain-group N with a Q-labeling μ if N' is a minor of N and $\mu'(v) \preceq \mu(v)$ for all $v \in V'$.

Theorem 6.1 (Labeled version)**.** *Let Q be a well-quasi-ordered set with an ordering $\preceq$. Let k be a constant. Let $\mathbb{F}$ be a finite field. Let $N_1, N_2, \ldots$ be an infinite sequence of Q-labeled Lagrangian chain-groups over $\mathbb{F}$ having branch-width at most k. Then there exist $i < j$ such that N_i is simply isomorphic to a Q-minor of N_j.*

Proof. We may assume that all bilinear forms $\langle\,,\rangle_K$ for all N_i's are the same bilinear form, that is either skew-symmetric or symmetric by taking a subsequence. Let V_i be the ground set of N_i. Let $\mu_i : V_i \to Q$ be the Q-labeling of N_i. We may assume that $|V_i| > 1$ for all i. By Lemma 6.9, there is a linked branch-decomposition $(T_i, \mathcal{L}_i)$ of N_i of width at most k for each i. Let T be a forest such that the i-th component is T_i. To make T a binary forest, for each T_i, we create a vertex r_i of degree 1, called a *root*, create a vertex of degree 3 by subdividing an edge of T_i and making it adjacent to r_i, and direct every edge of T_i so that each leaf has a directed path from the root r_i.

We now define a k-edge labeling λ of T, necessary for Lemma 6.10. For each edge e of T_i, let X_e be the set of leaves of T_i having a directed path from e. Let $A_e = \mathcal{L}_i^{-1}(X_e)$. We let $\lambda(e) = \lambda_{N_i}(A_e)$.

We want to associate each edge e of T_i with a Q-labeled boundaried chain-group $P_e = (A_e, N_i \times A_e, B_e)$ with a Q-labeling $\mu_e = \mu_i|_{A_e}$ and some boundary B_e satisfying the following property:

$$\text{if } f \text{ is } \lambda\text{-linked to } e\text{, then } P_e \text{ is a } Q\text{-minor of } P_f. \tag{6}$$

We note that $\mu_i|_{A_e}$ is a function on A_e such that $\mu_i|_{A_e}(x) = \mu_i(x)$ for all $x \in A_e$.

We claim that we can assign B_e to satisfy (6). We prove it by induction on the length of the directed path from the root edge of T_i to an edge e of T_i. If no other edge is λ-linked to e, then let B_e be an arbitrary boundary of $N_i \times A_e$. If f, other than e, is λ-linked to e, then choose f such that the distance between e and f is minimal. We claim that we can obtain B_e from B_f by Corollary 5.4 (Tutte's linking theorem) as follows; since T_i is a linked branch-decomposition, for all Z, if $A_e \subseteq Z \subseteq A_f$, then $\lambda_{N_i}(Z) \geq \lambda_{N_i}(A_e)$. By Corollary 5.4, there exist disjoint subsets C and D of $A_f \setminus A_e$ such that $N \times A_e = N \times A_f \mathbin{/\!\!/} C \mathbin{\backslash\!\!\backslash} D$. Since $|A_e| - \dim N_i \times A_e = |A_f| - \dim N_i \times A_f$, $B_e = B_f \mathbin{/\!\!/} C \mathbin{\backslash\!\!\backslash} D$ is well defined. This proves the claim.

For $e, f \in E(T)$, we write $e \leq f$ when a Q-labeled boundaried chain-group P_e is simply isomorphic to a Q-minor of P_f. Clearly $\leq$ has no infinitely strictly descending sequences, because there are finitely many boundaried chain-groups on bounded number of elements up to simple isomorphisms and furthermore Q is well-quasi-ordered. By construction, if f is λ-linked to e, then $e \leq f$.

The leaf edges of T are well-quasi-ordered because there are only finite many distinct boundaried chain-groups on one element up to simple isomorphisms and $\mathcal{Q}$ is well-quasi-ordered.

Suppose that the root edges are not well-quasi-ordered by the relation $\leq$. By Lemma 6.10, T contains an infinite sequence $e_0, e_1, \ldots$ of non-leaf edges such that

(i) $\{e_0, e_1, \ldots\}$ is an antichain with respect to $\leq$,
(ii) $l(e_0) \leq l(e_1) \leq \cdots$,
(iii) $r(e_0) \leq r(e_1) \leq \cdots$.

Since $\lambda(e_i) \leq k$ for all i, we may assume that $\lambda(e_i)$ is a constant for all i, by taking a subsequence.

The boundaried chain-group P_{e_i} is the sum of $P_{l(e_i)}$ and $P_{r(e_i)}$. The number of possible distinct connection types for this sum is finite, because $\mathbb{F}$ is finite and k is fixed, Therefore, we may assume that the connection types for all sums for all e_i are same for all i, by taking a subsequence.

Since $l(e_0) \leq l(e_1)$, there exists a simple isomorphism s_l from $A_{l(e_0)}$ to a subset of $A_{l(e_1)}$. Similarly, there exists a simple isomorphism s_r from $A_{r(e_0)}$ to a subset of $A_{r(e_1)}$ in $r(e_0) \leq r(e_1)$. Let s be a function on $A_{e_0} = A_{l(e_0)} \cup A_{r(e_0)}$ such that $s(v) = s_l(v)$ if $v \in A_{l(e_0)}$ and $s(v) = s_r(v)$ otherwise. By Proposition 6.8, P_{e_0} is simply isomorphic to a Q-minor of P_{e_1} with the simple isomorphism s. Since $l(e_0) \leq l(e_1)$ and $r(e_0) \leq r(e_1)$, we deduce that P_{e_0} is simply isomorphic to a Q-minor of P_{e_1} and therefore $e_0 \leq e_1$. This contradicts to (i). Hence we conclude that the root edges are well-quasi-ordered by $\leq$. So there exist $i < j$ such that N_i is simply isomorphic to a Q-minor of N_j. $\square$

7. Well-quasi-ordering of skew-symmetric or symmetric matrices

In this section, we will prove the following main theorem for skew-symmetric or symmetric matrices from Theorem 6.1.

Theorem 7.1. *Let $\mathbb{F}$ be a finite field and let k be a constant. Every infinite sequence M_1, M_2, ... of skew-symmetric or symmetric matrices over $\mathbb{F}$ of rank-width at most k has a pair $i < j$ such that M_i is isomorphic to a principal submatrix of (M_j/A) for some nonsingular principal submatrix A of M_j.*

To move from the principal pivot operation given by Theorem 4.9 to a Schur complement, we need a finer control how we obtain a matrix representation under taking a minor of a Lagrangian chain-group.

Lemma 7.2. *Let M_1, M_2 be skew-symmetric or symmetric matrices over a field $\mathbb{F}$. For $i = 1, 2$, let N_i be a Lagrangian chain-group with a special matrix representation (M_i, a_i, b_i) where $a_i(v) = \binom{1}{0}$, $b_i(v) = \binom{0}{1}$ for all v. If $N_1 = N_2 \mathbin{/\!\!/} X \mathbin{\backslash\!\!\backslash} Y$, then M_1 is a principal submatrix of the Schur complement (M_2/A) of some nonsingular principal submatrix A in M_2.*

Proof. For $i = 1, 2$, let V_i be the ground set of N_i. We may assume that X is a minimal set having some Y such that $N_1 = N_2 \mathbin{/\!\!/} X \mathbin{\backslash\!\!\backslash} Y$. We may assume $X \neq \emptyset$, because otherwise we apply Lemma 4.8. Note that the Schur complement of a $\emptyset \times \emptyset$ submatrix in M_2 is M_2 itself.

Suppose that $M_2[X]$ is singular. Let a_X be a chain on V_2 to $K = \mathbb{F}^2$ such that $a_X(v) = \binom{1}{0}$ if $v \notin X$ and $a_X(v) = \binom{0}{1}$ if $v \in X$. By Proposition 4.4, a' is not an eulerian chain of N_2. Therefore there exists a nonzero chain $f \in N_2$ such that $\langle f(v), a_X(v)\rangle_K = 0$ for all $v \in V_2$. Then $f \cdot V_1 = 0$ because $f \cdot V_1 \in N_1$ and a_1 is an eulerian chain of $N_1 = N_2 \mathbin{/\!\!/} X \mathbin{\backslash\!\!\backslash} Y$. There exists $w \in X$ such that $f(w) \neq 0$ because a_2 is an eulerian chain of N_2. For every chain $g \in N_2$, if $\left\langle g(v), \binom{1}{0}\right\rangle_K = 0$ for $v \in Y$ and $\left\langle g(v), \binom{0}{1}\right\rangle_K = 0$ for $v \in X$, then $g(w) = c_g f(w)$ for some $c_g \in \mathbb{F}$ by Lemma 3.2 and therefore $g \cdot V_1 = (g - c_g f) \cdot V_1 \in N_2 \mathbin{/\!\!/} (X \setminus \{w\}) \mathbin{\backslash\!\!\backslash} (Y \cup \{w\})$. This implies that $N_2 \mathbin{/\!\!/} X \mathbin{\backslash\!\!\backslash} Y \subseteq N_2 \mathbin{/\!\!/} (X \setminus \{w\}) \mathbin{\backslash\!\!\backslash} (Y \cup \{w\})$. Since $\dim(N_2 \mathbin{/\!\!/} X \mathbin{\backslash\!\!\backslash} Y) = \dim(N_2 \mathbin{/\!\!/} (X \setminus \{w\}) \mathbin{\backslash\!\!\backslash} (Y \cup \{w\})) = |V_1|$, we have $N_2 \mathbin{/\!\!/} X \mathbin{\backslash\!\!\backslash} Y = N_2 \mathbin{/\!\!/} (X \setminus \{w\}) \mathbin{\backslash\!\!\backslash} (Y \cup \{w\})$, contradictory to the assumption that X is minimal. This proves that $M_2[X]$ is nonsingular.

By Proposition 4.5, (M', a', b') is another special matrix representation of N_1 where $M' = M * X$ if $\langle\,,\,\rangle_K$ is symmetric or $M' = I_X(M * X)$ if $\langle\,,\,\rangle_K$ is skew-symmetric and a', b' are given in Proposition 4.5. We observe that $a' \cdot V_1 = a_1$ and $b' \cdot V_1 = b_1$. We apply Lemma 4.8 to deduce that $(M'[V_1], a_1, b_1)$ is a matrix representation of N_1. This implies that $M'[V_1] = M_1$. Let $A = M_2[X]$. Notice that $M'[V_1] = (M_2/A)[V_1]$. This proves the lemma. □

Proof of Theorem 7.1. By taking an infinite subsequence, we may assume that all of the matrices in the sequence are skew-symmetric or symmetric. Let $K = \mathbb{F}^2$ and assume $\langle\, ,\, \rangle_K$ is a bilinear form that is symmetric if the matrices are skew-symmetric and skew-symmetric if the matrices are symmetric. Let N_i be the Lagrangian chain-group represented by a matrix representation (M_i, a_i, b_i) where $a_i(x) = \binom{1}{0}$, $b_i(x) = \binom{0}{1}$ for all x. Then by Theorem 6.1, there are $i < j$ such that N_i is simply isomorphic to a minor of N_j. By Lemma 7.2, we deduce the conclusion. □

Now let us consider the notion of delta-matroids, a generalization of matroids. Delta-matroids lack the notion of the connectivity and hence it is not clear how to define the branch-width naturally for delta-matroids. We define the branch-width of a $\mathbb{F}$-representable delta-matroid as the minimum rank-width of all skew-symmetric or symmetric matrices over $\mathbb{F}$ representing the delta-matroid. Then we can deduce the following theorem from Theorem 4.12 and Proposition 4.10.

Theorem 7.3. *Let $\mathbb{F}$ be a finite field and k be a constant. Every infinite sequence $\mathcal{M}_1$, $\mathcal{M}_2$, … of $\mathbb{F}$-representable delta-matroids of branch-width at most k has a pair $i < j$ such that $\mathcal{M}_i$ is isomorphic to a minor of $\mathcal{M}_j$.*

Proof. Let M_1, M_2, … be an infinite sequence of skew-symmetric or symmetric matrices over $\mathbb{F}$ such that the rank-width of M_i is equal to the branch-width of $\mathcal{M}_i$ and $\mathcal{M}_i = \mathcal{M}(M_i)\Delta X_i$. We may assume that $X_i = \emptyset$ for all i. By Theorem 7.1, there are $i < j$ such that M_i is isomorphic to a principal submatrix of the Schur complement of a nonsingular principal submatrix in M_j. This implies that $\mathcal{M}_i$ is a minor of $\mathcal{M}_j$ as a delta-matroid. □

In particular, when $\mathbb{F} = GF(2)$, then binary skew-symmetric matrices correspond to adjacency matrices of simple graphs. Then taking a pivot on such matrices is equivalent to taking a sequence of graph pivots on the corresponding graphs. We say that a simple graph H is a *pivot-minor* of a simple graph G if H is obtained from G by applying pivots and deleting vertices. As a matter of a fact, a pivot-minor of a simple graph corresponds to a minor of an even binary delta-matroid. The *rank-width* of a simple graph is defined to be the rank-width of its adjacency matrix over $\mathbb{F}$. Then Theorem 7.1 or 7.3 implies the following corollary, originally proved by Oum [11].

Corollary 7.4 (Oum [11]). *Let k be a constant. Every infinite sequence G_1, G_2, … of simple graphs of rank-width at most k has a pair $i < j$ such that G_i is isomorphic to a pivot-minor of G_j.*

8. Corollaries to matroids and graphs

In this section, we will show how Theorem 6.1 implies the theorem by Geelen et al. [8] on well-quasi-ordering of $\mathbb{F}$-representable matroids of bounded branch-width for a finite field $\mathbb{F}$ as well as the theorem by Robertson and Seymour [14] on well-quasi-ordering of graphs of bounded tree-width.

We will briefly review the notion of matroids in the first subsection. In the second subsection, we will discuss how Tutte chain-groups are related to representable matroids and Lagrangian chain-groups. In the last subsection, we deduce the theorem of Geelen et al. [8] on matroids which in turn implies the theorem of Robertson and Seymour [14] on graphs.

8.1. **Matroids.** Let us review matroid theory briefly. For more on matroid theory, we refer readers to the book by Oxley [13].

A matroid $M = (E, r)$ is a pair formed by a finite set E of *elements* and a *rank* function $r : 2^E \to \mathbb{Z}$ satisfying the following axioms:

i) $0 \le r(X) \le |X|$ for all $X \subseteq E$.
ii) If $X \subseteq Y \subseteq E$, then $r(X) \le r(Y)$.
iii) For all $X, Y \subseteq E$, $r(X) + r(Y) \ge r(X \cap Y) + r(X \cup Y)$.

A subset X of E is called *independent* if $r(X) = |X|$. A *base* is a maximally independent set. We write $E(M) = E$. For simplicity, we write $r(M)$ for $r(E(M))$. For $Y \subseteq E(M)$, $M \setminus Y$ is the matroid $(E(M) \setminus Y, r')$ where $r'(X) = r(X)$. For $Y \subseteq E(M)$, M/Y is the matroid $(E(M) \setminus Y, r')$ where $r'(X) = r(X \cup Y) - r(Y)$. If $Y = \{e\}$, we denote $M \setminus e = M \setminus \{e\}$ and $M/e = M/\{e\}$. It is routine to prove that $M \setminus Y$ and M/Y are matroids. Matroids of the form $M \setminus X/Y$ are called a *minor* of the matroid M.

Given a field $\mathbb{F}$ and a set of vectors in $\mathbb{F}^m$, we can construct a matroid by letting $r(X)$ be the dimension of the vector space spanned by vectors in X. If a matroid permits this construction, then we say that the matroid is $\mathbb{F}$*-representable* or *representable* over $\mathbb{F}$.

The *connectivity function* of a matroid $M = (E, r)$ is $\lambda_M(X) = r(X) + r(E \setminus X) - r(E) + 1$. A *branch-decomposition* of a matroid $M = (E, r)$ is a pair $(T, \mathcal{L})$ of a subcubic tree T and a bijection $\mathcal{L}$:

$E \to \{t : t \text{ is a leaf of } T\}$. For each edge $e = uv$ of the tree T, the connected components of $T \setminus e$ induce a partition (X_e, Y_e) of the leaves of T and we call $\lambda_M(\mathcal{L}^{-1}(X_e))$ the *width* of e. The *width* of a branch-decomposition $(T, \mathcal{L})$ is the maximum width of all edges of T. The *branch-width* $\mathrm{bw}(M)$ of a matroid $M = (E, r)$ is the minimum width of all its branch-decompositions. (If $|E| \leq 1$, then we define that $\mathrm{bw}(M) = 1$.)

8.2. **Tutte chain-groups.** We review Tutte chain-groups [24]. For a finite set V and a field $\mathbb{F}$, a *chain* on V to $\mathbb{F}$ is a mapping $f : V \to \mathbb{F}$. The *sum* $f + g$ of two chains f, g is the chain on V satisfying

$$(f + g)(x) = f(x) + g(x) \quad \text{for all } x \in V.$$

If f is a chain on V to $\mathbb{F}$ and $\lambda \in \mathbb{F}$, the *product* λf is a chain on V such that

$$(\lambda f)(x) = \lambda f(x) \quad \text{for all } x \in V.$$

It is easy to see that the set of all chains on V to $\mathbb{F}$, denoted by $\mathbb{F}^V$, is a vector space. A *Tutte chain-group* on V to $\mathbb{F}$ is a subspace of $\mathbb{F}^V$. The *support* of a chain f on V to $\mathbb{F}$ is $\{x \in V : f(x) \neq 0\}$.

Theorem 8.1 (Tutte [22])**.** *Let N be a Tutte chain-group on a finite set V to a field $\mathbb{F}$. The minimal nonempty supports of N form the circuits of a $\mathbb{F}$-representable matroid $M\{N\}$ on V, whose rank is equal to $|V| - \dim N$. Moreover every $\mathbb{F}$-representable matroid M admits a Tutte chain-group N such that $M = M\{N\}$.*

Let S be a subset of V. For a chain f on V to $\mathbb{F}$, we denote $f \cdot S$ for a chain on S to $\mathbb{F}$ such that $(f \cdot S)(v) = f(v)$ for all $v \in S$. For a Tutte chain-group N on V to $\mathbb{F}$, we let $N \cdot S = \{f \cdot S : f \in N\}$, $N \times S = \{f \cdot S : f \in N, f(v) = 0 \text{ for all } v \notin S\}$, and $N^\perp = \{g : g \text{ is a chain on } V \text{ to } \mathbb{F}, \sum_{v \in V} f(v)g(v) = 0 \text{ for all } f \in N\}$.

A *minor* of a Tutte chain-group N on V to $\mathbb{F}$ is a Tutte chain-group of the form $(N \times S) \cdot T$ where $T \subseteq S \subseteq V$. By definition, it is easy to see that $M\{N\} \setminus X = M\{N \times (V \setminus X)\}$ and $M\{N\}/X = M\{N \cdot (V \setminus X)\}$. So the notion of representable matroid minors is equivalent to the notion of Tutte chain-group minors.

Tutte [25, Theorem VIII.7.] showed the following theorem. The proof is basically equivalent to the proof of Theorem 3.4.

Lemma 8.2 (Tutte [25, Theorem VIII.7.])**.** *If N is a Tutte chain-group on V to $\mathbb{F}$ and $X \subseteq V$, then $(N \cdot X)^\perp = N^\perp \times X$.*

We now relate Tutte chain-groups to Lagrangian chain-groups. For a chain f on V to $\mathbb{F}$, let f^*, f_* be chains on V to $K = \mathbb{F}^2$ such that

$f^*(v) = \binom{f(v)}{0} \in K$, $f_*(v) = \binom{0}{f(v)} \in K$ for every $v \in V$. For a Tutte chain-group N on V to $\mathbb{F}$, we let $\widetilde{N}$ be a Tutte chain-group on V to K such that $\widetilde{N} = \{f^* + g_* : f \in N, g \in N^\perp\}$. Assume that $\langle\,,\,\rangle_K$ is symmetric.

Lemma 8.3. *If N is a Tutte chain-group on V to $\mathbb{F}$, then $\widetilde{N}$ is a Lagrangian chain-group on V to $K = \mathbb{F}^2$.*

Proof. By definition, for all $f \in N$ and $g \in N^\perp$, $\langle f^*, f^* \rangle = \langle g_*, g_* \rangle = 0$ and $\langle f^*, g_* \rangle = \sum_{v \in V} f(v)g(v) = 0$. Thus, $\widetilde{N}$ is isotropic. Moreover, $\dim N + \dim N^\perp = \dim \mathbb{F}^V = |V|$ and therefore $\dim \widetilde{N} = |V|$. (Note that $\widetilde{N}$ is isomorphic to $N \oplus N^\perp$ as a vector space.) So $\widetilde{N}$ is a Lagrangian chain-group. $\square$

Lemma 8.4. *Let N_1, N_2 be Tutte chain-groups on V_1, V_2 (respectively) to $\mathbb{F}$. Then N_1 is a minor of N_2 as a Tutte chain-group if and only if $\widetilde{N}_1$ is a minor of $\widetilde{N}_2$ as a Lagrangian chain-group.*

Proof. Let N be a Tutte chain-group on V to $\mathbb{F}$ and let S be a subset of V. It is enough to show that $\widetilde{N \cdot S} = \widetilde{N} /\!\!/ (V \setminus S)$ and $\widetilde{N \times S} = \widetilde{N} \setminus\!\!\setminus (V \setminus S)$.

Let us first show that $\widetilde{N \cdot S} = \widetilde{N} /\!\!/ (V \setminus S)$. Since $\dim \widetilde{N \cdot S} = \dim \widetilde{N} /\!\!/ (V \setminus S) = |S|$ by Lemma 8.3, it is enough to show that $\widetilde{N \cdot S} \subseteq \widetilde{N} /\!\!/ (V \setminus S)$. Suppose that $f \in N \cdot S$ and $g \in (N \cdot S)^\perp$. By Lemma 8.2, $(N \cdot S)^\perp = N^\perp \times S$. So there are $\bar{f}, \bar{g} \in N$ such that $\bar{f} \cdot S = f$, $\bar{g} \cdot S = g$, and $\bar{g}(v) = 0$ for all $v \in V \setminus S$. Now it is clear that $f^* + g_* = (\bar{f}^* + \bar{g}_*) \cdot S \in N /\!\!/ (V \setminus S)$.

Now it remains to show that $\widetilde{N \times S} = \widetilde{N} \setminus\!\!\setminus (V \setminus S)$. Let $f \in N \times S$, $g \in (N \times S)^\perp = N^\perp \cdot S$. A similar argument shows that $f^* + g_* \in \widetilde{N} \setminus\!\!\setminus S$ and therefore $\widetilde{N \times S} \subseteq \widetilde{N} \setminus\!\!\setminus (V \setminus S)$. This proves our claim because these two Lagrangian chain-groups have the same dimension. $\square$

Now let us show that for a Tutte chain-group N on V to $\mathbb{F}$, the branch-width of a matroid $M\{N\}$ is exactly one more than the branch-width of the Lagrangian chain-group $\widetilde{N}$. It is enough to show the following lemma.

Lemma 8.5. *Let N be a Tutte chain-group on V to $\mathbb{F}$. Let X be a subset of V. Then,*

$$\lambda_{M\{N\}}(X) = \lambda_{\widetilde{N}}(X) + 1.$$

Proof. Recall that the connectivity function of a matroid is $\lambda_{M\{N\}}(X) = r(X) + r(V \setminus X) - r(V) + 1$ and the connectivity function of a Lagrangian chain-group is $\lambda_{\widetilde{N}}(X) = |X| - \dim(\widetilde{N} \times X)$. Let $Y = V \setminus X$. Let r be the rank function of the matroid $M\{N\}$. Then $r(X)$ is equal to the rank of the matroid $M\{N\} \setminus Y = M\{N \times X\}$. So by Theorem 8.1, $r(X) = |X| - \dim(N \times X)$. Therefore

$$\lambda_{M\{N\}}(X) = \dim N - \dim(N \times X) - \dim(N \times Y) + 1.$$

From our construction, $\lambda_{\widetilde{N}}(X) = |X| - \dim(\widetilde{N} \times X) = |X| - (\dim(N \times X) + \dim(N^{\perp} \times X)) = |X| - \dim N \times X - \dim(N \cdot X)^{\perp} = |X| - \dim N \times X - (|X| - \dim N \cdot X) = \dim N \cdot X - \dim N \times X$. It is enough to show that $\dim N = \dim N \times Y + \dim N \cdot X$. Let $L : N \to N \cdot X$ be a surjective linear transformation such that $L(f) = f \cdot X$. Then $\dim \ker L = \dim(\{f \in N : f \cdot X = 0\}) = \dim(N \times Y)$. Thus, $\dim N \cdot X = \dim N - \dim N \times Y$. □

8.3. **Application to matroids.** We are now ready to deduce the following theorem by Geelen, Gerards, and Whittle [8] from Theorem 6.1.

Theorem 8.6 (Geelen, Gerards, and Whittle [8])**.** *Let k be a constant and let $\mathbb{F}$ be a finite field. If $M_1, M_2, \ldots$ is an infinite sequence of $\mathbb{F}$-representable matroids having branch-width at most k, then there exist i and j with $i < j$ such that M_i is isomorphic to a minor of M_j.*

To deduce this theorem, we use Tutte chain-groups.

Proof. Let N_i be the Tutte chain-group on $E(M_i)$ to $\mathbb{F}$ such that $M\{N_i\} = M_i$. By Lemma 8.5, the branch-width of the Lagrangian chain-group $\widetilde{N}_i$ is at most $k - 1$. By Theorem 6.1, there are $i < j$ such that $\widetilde{N}_i$ is simply isomorphic to a minor of $\widetilde{N}_j$. This implies that $M_i = M\{N_i\}$ is isomorphic to a minor of $M_j = M\{N_j\}$ by Lemma 8.4. □

Geelen et al. [8] showed that Theorem 8.6 implies the following theorem. (We omit the definition of tree-width.) Thus our theorem also implies the following theorem of Robertson and Seymour.

Theorem 8.7 (Robertson and Seymour [14])**.** *Let k be a constant. Every infinite sequence $G_1, G_2, \ldots$ of graphs having tree-width at most k has a pair $i < j$ such that G_i is isomorphic to a minor of G_j.*

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Department of Mathematical Sciences, KAIST, 335 Gwahangno, Yuseong-gu, Daejeon 305-701, Republic of Korea
E-mail address: `sangil@kaist.edu`