Vertex-minors and Pivot-minors

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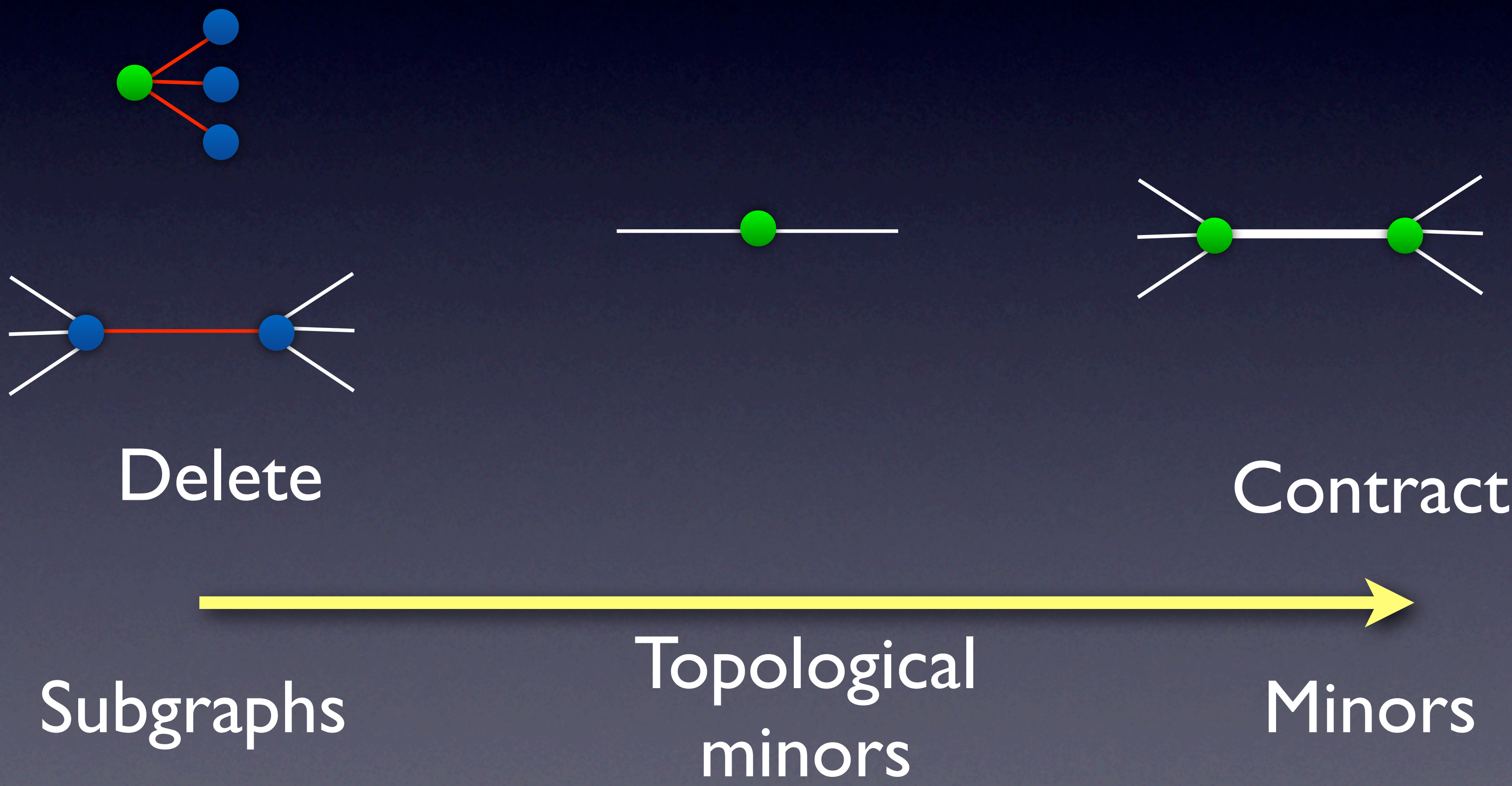
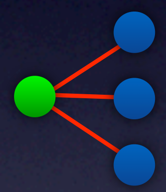
Subgraphs and minors

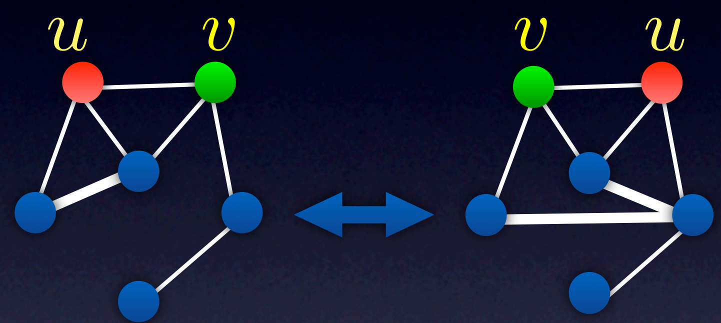


Induced subgraphs and ?



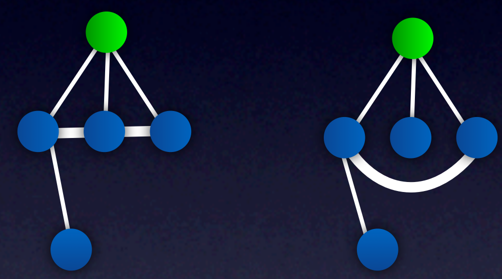
$G \setminus v$

Delete vertices



$G \wedge uv$

Pivot



$G * v$

Local Complementation

Induced Subgraphs

Pivot-minors

Vertex-minors

Minors	Vertex-minors, Pivot-minors
Planar graphs	Circle graphs
k-connected	k-rank-connected
Branch-width	Rank-width
Matroids	Delta-matroids, Isotropic Systems

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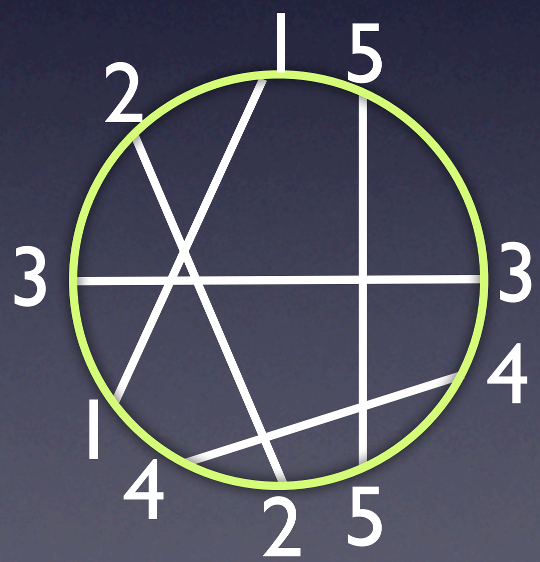
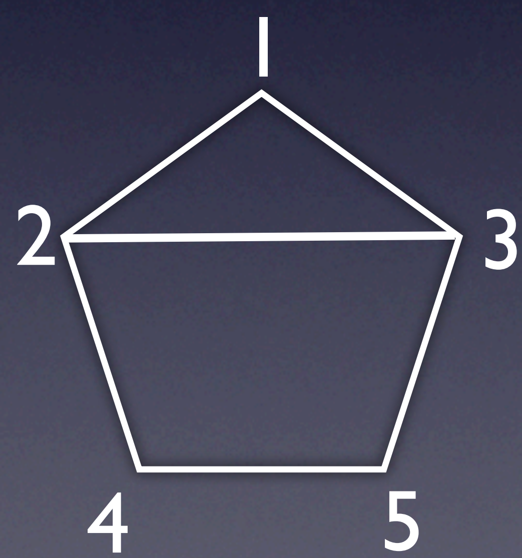
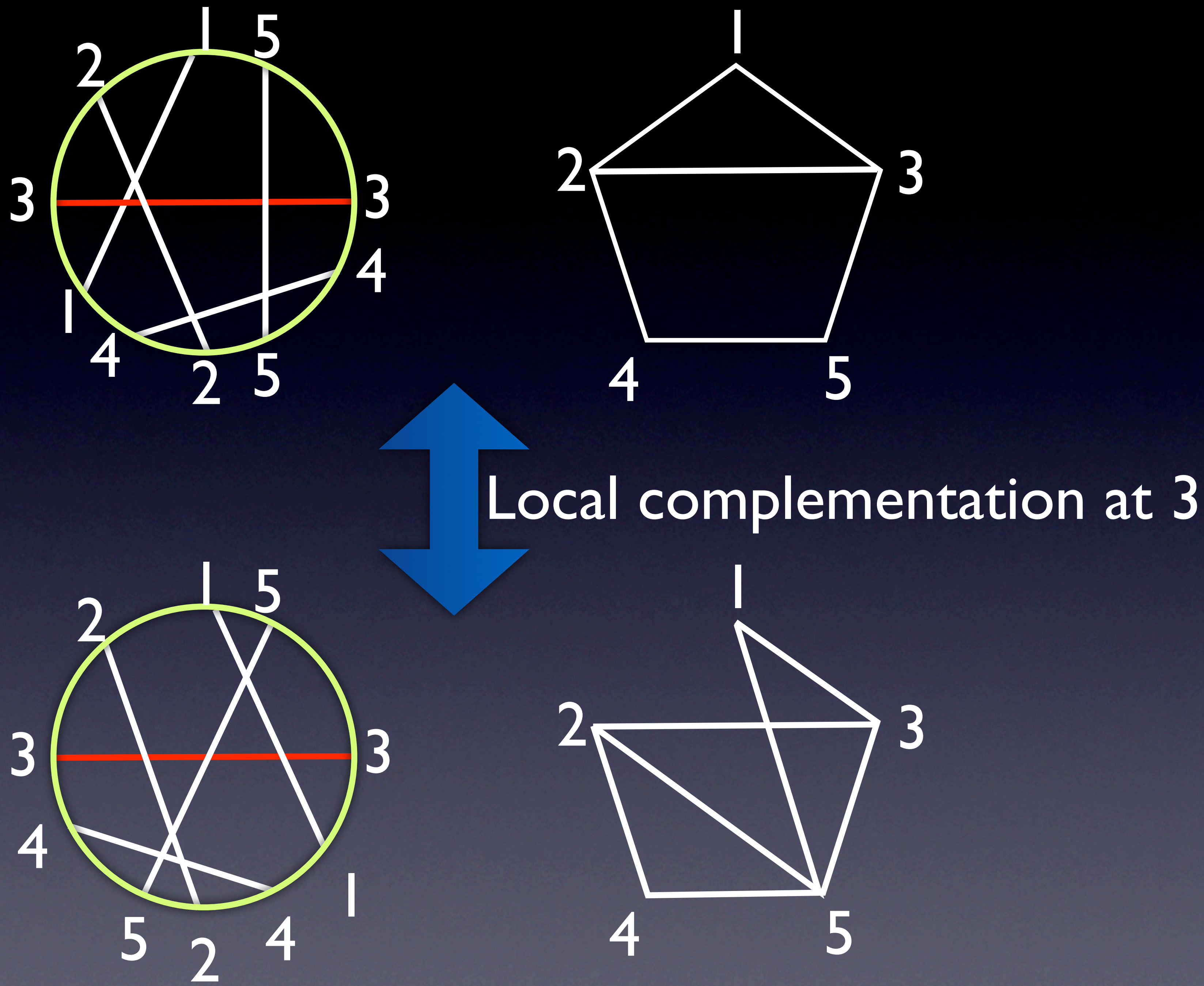
Circle graphs

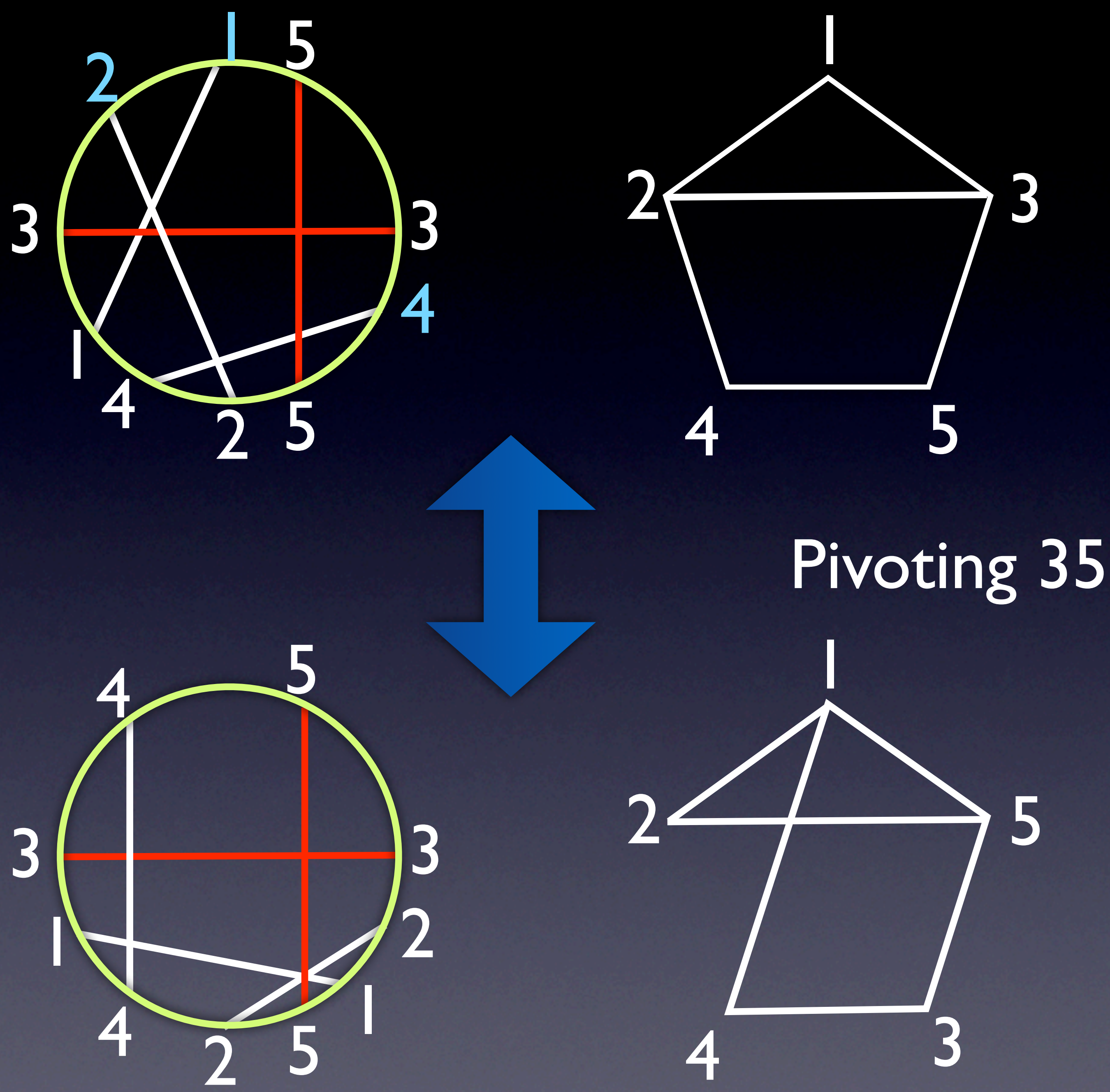
- Intersection graphs of chords of a circle
- Overlap graphs of intervals on a line



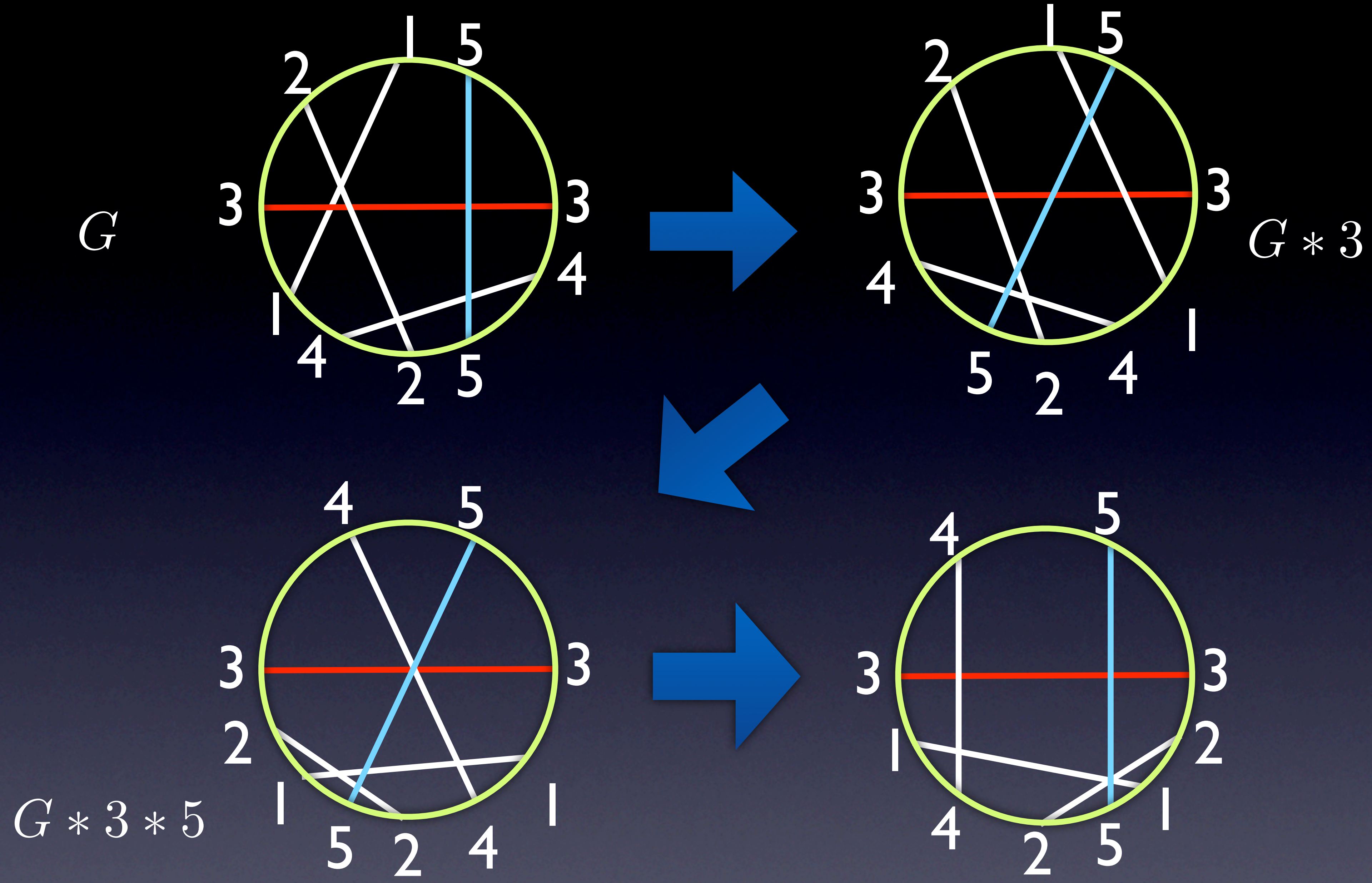




Circle graphs are closed under taking vertex-minors



Circle graphs are closed under taking pivot-minors



$$G * 3 * 5 * 3 = G \wedge 35$$

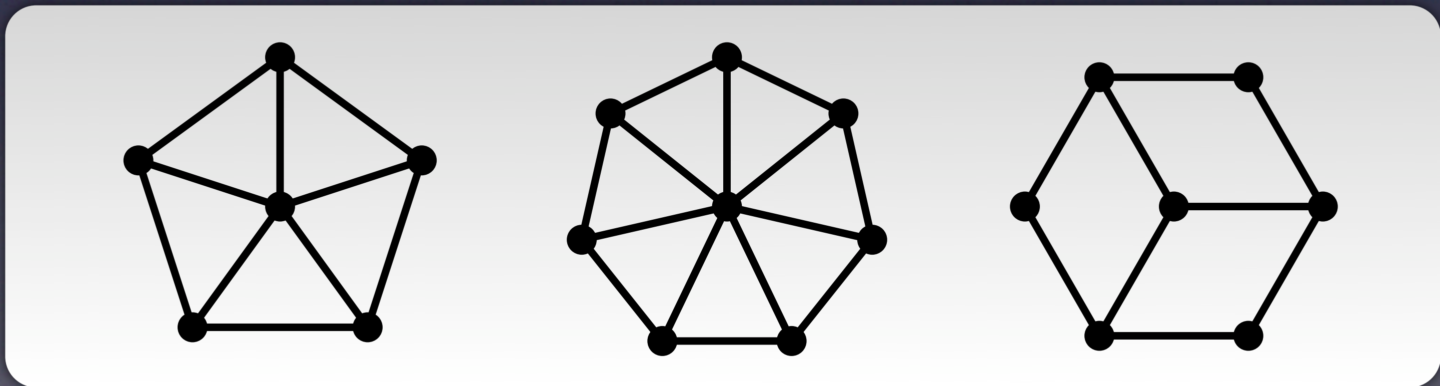
$$G * u * v * u = G \wedge uv$$

In general, every pivot-minor is a vertex-minor.

Forbidden vertex-minors for circle graphs

- A graph is a circle graph
  iff
  it has no vertex-minors isomorphic to

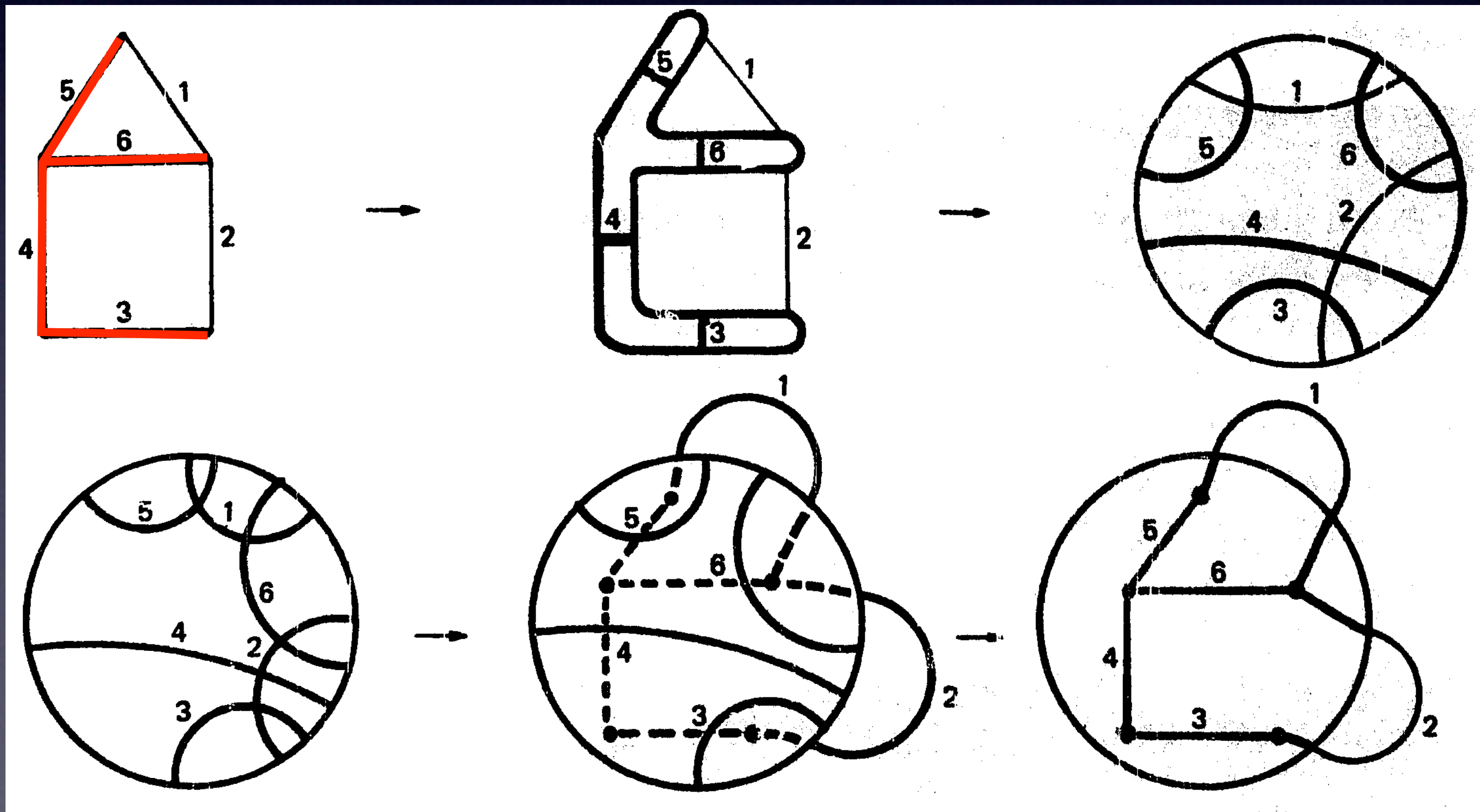
W_5 W_7 BW_3



Bouchet '94

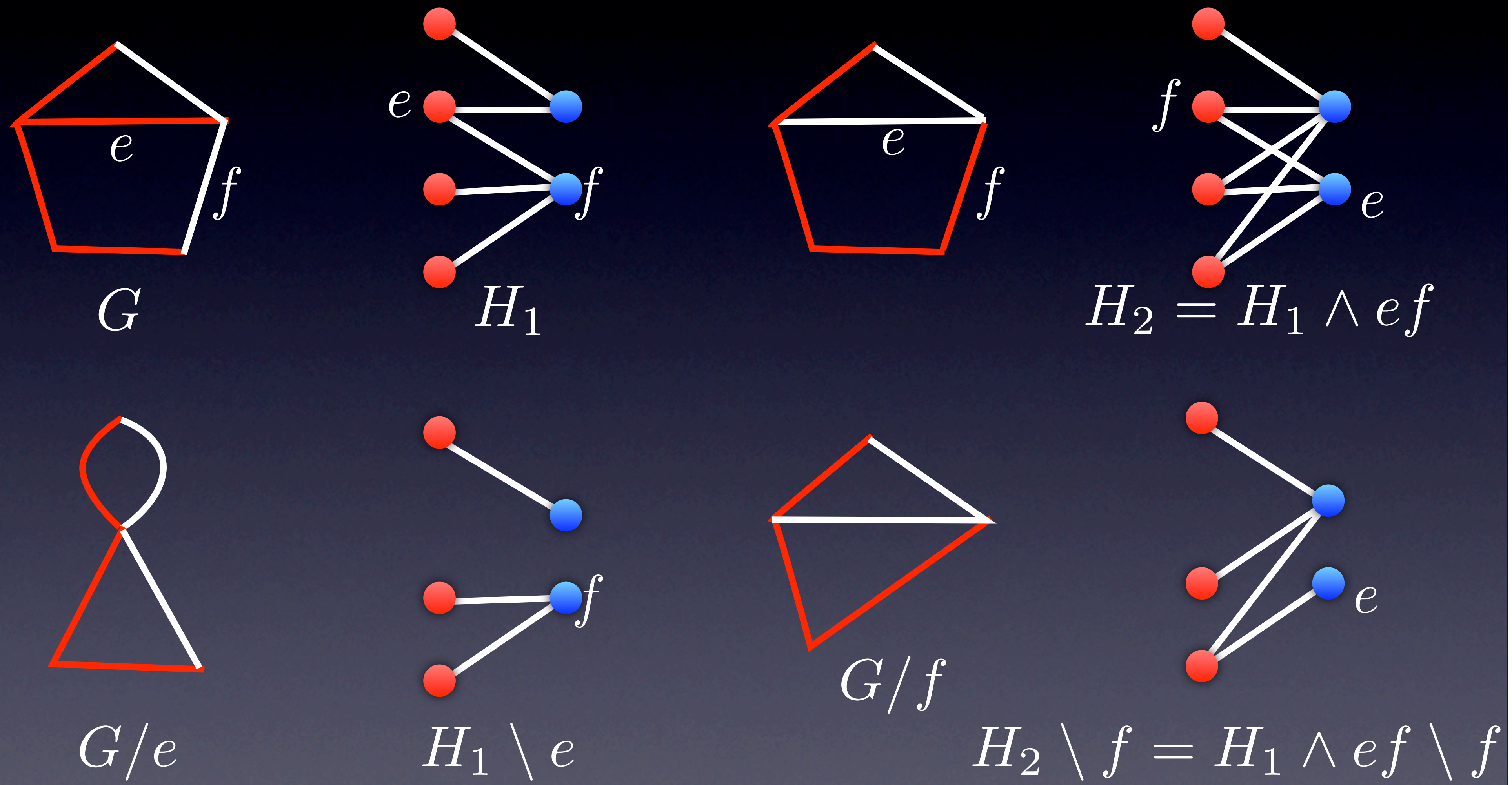
Circle v.s. Planar

- A bipartite graph is a circle graph
  iff
  it is a fundamental graph of a planar graph.



de Fraysseix '81

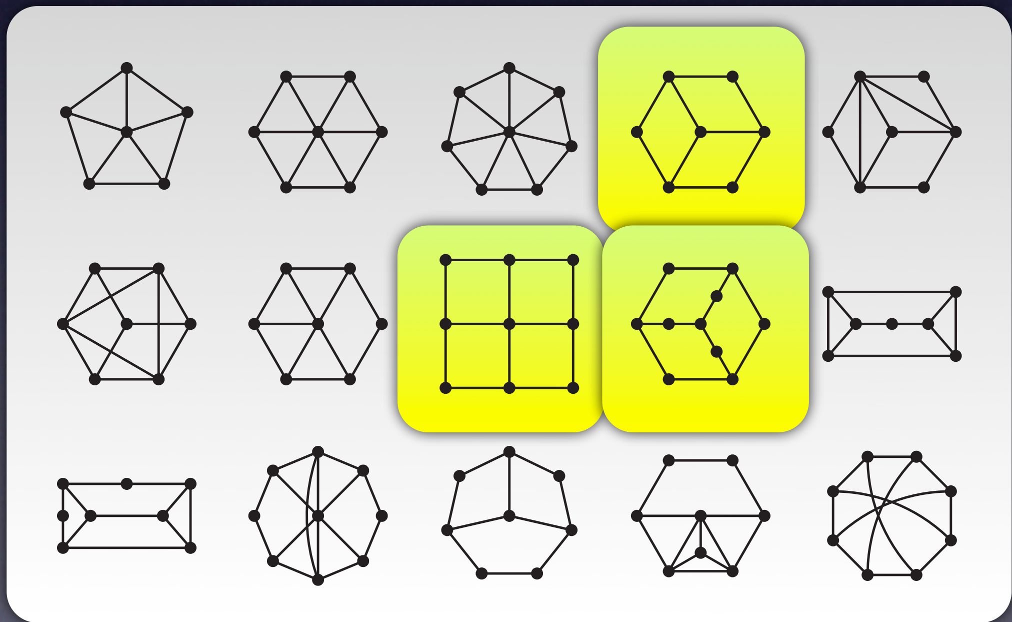
Minors & Pivot-minors



Minors correspond to pivot-minors of fundamental graphs.

Forbidden pivot-minors for circle graphs

- A graph is a circle graph
  iff
  it has no pivot-minors isomorphic to



FG of Fano matroid

$M(K_5)$

$M(K_{3,3})$

Implies Kuratowski's theorem!

Geelen, O. '09

Distance-hereditary

- Distance-hereditary: graphs that can be generated from a graph with no edges by
  - creating twins
  - creating pendant vertices
- Closed under taking vertex-minors
- "series-parallel graphs" for vertex-minors
- Distance-hereditary
  iff no C_5 vertex-minors
  iff no C_5, C_6 pivot-minors

Bandelt, Mulder '86 Bouchet'87,'88

Finitely many forbidden graphs

- Thm: Every minor-closed class of graphs has finitely many forbidden minors.
- Conj: Every pivot-minor-closed class of graphs has finitely many forbidden pivot-minors.
- Weak Conj: Every vertex-minor-closed class of graphs has finitely many forbidden vertex-minors.
  - If Weak Conj is true then: every vertex-minor-closed class of graphs has finitely many forbidden pivot-minors. (Geelen, O.'09)

Well-quasi-ordering

- Equivalently: Every infinite set of graphs contains a pair of graphs H, G such that H is isomorphic to a vertex-minor of G.
- Known to be true when graphs are:
  - bipartite graphs (by binary matroids)
  - line graphs (by group-labelled graphs)
  - bounded rank-width (O.'08)
  - circle graphs (by Graph Minors XXIII, immersion order of 4-regular graphs)

pivot-minors

?

Minors	Vertex-minors, Pivot-minors
Planar graphs	Circle graphs
k-connected	k-rank-connected
Branch-width	Rank-width
Matroids	Delta-matroids, Isotropic Systems

Connectivity

- For a subset X of E,
  mid(X)=set of vertices meeting X and E-X.
- k-connected:
  If |mid(X)|<k, then X or E-X is “small”.
  (“small”: no vertices meet edges in X only)
- mid(X) can only decrease
  if we take a minor

Rank connectivity

X

$$\rho_G(X) = \text{rank}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

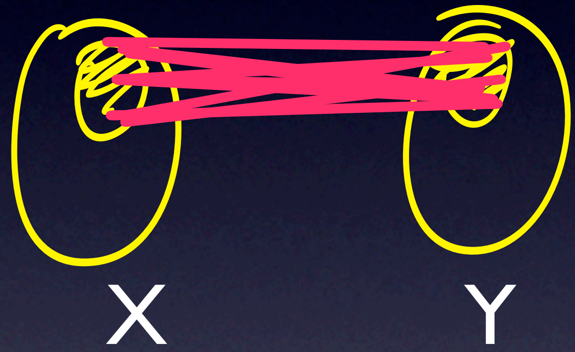
Y

cut-rank function

- k-rank-connected:
  If $\rho_G(X) < k$
  then $\min(|X|, |V - X|) \leq \rho(X)$
- All graphs are 0-rank-connected.
- All connected graphs are 1-rank-connected.
- k-rank-connected$\rightarrow(k-1)$-rank-connected

2-rank-connected graphs

- 2-rank-connected=
  $$\min(|X|, |V - X|) \le 1$$
  whenever $\rho(X) < 2$

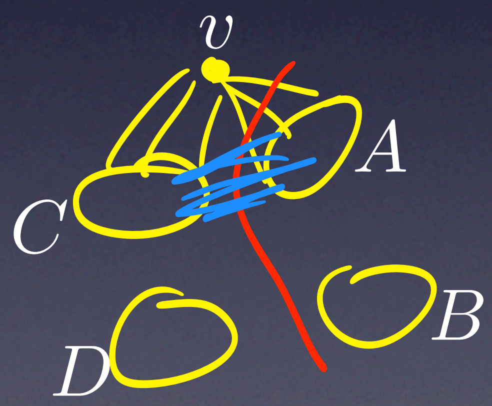


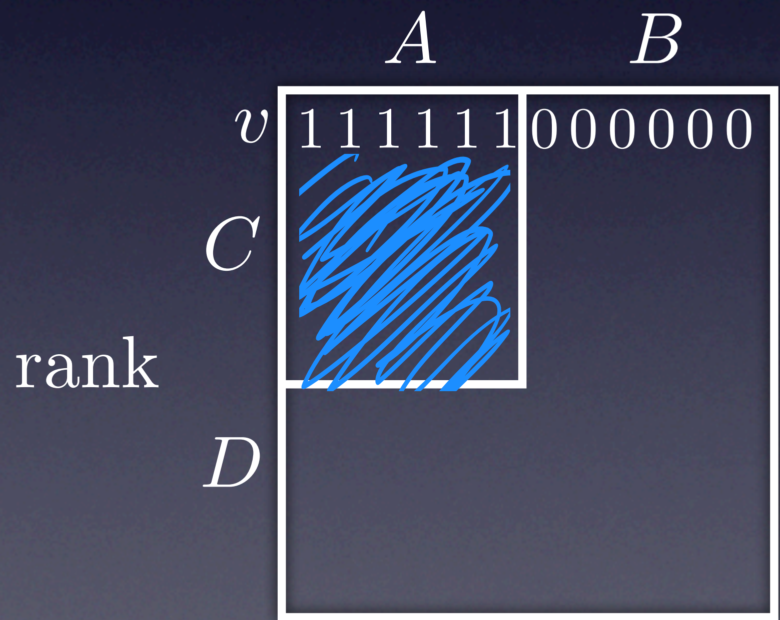
- Split (X,Y) (|X|,|Y|>1)
- 2-rank-connected = no splits (1-join)
- 2-rank-connected = prime w.r.t. split decompositions

Cunningham ’82

Rank-connectivity and vertex-minors

- Cut-rank is invariant under taking local complementation.





Chain theorems

- If G is simple 3-connected, then G has a simple simple 3-connected minor with one fewer edges, unless G=wheel.
- If G is 2-rank-connected with |V|>4, then G has a 2-rank-connected pivot-minor with one fewer vertices, unless G=cycle.
- If G is 2-rank-connected with |V|>5, then G has a 2-rank-connected vertex-minor with one fewer vertices.

Tutte'61
Bouchet'87, Allys'94

Splitter theorems

- If H is a simple 3-connected minor of a simple 3-connected graph G,
  then G has a simple 3-connected minor with one fewer edges having a minor isomorphic to H unless |V(G)|=|V(H)| or H=wheel.
- If H is a 2-rank-connected pivot-minor of a 2-rank-connected graph G and |V(H)|>4,
  then G has a 2-rank-connected pivot-minor with one fewer vertices having a pivot-minor isomorphic to H, unless |V(G)|=|V(H)| or H=cycle.

Seymour '80, Negami '82
Bouchet (unpublished), Geelen '95

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- If H is a 2-rank-connected vertex-minor of a 2-rank-connected graph G and |V(H)|>4,
  then G has a 2-rank-connected vertex-minor with one fewer vertices having a vertex-minor isomorphic to H, unless |V(G)|=|V(H)|.

Seymour '80, Negami '82
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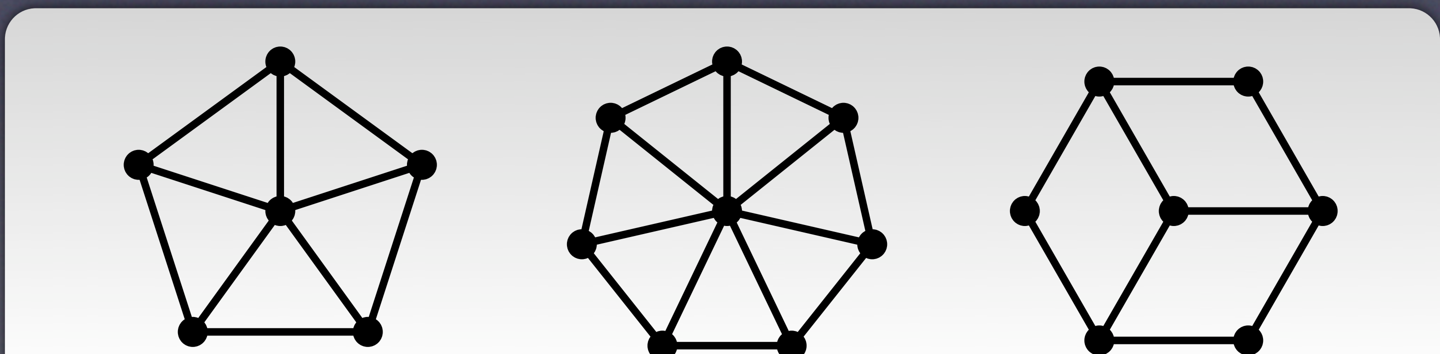
Applications of Splitter Theorems

- Graphs with no K_5 minor= Planar graphs+$K_{3,3}$+V_8 +their 0-,1-,2-,3-sums
- Graphs with no W_5 vertex-minor= circle graphs+W_7 +BW_3+cube +their disjoint unions + their 1-joins

W_5 W_7 BW_3



Wagner'3?

Geelen'95

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Matroids	Delta-matroids, Isotropic Systems

Rank-width

- "Branch-width measured by cut-rank"
- Rank-decomposition= A subcubic tree T whose leaves are labeled bijectively by V(G)
- Width of an edge of T=cutrank of the partition given by an edge of T
- Width(T,L)= max width of all edges
- Rank-width= min Width (T,L)
- Rank-width(H)≤Rank-width(G) if H=vertex-minor

O., Seymour '06

O.'05

Rank-width

- Related to branch-width of matroids and graphs
- Poly-time algorithm to construct a decomposition of width$\leq$k if it exists, for fixed k
- Thm: Graphs of bounded rank-width are well-quasi-ordered by the pivot-minor relation.

Hineny, O. ’08
O.’05, O.’08

Minors	Vertex-minors, Pivot-minors
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Delta-matroids

- If A is a skew-symmetric matrix over F, then

$$\mathcal{F} = \{X : A[X] \text{ is nonsingular}\}$$

satisfies the following axioms:

(1) $\mathcal{F}$ is nonempty.

(2) If $X, Y \in \mathcal{F}$ and $a \in X \Delta Y$

then there exists $b \in X \Delta Y$ such that

$$X \Delta \{a, b\} \in \mathcal{F}$$

- Delta-matroid: set-system satisfying (1), (2)
- Binary: represented by a matrix over GF(2)

Bouchet'87

Twisting

(1) $\mathcal{F}$ is nonempty.

(2) If $X, Y \in \mathcal{F}$ and $a \in X \Delta Y$ then there exists $b \in X \Delta Y$ such that $X \Delta \{a, b\} \in \mathcal{F}$

- Twisting: Replacing A by AΔX for some X
- Thm: If M is binary, then MΔX is binary

$$A = \begin{array}{c} \\ X \\ Y \end{array} \begin{array}{c} \begin{array}{cc} X & Y \end{array} \\ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \end{array} \qquad A * X = \begin{array}{c} \\ X \\ Y \end{array} \begin{array}{c} \begin{array}{cc} X & Y \end{array} \\ \begin{pmatrix} \alpha^{-1} & \alpha^{-1}\beta \\ -\gamma\alpha^{-1} & \delta - \gamma\alpha^{-1}\beta \end{pmatrix} \end{array}$$

principal pivot

(A*X)*Y=A*(XΔY)

Tucker'60:
A*X[Y] is nonsingular
iff
A[XΔY] is nonsingular

If X={u,v}, then $A(G) * X = A(G \wedge uv)$

Twisting

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(A*X)*Y=A*(XΔY)

Tucker'60:
A*X[Y] is nonsingular
iff
A[XΔY] is nonsingular

If X={u,v}, then $A(G) * X = A(G \wedge uv)$

Binary even Delta-matroids up to twisting
= Graphs up to pivot equivalence
Binary even Delta-maroid minors
= Graph pivot-minors

Isotropic systems

- Introduced by Bouchet '87
- Linear-algebraic description of equivalent classes of graphs up to local complementations
- Minors of isotropic systems =Vertex-minors
- Powerful tool for vertex-minors

Other topics

Algorithmic Aspects

- Can we decide whether G has a H-pivot-minor in poly time for fixed H?
  - Yes if G has bounded rank-width
  - Yes if G is a bipartite graph
  - Yes if G is a line graph
  - Yes if G is a circle graph and H is bipartite

MSOL formula

matroid result

group-labelled graph

bounded rank-width

Structural Aspects

- Any interesting class of graphs closed under vertex-minors or pivot-minors?
  - circle graphs, graphs of bounded rank-width, distance-hereditary graphs, bipartite graphs (pivot-minors), pivot-minors of line graphs (pivot-minors)
- Structures of graphs with no H vertex-minors?

Fields other than GF(2)

- Pivot-minors and vertex-minors of graphs: generalizing minors of binary matroids
- One can define: delta-matroids representable over a field F : pivot-minors of edge-labelled directed graphs
- Thm: Delta-matroids of "bounded branch-width" over a finite field are well-quasi-ordered.
- Structural theory for skew-symmetric matrices over a finite field?

Interlace polynomials

$$q(G;x,y) = \sum_{S \subseteq V} (x-1)^{\text{rank}(G[S])} (y-1)^{\text{nullity}(G[S])}$$

Over GF(2)

- Reduction formula is given in terms of pivot-minor operations.

$$q(G) = q(G \setminus a) + q(G \wedge ab \setminus a) + ((x-1)^2 - 1) q(G \wedge ab \setminus a \setminus b)$$

$$q(n\text{-vertex graph with no edges}) = y^n$$

- If G is a graph and H is a FG of G, then q(H;2,y)=T(H;y,y) (Tutte polynomial)

Arratia, Bollobás, Sorkin'04

Aigner, van der Holst '04

More...

- Measurement based quantum computation "Graph States"
  - >10 papers in last 5 years in Physics journals using local complementations and pivoting
- Coding theory "Self-dual additive codes over GF(4)"
- Local complementation on directed graphs
  - "Eulerian systems" (Bouchet '87)
  - "Directed rank-width" (Kanté '09)
  - "Weakly $\mathbb{Z}_2^n$-equivariant homeomorphism classes of small covers of the n-dim cube" (Choi '08)

Thank you!

http://mathsci.kaist.ac.kr/~sangil/

minor	vertex-minor pivot-minor
graph planar graph series parallel Tutte poly. Tree-width k-connected cycle matroid totally unimodular	simple graph circle graph distance-hereditary Interlace poly. Rank-width rank-(k-1)-connected delta-matroids/iso. sys. principally unimodular