Perfect Matchings in Claw-free Cubic Graphs

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Abstract

Lovász and Plummer conjectured that there exists a fixed positive constant c such that every cubic n-vertex graph with no cutedge has at least 2^{cn} perfect matchings. Their conjecture has been verified for bipartite graphs by Voorhoeve and planar graphs by Chudnovsky and Seymour. We prove that every claw-free cubic n-vertex graph with no cutedge has more than $2^{n/12}$ perfect matchings, thus verifying the conjecture for claw-free graphs.

1 Introduction

A graph is *claw-free* if it has no induced subgraph isomorphic to $K_{1,3}$. A graph is *cubic* if every vertex has exactly three incident edges. A well-known classical theorem of Petersen [9] states that every cubic graph with no cutedge has a perfect matching. Sumner [10] and Las Vergnas [6] independently showed that every connected claw-free graph with even number of vertices has a perfect matching. Both theorems imply that every claw-free cubic graph with no cutedge has at least one perfect matching.

In 1970s, Lovász and Plummer conjectured that every cubic graph with no cutedge has exponentially many perfect matchings; see [7, Conjecture 8.1.8]. The best lower bound has been obtained by Esperet, Kardoš, and Král' [5]. They showed that the number of perfect matchings in a sufficiently large cubic graph with no cutedge always exceeds any fixed linear function in the number of vertices.

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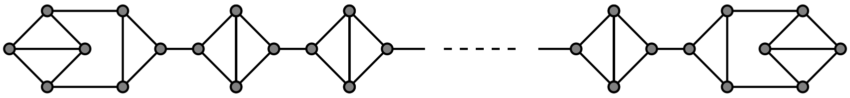

Figure 1: Claw-free cubic graphs with only 9 perfect matchings

So far the conjecture is known to be true for bipartite graphs and planar graphs. For bipartite graphs, Voorhoeve [11] proved that every *bipartite* cubic n-vertex graph has at least $6(4/3)^{n/2-3}$ perfect matchings. Recently, Chudnovsky and Seymour [2] proved that every *planar* cubic n-vertex graph with no cutedge has at least $2^{n/655978752}$ perfect matchings.

We prove that every claw-free cubic n-vertex graph with no cutedge has more than

$$2^{n/12}$$

perfect matchings. The graph should not have any cutedge; in Figure 1, we provide an example of a claw-free cubic graph with only 9 perfect matchings.

Our approach is to use the structure of 2-edge-connected claw-free cubic graphs. The *cycle space* $\mathcal{C}(H)$ of H is a collection of the edge-disjoint union of cycles of H. It is well known that $\mathcal{C}(H)$ forms a vector space over $GF(2)$ and

$$\dim \mathcal{C}(H) = |E(H)| - |V(H)| + 1$$

if H is connected, see Diestel [3]. Roughly speaking, almost all 2-edge-connected claw-free cubic graph G can be built from a 2-edge-connected cubic multigraph H by certain operations so that every member of $\mathcal{C}(H)$ can be extended to 2-factors of G. We will have two cases to consider; either H is big or small. If H is big, then $\mathcal{C}(H)$ is big enough to prove that G has many 2-factors. If H is small, then we find a 2-factor of H using many of the specified edges of H so that when transforming this 2-factor of H to that of G, each of those edges of H has many ways to make 2-factors of G.

2 Structure of 2-edge-connected claw-free cubic graphs

Graphs in this paper have no parallel edges and no loops, and multigraphs can have parallel edges and loops. We assume that a loop is counted twice when measuring a degree of a vertex in a multigraph. Every 2-edge-connected cubic multigraph can not have loops because if it has a loop, then it must have a cutedge.

We describe the structure of claw-free cubic graphs given by Palmer et al. [8]. A *triangle* of a graph is a set of three pairwise adjacent vertices. *Replacing a vertex*

v *with a triangle* in cubic graph is to replace v with three vertices v_1, v_2, v_3 forming a triangle so that if e_1, e_2, e_3 are three edges incident with v, then e_1, e_2, e_3 will be incident with v_1, v_2, v_3 respectively.

Every vertex in a claw-free cubic graph is in 1, 2, or 3 triangles. If a vertex is in 3 triangles, then the component containing the vertex is isomorphic to K_4. If a vertex is in exactly 2 triangles, then it is in an induced subgraph isomorphic to $K_4 \setminus e$ for some edge e of K_4. Such an induced subgraph is called a *diamond*. It is clear that no two distinct diamonds intersect.

A *string of diamonds* is a maximal sequence $D_1, D_2, \ldots, D_k$ of diamonds in which, for each $i \in \{1, 2, \ldots, k-1\}$, D_i has a vertex adjacent to a vertex in D_{i+1}. A string of diamonds has exactly two vertices of degree 2, which are called the *head* and the *tail* of the string. *Replacing an edge* $e = uv$ *with a string of diamonds* with the head x and the tail y is to remove e and add edges ux and vy.

A connected claw-free cubic graph in which every vertex is in a diamond is called a *ring of diamonds*. We require that a ring of diamonds contains at least 2 diamonds. It is now straightforward to describe the structure of 2-edge-connected claw-free cubic graphs as follows.

Proposition 1. *A graph G is 2-edge-connected claw-free cubic if and only if either*

(i) G is isomorphic to K_4,

(ii) G is a ring of diamonds, or

(iii) G can be built from a 2-edge-connected cubic multigraph H by replacing some edges of H with strings of diamonds and replacing each vertex of H with a triangle.

Proof. Let us first prove the "if" direction. It is easy to see that G is 2-edge-connected cubic and has no loops or parallel edges. If G is built as in (iii), then clearly G has neither loops nor parallel edges, and every vertex of G is in a triangle and therefore G is claw-free. Note that since H is 2-edge-connected, H can not have loops.

To prove the "only if" direction, let us assume that G is a 2-edge-connected claw-free cubic graph. We may assume that G is not isomorphic to K_4 or a ring of diamonds. We claim that G can be built from a 2-edge-connected cubic multigraph as in (iii). Suppose that G is a counter example with the minimum number of vertices.

If G has no diamonds, then every vertex of G is in exactly one triangle and therefore $V(G)$ can be partitioned into disjoint triangles. By contracting each triangle, we obtain a 2-edge-connected cubic multigraph H.

So G must have a string of diamonds. Let D be the set of vertices in the string of diamonds. Since G is cubic, G has two vertices not in D, say u and v, adjacent to D. If $u = v$, then because the degree of u is 3, u must have another incident edge e but e will be a cutedge of G. Thus $u \neq v$.

If u and v are adjacent in G, then u and v must has a common neighbor x, because otherwise G will have an induced subgraph isomorphic to $K_{1,3}$. However one of the edges incident with x will be a cutedge of G, a contradiction.

Thus u and v are nonadjacent in G. Let $G' = (G \setminus D) + uv$, that is obtained from G by deleting D and adding an edge uv. Then G' has no parallel edges or loops and moreover G' is 2-edge-connected claw-free cubic. Since G has a vertex not in a diamond, so does G' and therefore G' can be built from a 2-edge-connected cubic multigraph H by replacing some edges with strings of diamonds and replacing each vertex of H with a triangle. Since D is chosen maximally, u and v are not in diamonds and therefore H has the edge uv. So we can obtain G from H by doing all replacements to obtain G' and then replacing the edge uv with a string of diamonds. This completes the proof. □

We remark that Proposition 1 can be seen as a corollary of the structure theorem of quasi-line graphs by Chudnovsky and Seymour [1]. A graph is a *quasi-line* graph if the neighborhood of each vertex is expressible as the union of two cliques. It is obvious that every claw-free cubic graph is a quasi-line graph. Chudnovsky and Seymour [1] proved that every connected quasi-line graph is either a fuzzy circular interval graph or a composition of fuzzy linear interval strips. For 2-edge-connected claw-free cubic graphs, a fuzzy circular interval graph corresponds to a ring of diamonds and a composition of fuzzy linear interval strips corresponds to the construction (iii) of Proposition 1.

3 Main theorem

Theorem 2. *Every claw-free cubic n-vertex graph with no cutedge has more than $2^{n/12}$ perfect matchings.*

Proof. Let G be a claw-free cubic n-vertex graph with no cutedge. We may assume that G is connected. If G is isomorphic to K_4, then the claim is clearly true. If G is a ring of diamonds, then G has $2^{n/4}+1$ perfect matchings. Thus we may assume that G is obtained from a 2-edge-connected cubic multigraph H by replacing some edges of H with strings of diamonds and replacing each vertex of H with a triangle.

Let $k = |V(H)|$. In other words, $3k$ is the number of vertices not in a diamond of G.

Suppose that $k \geq n/6$. Since H has $3k/2$ edges, the cycle space of H has dimension $3k/2 - k + 1 = k/2 + 1$ and therefore $|\mathcal{C}(H)| = 2^{k/2+1}$. To obtain a 2-factor from $C \in \mathcal{C}(H)$, we transform C into a member $C' \in \mathcal{C}(G)$ so that it meets all 3 vertices of G corresponding to v for each vertex v of H incident with

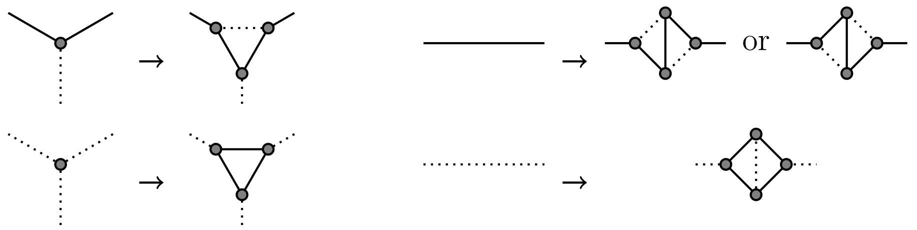

Figure 2: Transforming a member of $\mathcal{C}(H)$ into a 2-factor of G (Solid edges represent edges in a member of $\mathcal{C}(H)$ or a 2-factor of G.)

C as well as it meets all the vertices in each diamond that corresponds to an edge in C. Then for each vertex w of G unused yet in C', we add a cycle of length 3 or 4 depending on whether the vertex is in a diamond; see Figure 2. Then this is a 2-factor of G because it meets every vertex of G. Since the complement of the edge-set of a 2-factor is a perfect matching, we conclude that G has at least $2^{k/2+1} \geq 2^{n/12+1}$ perfect matchings.

Now let us assume that $k < n/6$. We know that G has $(n-3k)/4$ diamonds. The *length* of an edge e of H is the number of diamonds in the string of diamonds replaced with e. (If the edge e is not replaced with a string of diamonds, then the length of e is 0.)

Edmonds' characterization of the perfect matching polytope [4] implies that there exist a positive integer t depending on H and a list of $3t$ perfect matchings $M_1, M_2, \ldots, M_{3t}$ in H such that every edge of H is in exactly t of the perfect matchings. (In other words, H is fractionally 3-edge-colorable.) By taking complements, we have a list of $3t$ 2-factors of H such that each edge of H is in exactly $2t$ of the 2-factors in the list. Since G has $(n-3k)/4$ diamonds, the sum of the length of all edges of H is $(n-3k)/4$. Therefore there exists a 2-factor C of H whose length is at least $\frac{n-3k}{4}\frac{2}{3} = (n-3k)/6$.

We claim that G has at least $2^{(n-3k)/6}$ 2-factors corresponding to C. For each diamond in the string replacing an edge e of C, there are two ways to route cycles of C through the diamond, see Figure 2. Since C passes through at least $(n-3k)/6$ diamonds, G has at least $2^{(n-3k)/6}$ 2-factors. Since $k < n/6$, G has more than $2^{n/12}$ 2-factors. Thus G has more than $2^{n/12}$ perfect matchings. $\square$

We remark that every 3-edge-connected claw-free cubic n-vertex graph G has exactly $2^{n/6+1}$ perfect matchings, unless G is isomorphic to K_4. That is because G has no diamonds and so, from the idea of the above proof, there is a one-to-one correspondence between the set of all 2-factors of G and the cycle space of a multigraph H obtained by contracting each triangle of G.

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