Graphic delta-matroids

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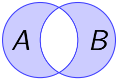
Overview of this talk

1. Introduce *graphic delta-matroids*.
   - Definition
2. Present some observations.
   - Minor-closed
   - Even binary
   - Why do I call them graphic?
3. Some questions.

"delta": $A \Delta B = (A \setminus B) \cup (B \setminus A)$.



Reference:

Oum. *Excluding a bipartite circle graph from line graphs*. Accepted to Journal of Graph Theory, 2008.

I. Definitions

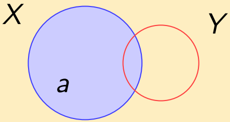
- Delta-matroids.
- Graphic delta-matroids.

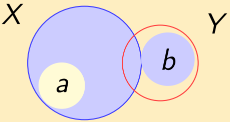
Delta-matroids

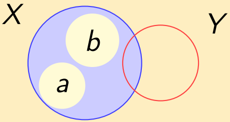
Bouchet, *Greedy algorithms and symmetric matroids*, Math. Programming, 1987.

$\mathcal{M} = (E, \mathfrak{F})$ is a *delta-matroid*
if $\mathfrak{F}$ is a *nonempty* collection of subsets of E such that

if $X, Y \in \mathfrak{F}$ and $a \in X \Delta Y$,
then $\exists b \in X\Delta Y$ such that $X \Delta \{a, b\} \in \mathfrak{F}$.

If 

, then 

or 

A set in $\mathfrak{F}$ is called a *feasible* set or a *basis* of the delta-matroid $\mathcal{M}$.

Matroids: if $X, Y \in \mathfrak{F}$ and $a \in X \setminus Y$,
then $\exists y \in Y \setminus X$ such that $X \Delta \{a, b\} \in \mathfrak{F}$.

Every matroid is a delta-matroid.

Delta-matroids

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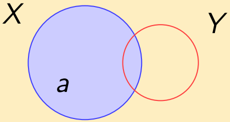
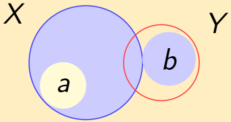
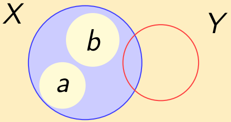
If 

, then 

or 

Twisting on a delta-matroid

$\mathcal{M} \Delta A = (E, \mathfrak{F} \Delta A)$ where $\mathfrak{F} \Delta A = \{X \Delta A : X \in \mathfrak{F}\}$

1. A twisting of a delta-matroid is a delta-matroid.
2. Every twisted matroid is a delta-matroid.

Graphic delta-matroids

- Graft: a pair (G, T) of a graph G and $T \subseteq V(G)$.
- A T-vertex is a vertex in T.

Delta-matroid $\mathfrak{G}(G, T)$ from a graft (G, T)

A set X of edges is feasible in $\mathfrak{G}(G, T)$ iff

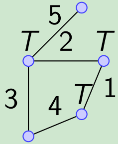
1. X induces no cycles,
2. each comp. of a spanning subgraph $(V(G), X)$ has odd #(T-vertices) (*) or it has no T-vertices but it spans a component of G.

Examples: Graphic matroids

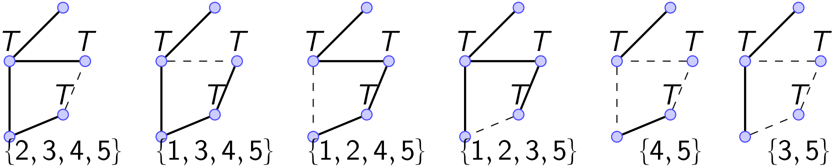
If G is connected and $|T| \leq 1$, then
X is feasible iff X is an edge-set of a spanning tree.
$\Rightarrow \mathfrak{G}(G, T)$ is a cycle matroid of G.

$\mathcal{M}$ is *graphic* iff $\mathcal{M} = \mathfrak{G}(G, T)\Delta X$ for some graft (G, T) and $X \subseteq E(G)$.

Examples: Graphic delta-matroids



What are the feasible sets in $\mathfrak{G}(G, T)$?



These are graphic delta-matroids. ($[5] = \{1, 2, 3, 4, 5\}$)

- $\mathcal{M}_1 = ([5], \{\{2, 3, 4, 5\}, \{1, 3, 4, 5\}, \{1, 2, 4, 5\}, \{1, 2, 3, 5\}, \{4, 5\}, \{3, 5\}\})$.
- $\mathcal{M}_2 = ([5], \{\{4, 5\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{1, 5\}, \{2, 3, 4, 5\}, \{2, 5\}\}) = \mathcal{M}_1 \Delta \{2, 3\}$.

When $|T| \le 2$

Let G be connected and $T \subseteq V(G)$.
If $|T| = 2$, then let G' be a graph obtained by identifying two vertices in T.
Then
$\mathfrak{G}(G, T) = \mathfrak{G}(G', \emptyset)$.

Consequence

If $|T| \le 2$, then $\mathfrak{G}(G, T)$ is a graphic matroid.

Why is $\mathfrak{G}(G, T)$ a delta-matroid?

1. X induces no cycles,
2. each comp. of a spanning subgraph $(V(G), X)$ has odd $\#(T\text{-vertices})$

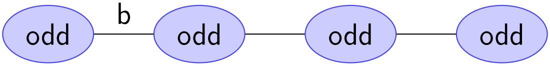
Assume X, Y are feasible, and let $a \in X \Delta Y$.

Claim: $\exists b \in X \Delta Y$ such that $X \Delta \{a, b\}$ is feasible.

Case 1: $a \in X$. If Y has an edge b joining even and odd comp. on $X \setminus \{a\}$,

Components of $X \setminus \{a\}$.

Otherwise, C_1 is partioned into odd components of Y with a tree structure.



Components inside C_1 on Y.

1. X induces no cycles,
2. each comp. of a spanning subgraph $(V(G), X)$ has odd $\#(T\text{-vertices})$

Assume X, Y are feasible, and let $a \in X \Delta Y$.

Claim: $\exists b \in X \Delta Y$ such that $X \Delta \{a, b\}$ is feasible.

Case 2: $a \in Y$.

Case 2.1: a joins two odd components C_1, C_2 of X. If Y has an edge b joining $C_1 \cup C_2$ to another odd component of X, then $X \cup \{a, b\}$ is feasible.

Components of X.

If Y has no such edge, then $Y \setminus \{a\}$ partitions $C_1 \cup C_2$ into components and either C_1 or C_2 has an even component D. Then we pick $b \in X$ joining D to others.

Case 2.2: a is inside a component C of X. Find $b \in X$ in the cycle of $X \cup \{a\}$. Then $X \Delta \{a, b\}$ is feasible. Q.E.D

II. Properties

- Minors of graphic delta-matroids are graphic.
- Graphic delta-matroids are even binary.
- Intersection of the set of graphic delta-matroids and the set of twisted matroids.

Minors of delta-matroids

Let $\mathcal{M} = (E, \mathfrak{F})$ be a delta-matroid.
Let $\mathfrak{F} \setminus X = \{F : F \cap X = \emptyset\}$ for $X \subseteq E$.
If $\mathfrak{F} \setminus X \neq \emptyset$, then define the *deletion*

$$\mathcal{M} \setminus X = (E, \mathfrak{F} \setminus X).$$

Obviously,

- $\mathcal{M} \setminus X$ is a delta-matroid.

$\mathcal{M}_2$ is a *minor* of $\mathcal{M}_1$
iff
$\mathcal{M}_2 = \mathcal{M}_1 \Delta X \setminus Y$ for some subsets X and Y of E.

Delta-matorid minors and graft minors

Theorem

Minors of graphic delta-matroids are graphic.

Enough to prove the following.

1. $\mathfrak{G}(G,T)\setminus e$ is graphic.
2. $\mathfrak{G}(G,T)\Delta\{e\}\setminus e$ is graphic.

In fact,

- $\mathfrak{G}(G,T)\setminus e = \mathfrak{G}(G\setminus e, T)$ unless every feasible set contains e.
- $\mathfrak{G}(G,T)\Delta\{e\}\setminus e = \mathfrak{G}(G/e, T')$ unless every feasible set doesn't contain e, where

$$\begin{cases} T' = T\setminus\{u,v\} & \text{if } u\notin T, v\notin T \text{ or } u,v\in T,\\ T' = T\setminus\{u,v\}\cup\{uv\} & \text{otherwise.}\end{cases}$$

Graft minors $\Longleftrightarrow$ Graphic delta-matroid minors.

Binary delta-matroids: definitions

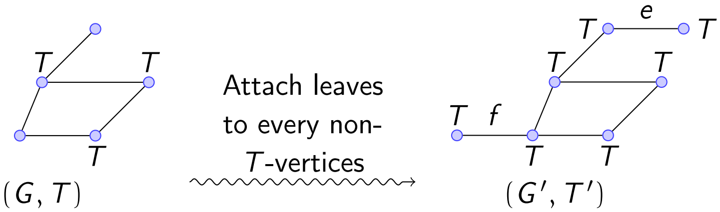
Theorem (Bouchet, Representability of Δ-matroids. 1988)

If A is an $E \times E$ skew-symmetric or symmetric matrix and let $\mathfrak{F} = \{X \subseteq E : A[X] \text{ is nonsingular}\}$, then $\mathcal{M}(A) = (E, \mathfrak{F})$ is a delta-matroid. (Note that $\emptyset \in \mathfrak{F}$.)

- A delta-matroid $\mathcal{M}$ is *representable* over a field F iff $\mathcal{M} = \mathcal{M}(A)\Delta X$ for some matrix A over F and $X \subseteq E$.
- $\mathcal{M}$ is *binary* iff it is representable over the binary field.
- $\mathcal{M}$ is *regular* iff it is representable over arbitrary field.
- $\mathcal{M}$ is *even* iff all feasible sets have the same parity.

- $\mathcal{M}$ is even binary iff $\mathcal{M} = \mathcal{M}(A)\Delta X$ for some $E \times E$ skew-symmetric matrix A over the binary field and $X \subseteq E$.
- For an even binary delta-matroid $\mathcal{M}$, the graph G whose adajcency matrix is A is called a *fundamental graph* of $\mathcal{M}$.

Claim: Graphic delta-matroids are even binary.



- $\mathcal{M}(G, T) = \mathcal{M}((G', T')/e/f)$
- $\mathcal{M}(G'/e/f, T')$ is representable by the adjacency matrix of $L(G')$.

Facts (Bouchet)

- Every minor of a binary delta-matroid is binary.
- Every minor of a even delta-matroid is even.

So, we may assume that $T = V(G)$.

Line graphs and graphic delta-matroids

Let A be the adjacency matrix of the line graph $L(G)$ over the binary field.
Let $A[X]$ be the principal submatrix of A indexed by $X \subseteq E(G)$.

Theorem (Kishi and Uetake, IEEE Trans. Circuit Theory, 1969)

Let G be simple. Then $A[X]$ is nonsingular (over the binary field) iff

1. X induces no cycle and
2. each comp. of the subgraph $(L(G), X)$ has odd #vertices.

Key Idea: If $I = \text{vertices} \overset{\text{edges}}{\left(\quad \right)}$ is the incidence graph, then $A = I^t I$.
Assume that G is connected. If A is nonsingular, then
$|V(G)| - 1 \leq |E(G)| = \text{rank}(A) \leq \text{rank}(I) \leq |V(G)| - 1$.
Moreover, $\text{rank}(A)$ is always even.

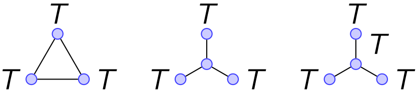
Theorem: Graphic delta-matroids are even binary.

$\{$Graphic delta-matroids$\}\cap\{$Twisted matroids$\}$=? (I)

Observation

A even binary delta-matroid is a twisted matroid iff
it has no MK_3-minor; $MK_3 = ([3], \{\emptyset, \{1,2\}, \{2,3\}, \{1,3\}\})$.

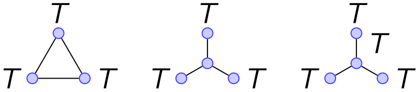
MK_3 has exactly 3 representations as a graphic delta-matroid:



$\mathfrak{G}(G, T)\Delta X$ is a twisted matroid iff
the graft (G, T) has no minor isomorphic to the three grafts.

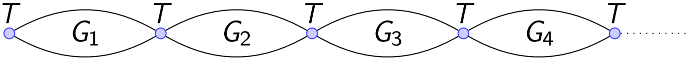
$\{$Graphic delta-matroids$\}\cap\{$Twisted matroids$\}$=? (II)

If G is connected, $|T| \geq 3$, and (G, T) has no minors isomorphic to any of



,

then (G, T) has the following structure.



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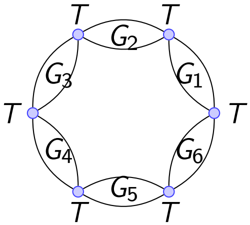
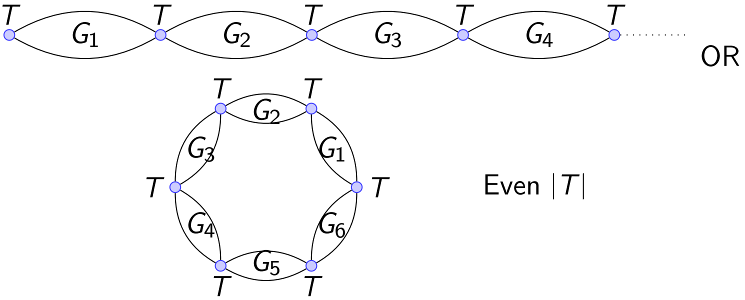


Even $|T|$

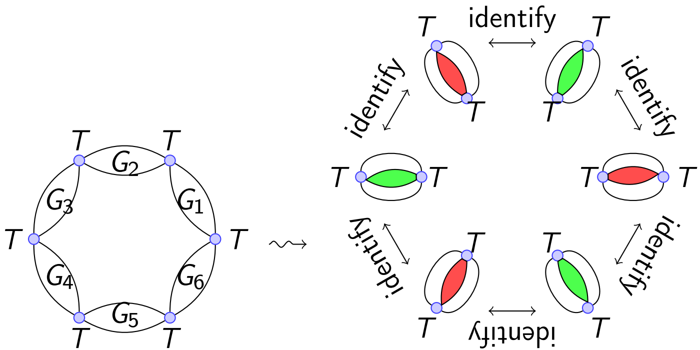
{Graphic delta-matroids}$\cap$ {Twisted matroids}=? (II)



A graphic delta-matroid is a twisted matroid iff

1. it is a twisted graphic matroid or
2. it is a twisting of the 2-sum of graphic matroids and cographic matroids in some special way (details omitted).
3. or ...

{Graphic delta-matroids}$\cap$ {Twisted matroids}=? (III)



red: cographic matroids (dual),

green: graphic matroids.

Branch-width & Rank-width

Let $\mathcal{M} = \mathfrak{G}(G, T)\Delta X$ be a graphic delta-matroid with a fundamental graph Γ.

Theorem

- Rankwidth(Γ) = Branchwidth(cycle matroid of G) $-$ (0 or 1 or 2).
- Rankwidth(Γ) = Branchwidth(G) $-$ (0 or 1 or 2), if G is not a forest.

Line graphs of large rank-width must contain a pivot-minor isomorphic to a fixed circle graph.

Sketch: Let Γ be a fundamental graph of $\mathfrak{G}(G, T)$. Suppose that Γ has sufficiently large rank-width. We claim that if (G, T) has a grid minor with no T vertices. Since G has large branch-width, Robertson and Seymour's theorem implies that G has a large grid minor. By partitioning the grid into 3×3 blocks and taking an appropriate minor, we get a grid with no T-vertices.

Testing a planar matroid minor

Let $\mathcal{N}$ be a fixed twisted planar matroid.

Input: A graphic delta-matroid $\mathcal{M}$ given by its fundamental graph Γ.
Output: Yes if $\mathcal{M}$ has a minor isomorphic to $\mathcal{N}$.

Sketch of the Algorithm: Let $\mathcal{M} = \mathfrak{G}(G, T)$ for a graph G and $T \subseteq V(G)$.
If Γ has large rank-width, then answer Yes.
If Γ has small rank-width, then use dynamic programming.

Questions

- Is this new? Any in the literatures?
- A graphic delta-matorid can have several distinct representations. Is it possible to find a generalization of Whitney's theorem on graphs with the same cycle spaces?
- Characterize graphic delta-matroids in terms of a list of excluded minors.
- How to recognize graphic delta-matroids?
- Let $\mathcal{N}$ be a fixed binary delta-matroid.
  Is it possible to decide whether an input binary delta-matroid has a $\mathcal{N}$-minor?
  Is it possible to decide whether an input graphic delta-matroid has a $\mathcal{N}$-minor?

Thank you for your attention!

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