CIRCLE GRAPH OBSTRUCTIONS UNDER PIVOTING

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Abstract. A circle graph is the intersection graph of a set of chords of a circle. The class of circle graphs is closed under pivot-minors. We determine the pivot-minor-minimal non-circle-graphs; there 15 obstructions. These obstructions are found, by computer search, as a corollary to Bouchet's characterization of circle graphs under local complementation. Our characterization generalizes Kuratowski's Theorem.

1. Introduction

The class of circle graphs is closed with respect to vertex-minors and hence also pivot-minors. (Definitions are postponed until Section 2.) Bouchet [5] gave the following characterization of circle graphs; the graphs W_5, F_7, and W_7 are defined in Figure 1.

Theorem 1.1 (Bouchet)**.** *A graph is a circle graph if and only it has no vertex-minor that is isomorphic to W_5, F_7, or W_7.*

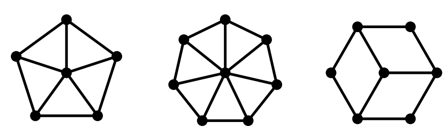

Figure 1. W_5, W_7, and F_7: Excluded vertex-minors for circle graphs.

As a corollary to Bouchet's theorem we prove the following result.

Theorem 1.2. *A graph is a circle graph if and only it has no pivot-minor that is isomorphic to any of the graphs depicted in Figure 2.*

In addition we prove the following related theorem.

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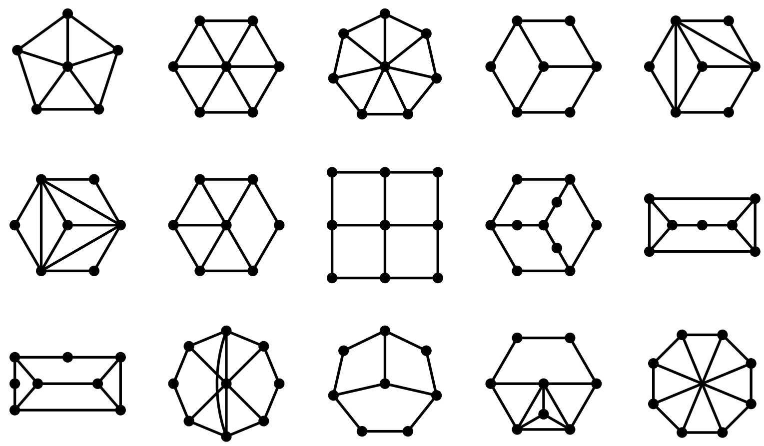

FIGURE 2. Excluded pivot-minors for circle graphs

Theorem 1.3. *Let $\mathcal{G}$ be a class of simple graphs closed under vertex-minors. If the excluded vertex-minors for $\mathcal{G}$ each have at most k vertices, then the excluded pivot-minors for $\mathcal{G}$ each have at most $2^k - 1$ vertices.*

The bounds in Theorem 1.3 are not tight enough to be of practical use in proving Theorem 1.2. We show that the excluded pivot-minors can be determined from the excluded vertex-minors by a simple inductive search. Before we discuss this method further, we will briefly discuss the motivation.

De Fraysseix [7] showed that bipartite circle graphs are fundamental graphs of planar graphs. It is then straightforward to show that Theorem 1.2 is a generalization of Kuratowski's Theorem. In fact, Theorem 1.2 applied to bipartite circle graphs is equivalent to the following result, initially due to Tutte [11]: *a binary matroid is the cycle matroid of a planar graph if and only if it does not contain a minor isomorphic to F_7, $M(K_5)$, $M(K_{3,3})$, or to the dual of any of these matroids.* The fundamental graphs of matroids are bipartite and it is straightforward to verify that a pivot-minor of a fundamental graph of a binary matroid (or graph) is a fundamental graph of a minor of the given matroid (or graph). Finally, the graphs H_1, H_2, and F_7 are fundamental graphs of $K_{3,3}$, K_5, and F_7 respectively. (See Figure 3 for drawings of H_1 and H_2.)

The primary motivation for Theorem 1.2 is as a step towards characterizing PU-orientable graphs (defined in Section 2). Bipartite PU-orientable graphs are the fundamental graphs of regular matroids. Seymour's decomposition theorem [10] provides a good characterization

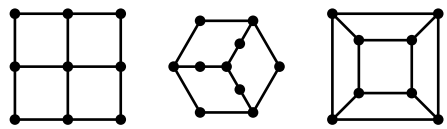

FIGURE 3. H_1, H_2, and Q_3

and a recognition algorithm for regular matroids and we hope to obtain similar results for PU-orientable graphs. Bouchet [2] proved that circle graphs admit PU-orientations and we hope that the class of circle graphs will play a central role in a decomposition theorem for PU-orientable graphs. The class of PU-orientable graphs is closed under pivot-minors but not under vertex-minors, and hence it is desirable to have the excluded pivot minor for the class of circle graphs. Although the class of PU-orientable graphs is not closed under local complementation, Bouchet's theorem does imply the following curious connection between PU-orientability and circle graphs: *a graph is a circle graph if and only if every locally equivalent graph is PU-orientable.*

We prove Theorem 1.2 by studying the graphs that are pivot-minor-minimal while containing a vertex-minor isomorphic to one of W_5, F_7, or W_7. We require the following two lemmas that are proved in Section 3. The proofs are direct but inelegant. These facts are transparent in the context of isotropic systems; see Bouchet [3]. However, the direct proofs are shorter than the requisite introduction to isotropic systems.

Lemma 1.4 (Bouchet [3, (9,2)]). *Let H be a vertex-minor of a simple graph G, let $v \in V(G) - V(H)$, and let w be a neighbour of v. Then H is a vertex-minor of one of the graphs $G - v$, $(G * v) - v$, and $(G \times vw) - v$.*

Note that the vertex w in Lemma 1.4 is an arbitrary neighbour of v. Indeed, if w_1 and w_2 are neighbours of v, then $G \times vw_1 = (G \times vw_2) \times w_1w_2$; see [8, Proposition 2.5]. (This fact is elementary and has been known for more than 20 years, but we could not find an earlier reference.) Therefore $(G \times vw_1) - v$ is pivot-equivalent to $(G \times vw_2) - v$. We let G/v denote the graph $(G \times vw) - v$ for some neighbour w of v; if v has no neighbours then we let G/v denote $G - v$. Thus G/v is well defined up to pivot-equivalence and, hence, also up to local-equivalence.

Let H be a graph. A graph G is called *H-unique* if G contains H as a vertex-minor and, for each vertex $v \in V(G)$, at most one of the graphs $G - v$, $(G * v) - v$, and G/v has a vertex-minor isomorphic to H. Note that if G is a graph that is pivot-minor-minimal with the property that it has a vertex-minor isomorphic to H, then G is H-unique.

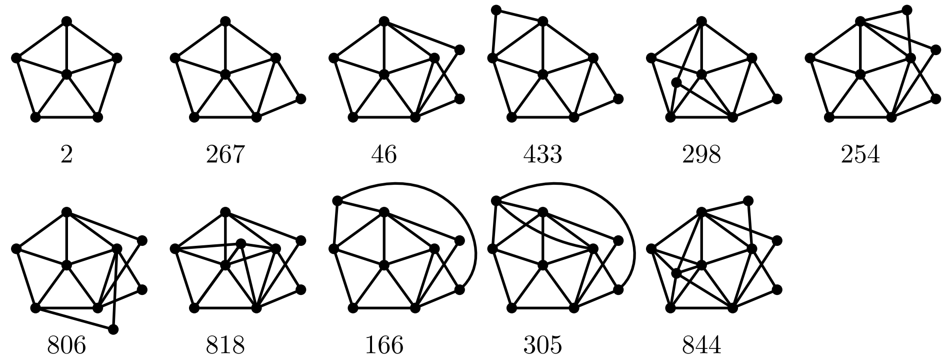

FIGURE 4. W_5-unique graphs

Lemma 1.5. *Let G be an H-unique graph and let G' be a vertex-minor of G that contains H as a vertex-minor. Then G' is H-unique.*

As an immediate corollary to Lemma 1.5 we obtain the following result.

Lemma 1.6. *Let H be a simple graph and let $k > |V(H)|$. If there is no H-unique graph on k vertices, then every H-unique graph has at most $k-1$ vertices.*

Using Lemma 1.6 and computer search we prove the following three results. The computation takes less than 3 minutes on a SUN Workstation; we use the package NAUTY for isomorphism-testing.

Lemma 1.7. *Every W_5-unique graph is isomorphic to one of the* 11 *graphs depicted in Figure 4.*

Lemma 1.8. *If G is W_7-unique then either G is locally equivalent to W_7 or G has a vertex-minor isomorphic to W_5.*

Lemma 1.9. *If G is F_7-unique then either G is locally equivalent to F_7 or Q_3, or G has a vertex-minor isomorphic to W_5. (The graph Q_3 is depicted in Figure 3.)*

Theorem 1.1 and the above lemmas imply that every pivot-minor-minimal non-circle-graph is locally-equivalent to W_7, F_7, Q_3, or to one of the 11 graphs depicted in Figure 4. The number below each of the graphs is the number of pair-wise non-isomorphic graphs that are locally equivalent to it; in total there are 4239 such graphs. In addition, there are $9 + 22 + 4$ graphs locally equivalent to F_7, W_7, and Q_3. To prove Theorem 1.2, it suffices to check which of these 4274 graphs is a pivot-minor-minimal non-circle-graph. This is also done by computer and takes less than 3 minutes. This includes 2.5 minutes to generate the 4274 graphs, 3 seconds to generate all circle graphs up to 9 vertices, and 2 seconds to test which of the 4274 graphs is a pivot-minor-minimal non-circle-graph.

In the context of delta-matroids, Theorem 1.2 is an excluded-minor characterization for the class of *even* eulerian delta-matroids. Using Lemmas 1.7, 1.8, and 1.9 one can prove that all excluded-minors for the class of eulerian delta-matroids have at most 10 elements. We discuss this further in Section 4.

We conclude the introduction by proving the following theorem that immediately implies Theorem 1.3.

Theorem 1.10. *Let H be a simple graph with $|V(H)| = k$. Then ever H-unique graph has at most $2^k - 1$ vertices.*

Proof. Let G be an H-unique graph. Up to local equivalence we may assume that H is an induced subgraph of G.

Consider any vertex $v \in V(G) - V(H)$. Let G_v denote the subgraph of G induced by the vertex set $V(H) \cup \{v\}$. By Lemma 1.5, G_v is H-unique. Note that $G_v - v = H$ and, hence, $(G_v * v) - v \neq H$. Therefore v has at least two neighbours in $V(H)$.

Now consider any two distinct vertices $u, v \in V(G) - V(H)$. Let G_{uv} denote the subgraph of G induced by the vertex set $V(H) \cup \{u, v\}$. By Lemma 1.5, G_{uv} is H-unique. Note that $G_{uv} - u - v = H$. Suppose that u and v both have the same neighbours among $V(H)$. If u and v are adjacent, then $G_{uv} \times uv = G_{uv}$ and, hence, both $G_{uv} - u$ and G_{uv}/u have H as a vertex-minor. If u and v are not adjacent, then $G_{uv} * u * v = G_{uv}$ and, hence, both $G_{uv} - u$ and $(G_{uv} * u) - u$ have H as a vertex-minor. In either case we contradict the fact that G_{uv} is H-unique, and hence u and v have distinct neighbours among $V(H)$.

In summary, each vertex in $V(G) - V(H)$ has at least 2 neighbours in $V(H)$ and no two vertices in $V(G) - V(H)$ have the same neighbours in $V(H)$. Therefore $|V(G)| \leq |V(H)| + 2^k - (k+1) = 2^k - 1$. □

We remark that we can slightly improve the above bound to $2^k - 2k - 1$ when the graph H has minimum degree at least 2 and H has no "twin" vertices. Two distinct vertices $u, v \in V(H)$ are *twins* if $N_H(u) - \{v\} = N_H(v) - \{u\}$); here $N_H(v)$ denotes the set of all neighbours of v.

2. Definitions

We assume that readers are familiar with elementary definitions in matroid theory including cycle matroids, binary matroids, regular matroids, duality, and minors; see Oxley [9]. However, all references to matroids are peripheral to the main results in the paper.

All graphs in this paper are finite. The following definitions are mostly well-known.

Circle graphs. A *chord* of a circle is a straight line segment whose two ends lie on the circle. Let V be a finite set of chords of a circle; the *intersection graph* of V is the simple graph $G = (V, E)$ where $uv \in E$ if and only if the chords u and v intersect. A *circle graph* is the intersection graph of a set of chords of a circle.

PU-orientable graphs. A *principally unimodular matrix* is a square matrix over the reals such that each non-singular principal submatrix has determinant ± 1. Let $G = (V, E)$ be an orientation of a simple graph. The *signed adjacency matrix* of G is the $V \times V$ matrix (a_{uv}) where $a_{uv} = 1$ when $uv \in E$, $a_{uv} = -1$ when $vu \in E$, and $a_{uv} = 0$ otherwise. A simple graph G is *PU-orientable* if it admits an orientation whose signed adjacency matrix is principally unimodular.

Local complementation and vertex-minors. Let v be a vertex of a simple graph G. The graph $G*v$ is the simple graph obtained from G by applying *local complementation* at v; that is, if u and w are distinct neighbours of v in G, then uw is an edge in exactly one of G and $G*v$. If G' can be obtained by a sequence of local complementations from G, then we say that G and G' are *locally equivalent*. A *vertex-minor* of G is an induced subgraph of any graph that is locally equivalent to G. (An *induced* subgraph is one that is obtained by vertex deletion.)

Pivot-minors. Let uv be an edge of a simple graph G. Let $G \times uv = G*u*v*u$; this operation is referred to as *pivoting*. It is straightforward to verify that $G*u*v*u = G*v*u*v$ and, hence, that pivoting is well defined. If G' can be obtained by a sequence of pivots from G, the we say that G and G' are *pivot equivalent*. A *pivot-minor* of G is an induced subgraph of any graph that is pivot equivalent to G.

Fundamental graphs. Let B be a basis of a matroid M. The *fundamental graph* of M with respect to B is the graph with vertex set $E(M)$ and edges uv where $u \in B$, $v \in E(M) - B$, and $(B - \{u\}) \cup \{v\}$ is a basis of M. Note that the fundamental graph is bipartite. A *fundamental graph* of a graph G is a fundamental graph of the cycle matroid of G.

3. Vertex-minors

In this section we prove Lemmas 1.4 and 1.5. As noted in the introduction, these results are easy in the context of isotropic systems [3], but the direct proofs given here avoid a lengthy introduction to isotropic systems. We start by proving the following key lemma.

Lemma 3.1. *Let $G = (V, E)$ be a simple graph and let $v, w \in V$.*

(1) *If $v \neq w$ and $vw \notin E$, then $(G * w) - v$, $(G * w * v) - v$, and $(G * w)/v$ are locally equivalent to $G - v$, $G * v - v$, and G/v respectively.*
(2) *If $v \neq w$ and $uv \in E$, then $(G * w) - v$, $(G * w * v) - v$, and $(G * w)/v$ are locally equivalent to $G - v$, G/v, and $(G * v) - v$ respectively.*
(3) *If $v = w$, then $(G * w) - v$, $((G * w) * v) - v$, and $(G * w)/v$ are locally equivalent to $(G * v) - v$, $G - v$, and G/v respectively.*

Proof. We first consider the case that $v \neq w$. It is obvious that $(G * w) - v = (G - v) * w$ and hence that $(G * w) - v$ is locally equivalent to $G - v$.

Suppose that $vw \in E$. Note that $(G*w*v)-v = (G*w*v*w*w)-v = ((G \times vw) - v) * w = (G/v) * w$ and hence $(G * w * v) - v$ is locally equivalent to G/v. Similarly, $(G * w)/v = ((G * w) \times vw) - v = (G*w*w*v*w)-v = ((G*v)-v)*w$ and hence $((G*w)/v$ is locally equivalent to $(G * v) - v$.

Now suppose that $vw \notin E$. Note that $(G * w * v) - v = (G * v * w) - v = ((G * v) - v) * w$ and hence $(G * w * v) - v$ is locally equivalent to $(G * v) - v$. Let u be a neighbour of v. If $uw \notin E$, then $((G*w)\times uv)-v = ((G\times uv)*w)-v = (G/v)*w$ and hence $((G*w)/v$ is locally equivalent to G/v. Hence we may assume that $uw \in E$. Now $(G * w)/v = (G * w * u * v * u) - v$ and $(G * w * u * v * u) - v$ is locally equivalent to $(G * w * u * v * w) - v = (G * w * u * w * w * v * w) - v = (G \times uw \times vw) - v = (G \times uv) - v = G/v$, as required.

Now suppose that $v = w$. Then $(G * w) - v = (G * v) - v$ and $(G * w * v) - v = G - v$. Moreover, if $uv \in E$, then $(G * w)/v = ((G*v)\times uv)-v = (G*v*v*u*v)-v = (G*u*v)-v = ((G\times uv)-v)*u$ and hence $(G * w) - v$ is locally equivalent to G/v. □

We now prove Lemma 1.4 which we restate here for convenience. This lemma appeared in [3, (9.2)].

Lemma 3.2. *Let H be a vertex-minor of a simple graph G and let $v \in V(G) - V(H)$. Then H is a vertex-minor of one of the graphs $G - v$, $(G * v) - v$, and G/v.*

Proof. If H is a vertex-minor of G, then there is a graph G' that is locally equivalent to G such that H is an induced subgraph of G. Now $G' - v$ contains H as a vertex-minor. Since G is locally equivalent to G' the result follows by Lemma 3.1. □

Finally we now prove Lemma 1.5 which again we restate for convenience.

Lemma 3.3. *Let G be an H-unique graph and let G' be a vertex-minor of G that contains H as a vertex-minor. Then G' is H-unique.*

Proof. By Lemma 3.1 every graph that is locally equivalent to G is H-unique. Then, inductively, it suffices to consider the case that $G' = G-v$ for some vertex v. If $G-v$ is not H-unique, then there is a vertex $w \neq v$ such that at least two of $(G - v) - w$, $((G - v) * w) - w$, and $(G-v)/w$ contain H as a vertex-minor. But then at least two of $G-w$, $(G * w) - w$, and G/w contain H as a vertex-minor, contradicting the fact that G is H-unique. □

4. Eulerian delta-matroids

In this section we prove the following theorem.

Theorem 4.1. *The excluded minors for the class of eulerian delta-matroids have at most* 10 *elements.*

The class of eulerian delta-matroids is contained in the class of binary delta-matroids. Bouchet and Duchamp [6] determined the excluded minors for the class of binary delta-matroids; the the largest of these has four elements. Then to prove Theorem 4.1, it suffices to consider binary delta-matroids. We give a terse introduction to binary delta-matroids and to eulerian delta matroids, for more detail the reader is referred to Bouchet [1, 4].

Delta-matroids and minors. For sets X and Y, we let $X\Delta Y$ denote the symmetric difference of X and Y. A *delta-matroid* is a pair $M = (V, \mathcal{F})$ of a finite set V and a nonempty set $\mathcal{F}$ of subsets of V, satisfying the *symmetric exchange axiom*: if $A, B \in \mathcal{F}$ and $x \in A\Delta B$, then there is $y \in A\Delta B$ such that $A\Delta\{x, y\} \in \mathcal{F}$. The sets in $\mathcal{F}$ are called *feasible sets* of M. For $X \subseteq V$, we abuse notation be letting $M\Delta X$ denote the set-system $(V, \mathcal{F}')$ where $\mathcal{F}' = \{F\Delta X : F \in \mathcal{F}\}$. It is straightforward to verify that $M\Delta X$ is a delta-matroid. Now let $M \setminus X$ denote the set-system $(V - X, \mathcal{F}'')$ where $\mathcal{F}'' = \{F \subseteq V - X : F \in \mathcal{F}\}$. If $M \setminus X$ has at lease one feasible set, then $M \setminus X$ is a delta-matroid. For any sets $X, Y \subseteq V$, if $(M\Delta X) \setminus Y$ has a feasible set, then we call it a *minor* of M. Two delta-matroids M_1, M_2 are *equivalent* if $M_1 = M_2\Delta X$ for some set X. A delta-matroid is *even* if its feasible sets either all have even cardinality or all have odd cardinality.

Binary delta-matroids. Let A be a symmetric $V \times V$ matrix over GF(2). For $X \subseteq V$, we let $A[X]$ denote the principal submatrix of A induced by X. A subset X of V is called *feasible* if $A[X]$ is non-singular. By convention, $A[\emptyset]$ is non-singular. We let $\mathcal{F}_A$ denote the set of all feasible sets and let $\mathrm{DM}(A) = (V, \mathcal{F}_A)$. Bouchet [4] proved that $\mathrm{DM}(A)$ is indeed a delta-matroid. A delta-matroid is *binary* if it is equivalent to $\mathrm{DM}(A)$ for some symmetric matrix A. We remark that $\mathrm{DM}(A)$ is even if and only if the diagonal of A is zero.

Eulerian delta-matroids. Let $G = (V, E)$ be a graph and let $X \subseteq V$. Let $A(G, X)$ denote the symmetric $V \times V$ matrix obtained from the adjacency matrix of G by changing the diagonal entries indexed by X from 0 to 1. Thus any symmetric binary matrix can be written as $A(G, X)$ for the appropriate choice of G and X. The binary delta-matroid $\mathrm{DM}(A(G, X))\Delta Y$ is *eulerian* if and only if G is a circle graph. This is the most convenient definition for the purpose of proving Theorem 4.1, but eulerian delta-matroids arise more naturally in relation to euler tours in a connected 4-regular graph; see Bouchet [1].

Bouchet and Duchamp [6] proved that the class of binary delta-matroids is minor-closed. The class of eulerian delta-matroids is also minor-closed, because the class of circle graphs is closed under local complementation.

If $v \in X$, then it is straightforward to prove that

$$\mathrm{DM}(A(G, X))\Delta\{v\} = \mathrm{DM}(A(G * v, X\Delta N_G(v)).$$

Similarly, if $uv \in E$ and $u, v \notin X$, then

$$\mathrm{DM}(A(G, X))\Delta\{u, v\} = \mathrm{DM}(A(G \times uv, X)).$$

The operations $A(G, X) \to A(G * x, X\Delta N_G(v))$, for $v \in X$, and $A(G, X) \to A(G \times uv, X)$, for $uv \in E$ and $u, v \notin X$, are referred to as *elementary pivots*. If $\mathrm{DM}(A(G_1, X_1)) = \mathrm{DM}(A(G_2, X_2))\Delta Y$, then we can obtain $A(G_2, X_2)$ from $A(G_1, X_1)$ via a sequence of elementary pivots, implied by the uniqueness of binary representation for binary delta-matroids; see Bouchet and Duchamp [6, Property 3.1].

Lemma 4.2. *Let $G = (V, E)$ be a graph, let $X \subseteq V$, and let $v \in V$. If $\mathrm{DM}(A(G, X))$ is an excluded minor for the class of eulerian delta-matroids, then at least two of the graphs $G - v$, $(G * v) - v$, and G/v are circle graphs.*

Proof. Suppose that $v \in X$. Then $\mathrm{DM}(A(G,X)) \setminus \{v\}$ and $(\mathrm{DM}(A(G,X))\Delta\{v\}) \setminus \{v\}$ are both eulerian. Thus $G - v$ and $(G * v) - v$ are both circle graphs, as required. Now suppose that $v \notin X$. Since $G - v$ is a circle graph but G is not, $N_G(v) \neq \emptyset$; let $w \in N_G(v)$. Now suppose that $w \notin X$. Then $\mathrm{DM}(A(G,X)) \setminus \{v\}$ and $(\mathrm{DM}(A(G,X))\Delta\{v,w\}) \setminus \{v\}$ are both eulerian. Thus $G - v$ and G/v are both circle graphs, as required. Finally suppose that $w \in X$. Now $\mathrm{DM}(A(G,X))\Delta\{w\} = \mathrm{DM}(A(G*w, X\Delta N_G(w)))$ is an excluded minor for the class of eulerian delta-matroids and $v \in X\Delta N_G(w)$. Then, by the first case in the proof, $(G*w)-v$ and $((G*w)*v)-v$ are both circle graphs. So, by Lemma 3.1, $G - v$ and G/v are both circle graphs. □

Lemma 4.2 and Theorem 1.1 imply that, if $\mathrm{DM}(A(G,X))$ is an excluded minor for the class of eulerian delta-matroids, then G is W_5-, W_7-, or F_7-unique. Then Theorem 4.1 follows immediately from Lemmas 1.7, 1.8 and 1.9.

By computer search, we found 166 non-equivalent binary excluded minors for the class of eulerian delta-matroids. Combined with the excluded minors for the class of binary delta-matroids, we conclude that there are exactly 171 excluded minors for the class of eulerian delta-matroids. This computation takes 14 minutes if the list of all W_5-unique graphs is given.

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