Rank-width is less than or equal to branch-width*

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Abstract

We prove that the rank-width of the incidence graph of a graph G is either equal to or exactly one less than the branch-width of G, unless the maximum degree of G is 0 or 1. This implies that rank-width of a graph is less than or equal to branch-width of the graph unless the branch-width is 0. Moreover, this inequality is tight.

Keywords: rank-width, branch-width, tree-width, clique-width, line graphs, incidence graphs.

1 Introduction

In this paper, graphs have no loops and no parallel edges. The *incidence graph* $I(G)$ of a graph $G = (V, E)$ is a graph on vertices in $V \cup E$ such that $x, y \in V \cup E$ are adjacent in $I(G)$ if one of x, y is a vertex of G, the other is an edge of G, and x is incident with y in G. In other words, $I(G)$ is the graph obtained from G by subdividing every edge exactly once.

We prove that the rank-width of a graph G is less than or equal to the branch-width of G, unless the branch-width of G is 0. Definitions of branch-width and rank-width are in Section 2. To show this, we prove a stronger theorem stating that the rank-width of $I(G)$ is equal to or exactly one less than the branch-width of G, or the branch-width of G is 0. Another corollary of this theorem is that the rank-width of the line graph of a graph is less than or equal to the branch-width of the graph.

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There are related works on this topic. In this paper we do not define clique-width [2] but rank-width is related to clique-width. Oum and Seymour [8] showed that $\mathrm{rw}(G) \leq \mathrm{cw}(G) \leq 2^{\mathrm{rw}(G)+1} - 1$, where $\mathrm{cw}(G), \mathrm{rw}(G)$ denote clique-width and rank-width respectively. Let $\mathrm{tw}(G)$ be the tree-width of G. Courcelle and Olariu [2] showed that clique-width is at most $2^{\mathrm{tw}(G)+1} + 1$ and later Corneil and Rotices [1] proved that clique-width is at most $3 \cdot 2^{\mathrm{tw}(G)-1}$. The previous results on clique-width imply that rank-width is smaller than or equal to $3 \cdot 2^{\mathrm{tw}(G)-1}$. Kanté [4] showed that the rank-width is at most $4\,\mathrm{tw}(G) + 2$. In this paper, we prove that rank-width is smaller than or equal to $\mathrm{tw}(G) + 1$.

2 Preliminaries

Branch-width [9] and rank-width [8] of graphs are defined in a similar way. We will describe more general branch-width of symmetric submodular functions, and then define branch-width of graphs and rank-width of graphs in terms of branch-width of symmetric submodular functions. For a finite set V, let 2^V be the set of subsets of V. Let $\mathbb{Z}$ be the set of integers. A function $f : 2^V \to \mathbb{Z}$ is *symmetric* if $f(X) = f(V \setminus X)$ for all $X \subseteq V$ and *submodular* if $f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$ for all $X, Y \subseteq V$. A tree is *subcubic* if every vertex has degree 1 or 3. A *branch-decomposition* of a symmetric submodular function $f : 2^V \to \mathbb{Z}$ is a pair (T, τ) of a subcubic tree T and a bijection $\tau : V \to \{t : t \text{ is a leaf of } T\}$. The *width* of an edge $e \in E(T)$ in a branch-decomposition (T, τ) is defined as $f(X_e)$ where (X_e, Y_e) is a partition of V from a partition of leaves of T induced by deleting e from T. The *width* of a branch-decomposition (T, τ) is the maximum width of all edges of T. The *branch-width* of f, denoted by $\mathrm{bw}(f)$, is the minimum width of all branch-decompositions of f. (If $|V| < 2$, then f has no branch-decomposition. In this case, we assume that $\mathrm{bw}(f) = f(\emptyset)$.)

Please be warned that in the above definition, V can be any finite set, not just the set of vertices of graphs. We define branch-width of a graph $G = (V, E)$ as branch-width of a certain symmetric submodular function η_G on the *set E of edges* as follows. For a subset X of E, let $\mathrm{mid}(X)$ be the set of vertices that are incident with both an edge in X and another edge in $E \setminus X$. Let $\eta_G(X) = |\mathrm{mid}(X)|$. Then $\eta : 2^E \to \mathbb{Z}$ is a symmetric submodular function and so the branch-width of η_G is well-defined. The *branch-width* $\mathrm{bw}(G)$ of a graph G is defined as the branch-width of η_G.

The rank-width is defined by the *cut-rank* function $\rho_G : 2^V \to \mathbb{Z}$ of a graph $G = (V, E)$. For a subset X of V, consider a 0-1 matrix M_X over the binary field GF(2), in which the number of rows is $|X|$ (so rows are indexed by X), the number of columns is $|V \setminus X|$, (so columns are indexed by $V \setminus X$), and the entry is 1 if and only if the corresponding vertices of the row and the column are adjacent. Let $\rho_G(X) = \mathrm{rank}(M_X)$ where rank is the matrix rank function. Then ρ_G is symmetric and submodular [8]. The *rank-width* $\mathrm{rw}(G)$ of a graph G is defined as the branch-width of ρ_G.

We will need a definition of matroid branch-width. A *matroid* is a pair $M = (E, \mathrm{r})$ of a finite set E and a rank function $\mathrm{r} : 2^E \to \mathbb{Z}$ satisfying the following axioms: $\mathrm{r}(\emptyset) = 0$, $\mathrm{r}(X) \leq |X|$ for all $X \subseteq E$, $\mathrm{r}(X) \leq \mathrm{r}(Y)$ if $X \subseteq Y$, and r is submodular. The *connectivity function* of a matroid $M = (E, \mathrm{r})$ is $\lambda_M(X) = \mathrm{r}(X) + \mathrm{r}(E \setminus X) - \mathrm{r}(E) + 1$. It is easy to see that λ_M is symmetric and submodular. The *branch-width* $\operatorname{bw}(M)$ of a matroid M is defined as the branch-width of λ_M.

Given a matrix A over GF(2) whose columns are indexed by E, let $\mathrm{r}_A(X) = \operatorname{rank} A_X$ where A_X is the submatrix of A obtained by removing columns not in X. Then r_A satisfies the matroid rank axiom and therefore $M = (E, \mathrm{r}_A)$ is a matroid. A matroid that has such a representation is called a *binary matroid*. Since the elementary row operations do not change r_A, every binary matroid has the *standard representation* A in which A_B is an identity matrix for some $B \subseteq E$. The *fundamental graph* $F(M)$ of a binary matroid with respect to the above standard representation is a bipartite graph on vertices E with a bipartition $(B, V \setminus B)$ such that $x \in B$ and $y \in V \setminus B$ are adjacent if and only if the row having 1 in the column vector of x in A has 1 in the column vector of y in A. Oum [6] showed the following.

Lemma 1 (Oum [6])**.** *The branch-width of a binary matroid M is exactly one more than the rank-width of its fundamental graph.*

The *cycle matroid* $M(G)$ of a graph $G = (V, E)$ is a binary matroid having the following standard representation: Let B be an edge set of the spanning forest F of G. Let A be a 0-1 matrix $(a_{ij})_{i \in B, j \in V}$ such that $a_{ij} = 1$ if and only if $i = j \in B$ or the fundamental circuit of $j \notin B$ with respect to F contains i. Then A is a (standard) representation of $M(G)$. It is well-known that $\lambda_{M(G)}(X) \leq \eta_G(X)$ for all nonempty $X \subset E(G)$. This implies that the branch-width of $M(G)$ is at most the branch-width of G if G has at least two edges. The following theorem was shown by Hicks and McMurray [3] and Mazoit and Thomassé [5] independently.

Theorem 2 (Hicks and McMurray [3]; Mazoit and Thomassé [5])**.** *The branch-width of a 2-connected graph G is equal to the branch-width of the cycle matroid $M(G)$.*

3 Main Theorem

Now let us prove the main theorem.

Theorem 3. *For a graph G, $\operatorname{rw}(I(G))$ is equal to $\operatorname{bw}(G) - 1$ or $\operatorname{bw}(G)$ unless the maximum degree of G is 0 or 1. (If the maximum degree of G is 0 or 1, then $\operatorname{rw}(I(G)) \leq 1$ and $\operatorname{bw}(G) = 0$.)*

Proof. This proof will work even if G has parallel edges. (If G has loops, then $I(G)$ has parallel edges, but $\operatorname{rw}(I(G))$ is defined only if $I(G)$ has no parallel edges and no loops.) Without loss of generality, we may assume that G is connected and has at least two vertices. If $|V(G)| = 2$ then $\operatorname{rw}(I(G)) = 1$ and $\operatorname{bw}(G) \leq |V(G)| = 2$.

Now we assume that $|V(G)| > 2$, $|E(G)| > 1$, and $\operatorname{bw}(G) \geq 1$ and so G admits rank-decompositions and branch-decompositions. Let us construct a graph $\hat{G}$ by adding a new vertex v to G that is adjacent to all vertices of G. Then $\hat{G}$ is 2-connected. Let F be a spanning tree of $\hat{G}$ consisting of all edges incident with v and $B = E(F)$. It is easy to see that the fundamental graph of $M(\hat{G})$ with respect to B is $I(G)$. Therefore by Lemma 1, $\mathrm{rw}(I(G)) = \mathrm{bw}(M(\hat{G})) - 1$. By Theorem 2, $\mathrm{bw}(M(\hat{G})) = \mathrm{bw}(\hat{G})$.

Now it is enough to show that $\mathrm{bw}(\hat{G}) = \mathrm{bw}(G)$ or $\mathrm{bw}(G)+1$ when G is connected and $|V(G)| > 2$. (This is false when G has a single edge; $\mathrm{bw}(G) = 0$ but $\mathrm{bw}(\hat{G}) = 2$.) Since G is a minor of $\hat{G}$, $\mathrm{bw}(G) \le \mathrm{bw}(\hat{G})$. Let (T, τ) be a branch-decomposition of G of width $\mathrm{bw}(G)$. For every vertex w of G, pick an edge f_w of G incident with w. Then let e_w be the unique edge of T incident to the leaf $\tau(f_w)$. We subdivide e_w and attach a new leaf corresponding to the edge vw of $\hat{G}$. (If an edge xy of G is chosen twice and exactly one of its ends, say x, has degree 1, then we apply the above operation for vy first and then apply for vx. This is to avoid having $\{xy, vy\}$ in one side of the branch-decomposition because $\eta_{\hat{G}}(\{xy, vy\}) = 3$.) It is easy to see that the obtained branch-decomposition of $\hat{G}$ has width at most $\mathrm{bw}(G)+1$. (Notice that $\mathrm{bw}(G) \ge 1$ and therefore $\mathrm{bw}(G) + 1 \ge 2$.) Consequently $\mathrm{bw}(\hat{G}) \le \mathrm{bw}(G) + 1$. This proves the theorem. □

If we use an easy inequality $\mathrm{bw}(M(G)) \le \mathrm{bw}(G)$ instead of Theorem 2, we can still prove that $\mathrm{rw}(I(G)) \le \mathrm{bw}(G)$ unless the maximum degree of G is 0 or 1. Actually this will be enough to prove the following two corollaries of Theorem 3.

We will need a definition of a vertex-minor. The *local complementation* at a vertex v of a graph G is an operation to obtain a graph $G * v$ on the vertices of G such that two distinct vertices x, y in $G * v$ are adjacent if and only if either (i) both x and y are neighbors of v and they are nonadjacent in G, or (ii) at least one of x or y is nonadjacent to v and x, y are adjacent in G. It is shown in [6] that local complementations preserve the cut-rank functions and rank-width. A *vertex-minor* of a graph is a graph obtainable by successive local complementations and vertex deletions.

Lemma 4 (Oum [6])**.** *If H is a vertex-minor of G, then* $\mathrm{rw}(H) \le \mathrm{rw}(G)$.

Corollary 5. *For a graph G,* $\mathrm{rw}(G) \le \max(\mathrm{bw}(G), 1)$.

Proof. A graph G is a vertex-minor of $I(G)$, because we get G by applying local complementations to vertices of $I(G)$ corresponding to edges of G and delete those vertices. Then by Lemma 4, $\mathrm{rw}(G) \le \mathrm{rw}(I(G))$. By Theorem 3, either $\mathrm{rw}(I(G)) \le \mathrm{bw}(G)$ or $\mathrm{rw}(I(G)) \le 1$ and $\mathrm{bw}(G) = 0$. □

The above corollary implies that $\mathrm{rw}(G) \le \mathrm{tw}(G) + 1$, because $bw(G) \le \mathrm{tw}(G) + 1$ [9].

Corollary 6. *Let G be a graph. The rank-width of the line graph $L(G)$ of G is at most the branch-width of G.*

Proof. If no two edges of G are adjacent, then $L(G)$ has no edges and therefore $\mathrm{rw}(L(G)) = \mathrm{bw}(G) = 0$. We may now assume that $\mathrm{bw}(G) > 0$. It is enough to show that $L(G)$ is a vertex-minor of $I(G)$. We apply local complementations to vertices in $I(G)$ corresponding to vertices of G and then remove those vertices. The remaining graph is a line graph of G. □

We remark that with different methods, Oum [7] showed that the rank-width of $L(G)$ is exactly one of $\mathrm{bw}(G)-2$, $\mathrm{bw}(G)-1$, or $\mathrm{bw}(G)$ if G is 2-connected.

Let us now show that the inequalities in Theorem 3 and Corollary 5 and 6 are tight. Robertson and Seymour [9] showed that $\mathrm{bw}(K_n) = \lceil 2n/3 \rceil$ for $n \geq 3$. It is straightforward to prove that $\mathrm{bw}(I(K_n)) = \mathrm{bw}(K_n) = \lceil 2n/3 \rceil$ for $n \geq 3$. (In fact, $\mathrm{bw}(G) = \mathrm{bw}(I(G))$ for any graphs G with branch-width at least 2.) We know from the proof of Theorem 3 that $\mathrm{rw}(I(K_n)) = \mathrm{bw}(M(\hat{K}_n)) - 1 = \mathrm{bw}(M(K_{n+1})) - 1 = \mathrm{bw}(K_{n+1}) - 1 = \lceil 2(n+1)/3 \rceil - 1 = \lceil (2n-1)/3 \rceil$. So if $n \geq 3$ and $n \equiv 0, 1 \pmod 3$, then $\mathrm{rw}(I(K_n)) = \mathrm{bw}(I(K_n))$. This proves that Theorem 3 and Corollary 5 are tight.

It remains to prove that Corollary 6 is tight. Let $P(G)$ be a graph obtained from G by attaching a pendant vertex to each vertex of G. (So, $|V(P(G))| = 2|V(G)|$, G is an induced subgraph of $P(G)$, and all vertices of $P(G)$ not in the subgraph G have degree 1 and have distinct neighbors in the subgraph G.) Oum [6] showed that $\mathrm{rw}(P(G)) = \mathrm{rw}(G)$ if G has at least one edge. We observe that if we apply local complementations to vertices of $L(P(K_n))$ corresponding to edges of $P(G)$ incident with pendant vertices, then we obtain $I(K_n)$. Therefore $\mathrm{rw}(L(P(K_n))) = \mathrm{rw}(I(K_n))$. It is routine to prove that $\mathrm{bw}(P(K_n)) = \mathrm{bw}(K_n)$ if $n \geq 3$. So if $n \geq 3$ and $n \equiv 0, 1 \pmod 3$, then $\mathrm{rw}(L(P(K_n))) = \mathrm{bw}(P(K_n))$. Therefore Corollary 6 is tight.

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