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Rank-width and Well-quasi-ordering of Symmetric or Skew-symmetric Matrices

Sang-il Oum

Program in Applied and Computational Mathematics
Princeton University

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Well-quasi-ordering

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A set Q is well-quasi-ordered by an ordering $\preceq$ if
$\forall$ infinite sequences $a_1, a_2, \ldots$ of Q,
$\exists i, j$ such that
$i < j$ and $a_i \preceq a_j$.
Equivalently, we say: $\leq$ is an well-quasi-ordering of Q.

Examples:

1. {positive integers} is well-quasi-ordered by $\leq$.
2. {integers} is NOT well-quasi-ordered by $\leq$.

Overview

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common generalization $\begin{cases} \text{symmetric or skew-symmetric matrices,} \\ \text{delta-matroids representable over a finite field.} \end{cases}$

(1) Robertson and Seymour (1990)

Graphs + minors
Tree-width and well-quasi-ordering

(2) Geelen, Gerards, and Whittle (2002)

Matroids representable over a finite field + minors
Branch-width and well-quasi-ordering

(3) Oum (2005)

Graphs + Vertex-minors
Rank-width (or Clique-width) and well-quasi-ordering

(3) implies (2) for $\mathrm{GF}(2)$. (2) or (3) implies (1).

Preview of main tools

Ph.D. thesis: An Algebraic Theory of Graphs

> *Develop the theory of Chain-groups (unaware of matroids at that time)*

A matroid is representable
⇔ There is a chain-group representing the matroid.

Delta-matroids: (sort of) relaxation of matroids.
A delta-matroid is representable (by def) if

> *it is equivalent to a delta-matroid represented by some symmetric or skew-symmetric matrices.*

We develop the theory of Lagrangian Chain-groups.
A delta-matroid is representable
⇔ There is a Lagrangian chain-group representing it.

Preliminaries

- Transpose of a matrix: A^t.
- Symmetric matrices: $A = A^t$.
- Skew-symmetric matrices:
  1. $A = -A^t$.
  2. 0 on the diagonal entries.
     (We require this for matrices over $\mathrm{GF}(2)$.)
- Matrices of symmetric type: either symmetric or skew-symmetric
- Submatrix of A: $A[X, Y]$ (rows in X, columns in Y)
- Principal Submatrix of A: $A[X] = A[X, X]$

(Principal) Pivoting

Let A: $V \times V$ matrix.
If $A[X]$ is nonsingular, then

$$A = \begin{array}{cc} & \begin{array}{cc} X & Y \end{array} \\ \begin{array}{c} X \\ Y \end{array} & \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \end{array} \Longrightarrow A * X = \begin{array}{cc} & \begin{array}{cc} X & Y \end{array} \\ \begin{array}{c} X \\ Y \end{array} & \begin{pmatrix} \alpha^{-1} & \alpha^{-1}\beta \\ -\gamma\alpha^{-1} & \delta - \gamma\alpha^{-1}\beta \end{pmatrix} \end{array}$$

1. If A is skew-symmetric, then $A * X$ is skew-symmetric.
2. If A is symmetric, we can obtain a symmetric matrix by negating columns of X in $A * X$.

Pivot-minors of a skew-symmetric matrix A : $A * X[Y]$
Pivot-minors of a symmetric matrix A: Symmetric matrices obtained by resigning of columns in $A * X[Y]$.

Rank-width

Let M: $V \times V$ matrices of symmetric type

- Rank-decomposition: a pair $(T, \mathcal{L})$ of
  1. a subcubic tree T
  2. a bijection $\mathcal{L} : V \to \{\text{leaves of } T\}$.
- Width of an edge e of T: $\operatorname{rank} M[\mathcal{L}^{-1}(A_e), \mathcal{L}^{-1}(B_e)]$
  (A_e, B_e): partition of leaves of T induced by $T \setminus e$.
- Width of a rank-decomposition $(T, \mathcal{L})$:
  Maximum width of edges of T.
- Rank-width of M:
  Minimum width of all its rank-decompositions

Our theorem (simplified, matrix version)

Rank-width and WQO

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- $A \times B$ Matrix over F: a mapping from $A \times B$ to F.
- Isomorphism
  from a $V \times V$ matrix M to a $W \times W$ matrix N:
  a bijection μ from V to W such that
  $M(x, y) = N(\mu(x), \mu(y))$.

Let k: constant, F: finite field.
Matrices over F of symmetric types having rank-width $\leq k$ are well-quasi-ordered by pivot-minors.

More precisely, if $M_1, M_2, \ldots$ are matrices of symmetric type over F having rank-width $\leq k$,
then there exist i and j such that
$i < j$ and M_i is isomorphic to a pivot-minor of M_j.

Chain-groups

Let V: finite set, F: field.
Let $K = F^2 = \{\binom{a}{b} : a, b \in F\}$: 2-dim vector space over F.

- Chain: function from V to K.
- Sum of chains: $f + g$
- Scalar product cf when $c \in F$.
- Chain-group: set of chains on V to K closed under sum and scalar product. (subspace of K^V)

Note:
Tutte's chain: function from V to F.

Lagrangian chain-groups

We give a bilinear form to chains.

- Let $\langle \binom{a}{b}, \binom{c}{d} \rangle_K = ad + bc$ (symmetric) or $\langle \binom{a}{b}, \binom{c}{d} \rangle_K = ad - bc$. (skew-symmetric)
- For chains f, g on V to K,

$$\langle f, g \rangle = \sum_{v \in V} \langle f(v), g(v) \rangle_K .$$

- A chain-group N on V to K is called isotropic:

$$\forall f, g \in N, \quad \langle f, g \rangle = 0.$$

- A chain-group N on V to K is called Lagrangian: isotropic and $\dim(N) = |V|$.

Idea from both
Chain-groups (Tutte) + Isotropic systems (Bouchet)

Minors of chain-groups

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For a chain f on V to K,
Restriction $f \cdot T$: chain on T to K such that

$$f \cdot T(x) = f(x) \text{ for all } x \in T.$$

For a chain-group N on V to K,

- Deletion $N \oslash T$:

$$\{f \cdot (V \setminus T) : f \in N, \langle f(x), \binom{1}{0} \rangle_K = 0 \ \forall x \in T\}.$$

- Contraction $N \oslash T$:

$$\{f \cdot (V \setminus T) : f \in N, \langle f(x), \binom{0}{1} \rangle_K = 0 \ \forall x \in T\}.$$

Minors of a chain-group N on V to K:

Chain-groups of the form $N \oslash X \oslash Y$.

Properties of Minors of chain-groups

Rank-width and WQO
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1. A minor of a minor of N is a minor of N.
2. A minor of an isotropic chain-group is isotropic.
3. A minor of a Lagrangian chain-group is Lagrangian.

Minor-closed classes of chain-groups.

Chain-groups

Isotropic Chain-groups

Lagrangian Chain-groups

Note: Bouchet 1987 introduced isotropic chain-groups.

Algebraic Duality

Let N: chain-group.
Let $N \cdot T = \{f \cdot T : f \in N\}$,
$N \times T = \{f \cdot T : f \in N, f(x) = 0\ \forall x \notin T\}$,
$N^\perp = \{g : \langle f, g \rangle = 0\ \forall f \in N\}$.

Theorem (Same proof by Tutte works)

$$(N \cdot T)^\perp = N^\perp \times T.$$

$$(N \odot T)^\perp = N^\perp \odot T,$$
$$(N \oslash T)^\perp = N^\perp \oslash T.$$

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Connectivity

For a chain-group N on V to K,

$$\lambda_N(U) = \frac{\dim N - \dim(N \times (V \setminus U)) - \dim(N \times U)}{2}$$

- Symmetric: $\lambda_N(X) = \lambda_N(V \setminus X)$.
- Submodular: $\lambda_N(X) + \lambda_N(Y) \geq \lambda_N(X \cap Y) + \lambda_N(X \cup Y)$.

If N is Lagrangian, then

$$\lambda_N(U) = |V| - \dim(N \times U).$$

Branch-width of chain-groups

For a chain-group on V to K,

- **Branch-decomposition**: a pair $(T, \mathcal{L})$ of
  1. a subcubic tree T
  2. a bijection $\mathcal{L} : V \to \{\text{leaves of } T\}$.
- **Width** of an edge e of T: $\lambda_N(\mathcal{L}^{-1}(A_e))$.
  (A_e, B_e): partition of leaves of T induced by $T \setminus e$.
- **Width** of a branch-decomposition $(T, \mathcal{L})$:
  Maximum width of edges of T.
- **Branch-width** of M:
  Minimum width of all its branch-decompositions

If M is a minor of N, then

$$(\text{branch-width of } M) \leq (\text{branch-width of } N).$$

Our theorem (simplified)

- Simple Isomorphism
  from a chain-group N on V to K to a chain-group M on W to K: a bijection μ from V to W such that $N = \{f \circ \mu : f \in M\}$.

Let k: constant, F: finite field. $K = F^2$.
Lagrangian chain-groups over F having branch-width $\leq k$ are well-quasi-ordered by minors.

More precisely, if $N_1, N_2, \ldots$ are Lagrangian chain-groups over F having branch-width $\leq k$,
then there exist i and j such that
$i < j$ and N_i is simply isomorphic to a minor of N_j.

Delta-matroids

Axiom for bases of matroids $(E, \mathcal{B})$

- for B_1 and $B_2 \in \mathcal{B}$,
  if $x \in B_1 \setminus B_2$, then
  $\exists y \in B_2 \setminus B_1$ such that $B_1 \Delta \{x, y\} \in \mathcal{B}$
- $\mathcal{B} \neq \emptyset$.

Axiom for feasible sets of delta-matroids $(V, \mathcal{F})$

- for F_1 and $F_2 \in \mathcal{F}$,
  if $x \in F_1 \Delta F_2$, then
  $\exists y \in F_1 \Delta F_2$ such that $F_1 \Delta \{x, y\} \in \mathcal{F}$
- $\mathcal{F} \neq \emptyset$.

Twisting, Deletion, and Minors

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If $\mathcal{M} = (V, \mathcal{F})$ is a delta-matroid,
then $\mathcal{M}' = (V, \{F \Delta X : F \in \mathcal{F}\})$ is also a delta-matroid.
This operation is called a twisting.
We write $\mathcal{M}' = \mathcal{M} \Delta X$.

Two delta-matroids are equivalent if one is obtained from another by twisting.

If there is a feasible set Y such that $Y \cap X = \emptyset$, then $\mathcal{M} \setminus X = (V, \{F \in \mathcal{F} : F \cap X = \emptyset\})$ is a delta-matroid. This operation is called a deletion.

A delta-matroid obtained by twisting and deletion from $\mathcal{M}$ is called a minor of $\mathcal{M}$.

Representable Delta-matroids

For a $V \times V$ matrix A of symmetric type,
Let $\mathcal{F} = \{X : A[X] \text{ is nonsingular}\}$.

Theorem (Bouchet, 1987)

$\mathcal{M}(A) = (V, \mathcal{F})$ is a delta-matroid.

Representable delta-matroids:

Delta-Matroids of the form $\mathcal{M}(A)\Delta X$

His proof:

Matrices of symmetric type
$\Rightarrow$ Isotropic chain-groups
$\Rightarrow$ Delta-matroids

Matrices to Chain-groups

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Let $A = (a_{ij})_{i,j\in V}$ be a matrix of symmetric type over F.
Two chains a, b on V are called supplementary if
$\langle a(v), b(v)\rangle_K = \langle a(v), a(v)\rangle_K = \langle b(v), b(v)\rangle_K = 0$.

$$\langle\,,\,\rangle_K = \begin{cases} \text{symmetric} & \text{if } A \text{ is skew-symmetric,} \\ \text{skew-symmetric} & \text{if } A \text{ is symmetric.} \end{cases}$$

Let f_i: a chain on V to K such that

$$f_i(j) = \begin{cases} a_{ij}\binom{1}{0} & \text{if } i \neq j, \\ a_{ij}\binom{1}{0} + \binom{0}{1} & \text{if } i = j. \end{cases}$$

Let N: a chain-group spanned by $\{f_i : i \in V\}$.

If $i \neq j$, then $\langle f_i, f_j\rangle = a_{ij}\langle\binom{1}{0}, \binom{0}{1}\rangle_K + a_{ji}\langle\binom{0}{1}, \binom{1}{0}\rangle_K = 0$.
If $i = j$, then $\langle f_i, f_i\rangle = 0$. N is isotropic.
Since $\dim(N) = |V|$, N is Lagrangian.

Matrices to Chain-groups

Let $A = (a_{ij})_{i,j \in V}$ be a matrix of symmetric type over F.
Two chains a, b on V are called supplementary if
$\langle a(v), b(v) \rangle_K = \langle a(v), a(v) \rangle_K = \langle b(v), b(v) \rangle_K = 0$.

$$\langle \, , \, \rangle_K = \begin{cases} \text{symmetric} & \text{if } A \text{ is skew-symmetric,} \\ \text{skew-symmetric} & \text{if } A \text{ is symmetric.} \end{cases}$$

Let f_i: a chain on V to K such that

$$f_i(j) = \begin{cases} a_{ij} a(v) & \text{if } i \neq j, \\ a_{ij} a(v) + b(v) & \text{if } i = j. \end{cases}$$

Let N: a chain-group spanned by $\{f_i : i \in V\}$.

If $i \neq j$, then $\langle f_i, f_j \rangle = a_{ij} \langle \binom{1}{0}, \binom{0}{1} \rangle_K + a_{ji} \langle \binom{0}{1}, \binom{1}{0} \rangle_K = 0$.
If $i = j$, then $\langle f_i, f_i \rangle = 0$. N is isotropic.
Since $\dim(N) = |V|$, N is Lagrangian.

Isotropic Chain-groups to Delta-matroids

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Let N: isotropic chain-groups on V to K.

Theorem (Bouchet)

Say $F \in \mathcal{F}$ if $f \equiv 0$ is the only chain in N satisfying

- $\langle f(x), a(v)\rangle_K = 0$ *for all* $x \notin F$,
- $\langle f(x), b(v)\rangle_K = 0$ *for all* $x \in F$.

$(V, \mathcal{F})$ *is a delta-matroid.*

Moreover, if N is from a matrix A of symmetric type, F is feasible if and only if $A[X]$ is nonsingular.

Matrices of symmetric type
$\Rightarrow$ Lagrangian chain-groups
$\Rightarrow$ Isotropic chain-groups
$\Rightarrow$ Delta-matroids

Reverse direction?

Twisting in Isotropic Chain-groups

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$$\begin{array}{ccc}
\text{Isotropic chain-group } N & \longrightarrow & \text{Delta-matroid } \mathcal{M} \\
\Vert & & \big\downarrow \text{twisting} \\
\text{Same isotropic chain-group} & \longleftarrow & \text{Delta-matroid } \mathcal{M}\Delta X.
\end{array}$$

If $\mathcal{M}$: represented by N with supplementary chains a and b, then $\mathcal{M}\Delta X$ is represented by N with a', b'.

	$x \notin X$	$x \in X$
$a'(x)$	$a(x)$	$\pm b(x)$
$b'(x)$	$b(x)$	$a(x)$

$+b(x)$ if $\langle\, ,\, \rangle_K$ is symmetric,
$-b(x)$ if $\langle\, ,\, \rangle_K$ is skew-symmetric.

Representable delta-matroids $\Rightarrow$ Lagrangian chain-groups

Lagrangian Chain-groups to Matrices

Lagrangian chain-groups to Representable delta-matroids

For a Lagrangian chain-group N,

- Choose supplementary chains a, b so that $\emptyset$ is feasible in the delta-matroid of N (by twisting).
- For each $v \in V$, there exists a unique chain $f_v \in N$ such that

$$\langle a(v), f_v(w)\rangle_K = \begin{cases} 0 & \text{if } v \neq w \\ 1 & \text{if } v = w. \end{cases}$$

- Construct a $V \times V$ matrix $A = (\langle f_i(j), b(j)\rangle_K : i, j \in V)$.

Then the matrix with a, b represents N.

Matrices of symmetric type
$\Leftrightarrow$ Lagrangian chain-groups
$\Leftrightarrow$ Representable Delta-matroids

Connectivity

If N: Lagrangian chain-group from a matrix A, then

$$\operatorname{rank} A[X, V \setminus X] = \lambda_N(X).$$

Reminder: $\lambda_N(X) = |X| - \dim(N \times X)$ for Lagrangian chain-group N.

Corollary

Rank-width of a matrix of symmetric type
= Branch-width of a Lagrangian chain-group from the matrix

Minors

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Theorem

For an isotropic chain-group N on V to K and $v \in V$, if

- *a, b: supplementary chains s.t. $a(v) \in \{\binom{1}{0}, \binom{-1}{0}\}$,*
- *$\mathcal{M}$: delta-matroid represented by N with a, b,*

then $N \oslash \{v\}$ with $a \cdot (V \setminus \{v\})$, $b \cdot (V \setminus \{v\})$ represents one of the following delta-matroids.

- *$\mathcal{M} \setminus \{v\}$ if there is a feasible set not containing v,*
- *$\mathcal{M} \Delta \{v\} \setminus \{v\}$ otherwise.*

$N \oslash \{v\}$ with $a \cdot (V \setminus \{v\})$, $b \cdot (V \setminus \{v\})$ represents one of the following delta-matroids.

- *$\mathcal{M} \Delta \{v\} \setminus \{v\}$ if there is a feasible set containing v,*
- *$\mathcal{M} \setminus \{v\}$ otherwise.*

Delta-matroid minors $\Leftrightarrow$ Isotropic Chain-group minors

Pivoting and Twisting

$$\begin{array}{ccc} \text{Matrix}A & \longrightarrow & \text{Lagrangian chain-group } N \\ \text{pivoting}\downarrow & & \| \\ \text{Matrix}A * X & \longrightarrow & \textcolor{red}{\text{Same Lagrangian chain-group}} \end{array}$$

If N: represented by A with supplementary chains a and b, then N is represented by $A * X$ with a', b'.

	$x \notin X$	$x \in X$
$a'(x)$	$a(x)$	$\pm b(x)$
$b'(x)$	$b(x)$	$a(x)$

$+b(x)$ if $\langle\,,\,\rangle_K$ is symmetric,
$-b(x)$ if $\langle\,,\,\rangle_K$ is skew-symmetric.

Pivot-minors $\Leftrightarrow$ Minors of Lagrangian Chain-groups

Our theorem

For a well-quasi-ordered set Q with an ordering $\preceq$,

- Q-labeling: a function from an element set to Q.
- Q-labeled chain-group on V: a chain-group with a Q-labeling.
- Q-minor of a Q-labeled chain-group N is a minor on V' such that (Q-labeling of minor)$(x) \preceq$ (Q-labeling of N)(x) for all $x \in V'$.

If $N_1, N_2, \ldots$ are Q-labeled Lagrangian chain-groups over F having branch-width $\leq k$,
then there exist i and j such that
$i < j$ and N_i is simply isomorphic to a Q-minor of N_j.

Proof Sketch (Vaguely)

- Tutte's linking theorem for Lagrangian chain-groups
- Existence of linked branch-decompositions
  Thomas (tree-decomposition), Geelen et al. (branch-decomposition).
- Boundary of an isotropic chain-group N: an ordered basis of $N^{\perp}/N$.
- Boundaried isotropic chain-group: An isotropic chain-group with a boundary.
- Sum of two boundaried isotropic chain-group.
- Connection types: describe the sum of two boundaried isotropic chain-group uniquely.
- Lemma on Trees by Robertson and Seymour.

(Vaguely) If branch-width is bounded, number of distinct connection types is finite. Using that, show that there is no "minimal" antichain.

Matroids Representable over a Finite Field

If $\mathcal{M}$ is a representable matroid, let B be a basis. There is a standard representation:

$$\begin{array}{cc} B & E(\mathcal{M}) \setminus B \\ \left(\begin{array}{c} 1 \\ \ddots \\ 1 \end{array}\right. & \left.\begin{array}{c} \\ A \\ \\ \end{array}\right) \end{array}$$

Take the following skew-symmetric matrix.

$$S = \begin{array}{c} \begin{array}{cc} B & E(\mathcal{M}) \setminus B \end{array} \\ \begin{pmatrix} 0 & A \\ -A^t & 0 \end{pmatrix} \end{array}$$

Theorem (Bouchet)

$\mathcal{M}(S)\Delta B$ is a delta-matroid whose feasible sets are bases of $\mathcal{M}$.

Moreover,
Matroid Branch-width
= (Rank-width of S) +1.

Matroid Minors
$\Rightarrow$ Pivot-minors
$\Leftrightarrow$ Delta-matroid minors

Representable Matroids of Bounded Branch-width

Let $Q = \{x, y\}$ and $x \not\leq y$ and $y \not\leq x$. We assign x for elements in the chosen base, y for other elements.

$$
\begin{tikzcd}
\text{Representable Matroid} \arrow[r] \arrow[d, "\text{minor?}"] & Q\text{-labeled Skew-symmetric Matrix} \arrow[r] & Q\text{-labeled Lagrangian Chain-group} \arrow[d, "Q\text{-minor}"'] \\
\text{Representable Matroid} \arrow[r] & Q\text{-labeled Skew-symmetric Matrix} \arrow[r] & Q\text{-Minor}
\end{tikzcd}
$$

As a corollary, we conclude that:

Matroids representable over a fixed finite field having bounded branch-width are well-quasi-ordered by matroid minors.

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Implications

- Matroids (Geelen, Gerards, and Whittle. 2002)

Matroids representable over a fixed finite field having bounded branch-width are well-quasi-ordered by matroid minors.

- Graphs (Robertson and Seymour. 1990)

Graphs representable over a fixed finite field having bounded branch-width are well-quasi-ordered by matroid minors.

- Graphs with Rank-width. (Oum. 2005)
  Rank-width of a graph: Rank-width of its adjacency matrix over $\mathrm{GF}(2)$.
  Pivot-minor of a graph: Pivot-minor of the adjacency matrix.

Open problems — Branch-width, Well-quasi-ordering

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- Suitable Connectivity functions for Representable Delta-matroids.

My definition of branch-width of delta-matroids representable over F:

> *Minimum branch-width of Lagrangian chain-groups over F representing the delta-matroid.*

Then, F-representable delta-matroids of bounded branch-width are well-quasi-ordered.

- Are delta-matroids representable over a finite field well-quasi-ordered?

This will imply the Graph Minor Theorem.

Open problems — isotropic chain-groups

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- Characterization of Delta-matroids obtained from isotropic chain-groups?

  - They are minor-closed.
  - They are either representable delta-matroids or something else(?).

- Are isotropic chain-groups of bounded branch-width well-quasi-ordered?
- Is Tutte's linking theorem true for isotropic chain-groups?

Thank you for your attention!