# Testing Branch-width

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Joint work with Paul Seymour.

A function $f : 2^V \to \mathbb{Z}$ is a connectivity function if

(i) $f(X) + f(Y) \geq f(X \cap Y) + f(X \cup Y)$, (submodular)

(ii) $f(X) = f(V \setminus X)$, (symmetric)

(iii) $f(\emptyset) = 0$.



$v(X)$ =number of vertices meeting both $X$ and $E \setminus X$.



$e(X)$ =number of edges meeting both $X$ and $V \setminus X$.

$\mathcal{M}$: matroid, $\lambda(X) = r(X) + r(E(\mathcal{M}) - X) - r(E(\mathcal{M}))$.

For a graph $G$, let $A$ = adjacency matrix. $\rho_G(X) = \text{rank } A[X, V \setminus X]$.

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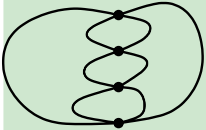

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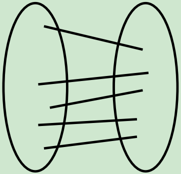

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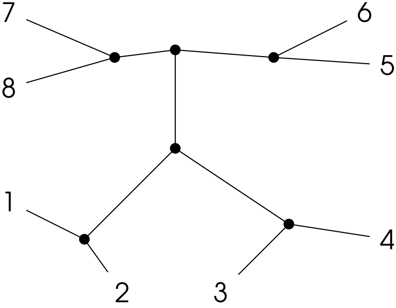
Branch-decomposition of $f$: a pair $(T, L)$ of
a *subcubic tree* $T$ and a *bijection* $L : V \rightarrow$ {leaves of T}.




Branch-width



Carving-width



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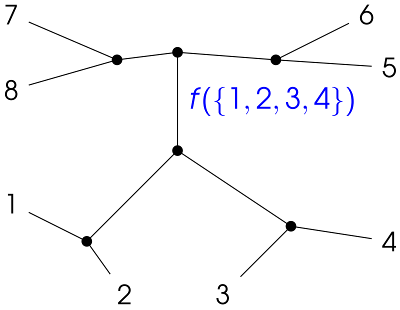


Width of an edge $e$ of $T$: $f(A_e)$

$(A_e, B_e)$ is a partition of $V$ given by deleting $e$.

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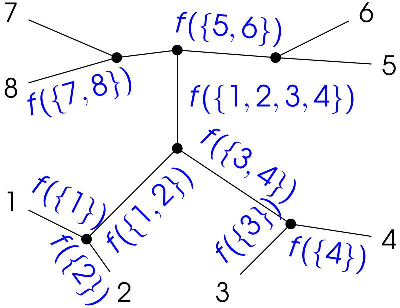


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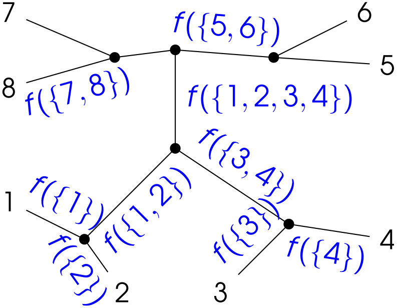


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Branch-width: $min_{(T,L)}$ width$(T, L)$.
(If $|V| \leq 1$, then branch-width=0)

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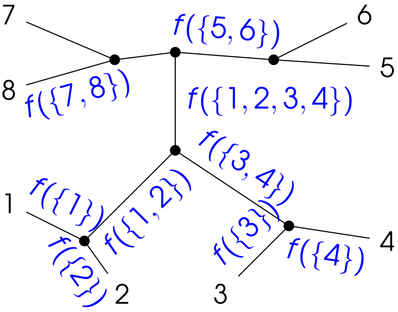


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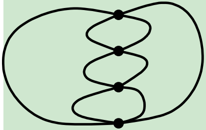

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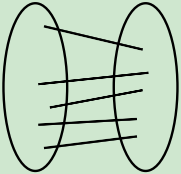

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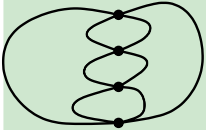

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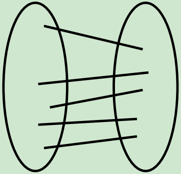

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## Testing Branch-width $\leq k$ for fixed $k$

- Branch-width of graphs: Linear (Bodlaender, Thilikos '97)
- Carving-width of graphs: Linear (Thilikos, Serna, Bodlaendar '00)
- Branch-width of matroids represented over a fixed finite field: $O(|E(\mathcal{M})|^3)$ (Hliněný '05)
- Rank-width of graphs: $O(|V(G)|^3)$ (Oum '05)

Poly-time algorithm to test branch-width$\leq k$ for any connectivity functions? *assuming that f is given by an oracle*.

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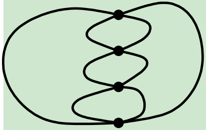

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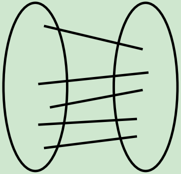

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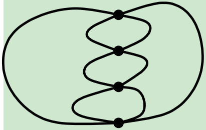

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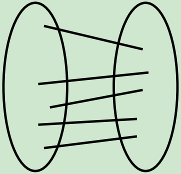

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**$f$-tangle of order $k+1$ (Robertson and Seymour)**

A set $\mathcal{T}$ of subsets of $V$ satisfying

(T1) If $f(X) \le k$, then $X \in \mathcal{T}$ or $V \setminus X \in \mathcal{T}$.

(T2) If $A, B, C \in \mathcal{T}$, then $A \cup B \cup C \ne V$.

(T3) $V \setminus \{v\} \notin \mathcal{T}$ for all $v \in V$.

**Robertson, Seymour ('91)**

Branch-width$\le k$ if and only if no $f$-tangle of order $k+1$ exists.

Naive algorithm: Choose one from $X$ or $V \setminus X$ if $f(X) \le k$ and see whether (T2) and (T3) are satisfied.

**loose $f$-tangle of order $k+1$**

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(L1) $V \notin \mathcal{T}$.

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THM: An $f$-tangle of order $k+1$ exists if and only if a loose $f$-tangle of order $k+1$ exists.

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## Naive algorithm to find a loose $f$-tangle

(1) Begin with $\mathcal{T} = \{X : |X| \leq 1, f(X) \leq k\}$.

(2) Test (L1).
If it fails, then no loose $f$-tangle of order $k+1$.

(3) Test (L2).
If it fails, then find $C$ and add it to $\mathcal{T}$. Go back to 2.

(4) $\mathcal{T}$ is a loose $f$-tangle of order $k+1$.

Problem: $|\mathcal{T}|$ can be exponentially large.

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(L1) $V \notin \mathcal{T}$.

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(L3) If $|X| \leq 1$ and $f(X) \leq k$, then $X \in \mathcal{T}$.

Let $f_{\min}(A,B) = \min\{f(X) : A \subseteq X \subseteq V \setminus B\}$ for $A, B \subseteq V$, $A \cap B = \emptyset$.

## loose $f$-tangle kit of order $k+1$

A pair $(P, \mu)$ where
$P = \{(A,B) : A \cap B = \emptyset, \max(|A|,|B|) \leq f_{\min}(A,B) \leq k.\}$
and $\mu : P \to 2^V$ is a function satisfying the following.

(K1) $\mu(\emptyset,\emptyset) \neq V$ if $(\emptyset,\emptyset) \in P$.

(K2) If $(A,B), (C,D), (E,F) \in P$, $E \subseteq X \subseteq \mu(A,B) \cup \mu(C,D) - F$, and $f_{\min}(E,F) = f(X)$, then $X \subseteq \mu(E,F)$.

(K3) If $|X| \leq 1$, $f(X) \leq 1$,
then there exists $(A,B) \in P$ such that $A \subseteq X \subseteq V \setminus B$,
$f(X) = f_{\min}(A,B)$, and $X \subseteq \mu(A,B)$.

(K1) $\mu(\emptyset,\emptyset) \neq V$ if $(\emptyset,\emptyset) \in P$.

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## Poly-time algorithm to find a loose $f$-tangle

(A1) Let $P = \{(A,B) : A \cap B = \emptyset, \max(|A|,|B|) \leq f_{\min}(A,B) \leq k\}$.

(A2) For each $v \in V$, if $0 < f(\{v\}) \leq k$, then find $B \subseteq V \setminus \{v\}$ such that $|B| \leq f_{\min}(\{v\},B) \leq k$. Let $\mu(\{v\},B) = \{v\}$.
Let $\mu(\emptyset,\emptyset) = \{v \in V : f(\{v\}) = 0\}$ if $(\emptyset,\emptyset) \in P$.
For all other $(A,B) \in P$, let $\mu(A,B) = \emptyset$.

(A3) Test (M1). If it fails, then no loose $f$-tangle kit of order $k+1$.

(A4) Test (M2).
If it fails, then find $X$ and enlarge $\mu(E,F)$. Go back to (A3).

(A5) $(P,\mu)$ is a loose $f$-tangle kit of order $k+1$.

(K1) $\mu(\emptyset, \emptyset) \neq V$ if $(\emptyset, \emptyset) \in P$.

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## Poly-time algorithm to find a loose $f$-tangle

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(A2) For each $v \in V$, if $0 < f(\{v\}) \leq k$, then find $B \subseteq V \setminus \{v\}$ such that $|B| \leq f_{\min}(\{v\}, B) \leq k$. Let $\mu(\{v\}, B) = \{v\}$.
Let $\mu(\emptyset, \emptyset) = \{v \in V : f(\{v\}) = 0\}$ if $(\emptyset, \emptyset) \in P$.
For all other $(A, B) \in P$, let $\mu(A, B) = \emptyset$.

(A3) Test (M1). If it fails, then no loose $f$-tangle kit of order $k + 1$.

(A4) Test (M2).
If it fails, then find $X$ and enlarge $\mu(E, F)$. Go back to (A3).

(A5) $(P, \mu)$ is a loose $f$-tangle kit of order $k + 1$.

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Time Complexity: $O(n^{2k} n n^{6k+1} n n^5 \log n)$

## Consequence to Matroids

Poly-time algorithm to test matroid branch-width $\le k$ for fixed $k$, when the input matroid is given by an independence oracle.

# Constructing Branch-decomposition of width $\leq k$

Is it possible to construct the branch-decomposition of width $\leq k$ if there exists one in polynomial time (in $|V|$)?
So far, we can only show that there is no loose $f$-tangle kit of order $k+1$.

Jim Geelen (2005) observed a simple way.
For a pair $(a, b)$ of elements, let $f_{(a,b)}$ be a connectivity function on $(V \setminus \{a, b\}) \cup \{(a, b)\}$ such that

$$f_{(a,b)}(X) = \begin{cases} f(X) & \text{if } (a, b) \notin X, \\ f(X \cup \{a, b\}) & \text{otherwise.} \end{cases}$$

Find a pair $(a, b)$ such that branch-width of $f_{(a,b)}$ is at most $k$.
(There always exists such a pair if branch-width of $f$ is at most $k$.)
Then by splitting the leaf in the branch-decomposition of $f_{(a,b)}$, we obtain the branch-decomposition of $f$.
We only need $O(n^3)$ calls to testing branch-width at most $k$.

# Constructing Branch-decomposition of width $\leq k$

Is it possible to construct the branch-decomposition of width $\leq k$ if there exists one in polynomial time (in $|V|$)?
So far, we can only show that there is no loose $f$-tangle kit of order $k+1$.

Jim Geelen (2005) observed a simple way.
For a pair $(a, b)$ of elements, let $f_{(a,b)}$ be a connectivity function on $(V \setminus \{a, b\}) \cup \{(a, b)\}$ such that

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# Further topics

**Fixed Parameter Tractable?**

Is it possible to have a running time $O(f(k)|V|^c)$ for all $k$?

Thank you!

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