Unavoidable vertex-minors in large prime graphs

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Unavoidable structures
$\forall n, \exists N$ s.t.

- every graph on $\geq N$ vertices has K_n or its complement as a **subgraph**. Ramsey
- every **connected** graph on $\geq N$ vertices has K_n, $K_{1,n}$, or P_n as an induced **subgraph**.
- every **2-connected** graph on $\geq N$ vertices has C_n or $K_{2,n}$ as a **topological minor**.
- every **3-connected** graph on $\geq N$ vertices has a k-spoke wheel or $K_{3,k}$ as a **minor**. (Oporowski, Oxley, Thomas 1993)

Further generalization (Matroids - Ding, Oporowski, Oxley, Vertigan)

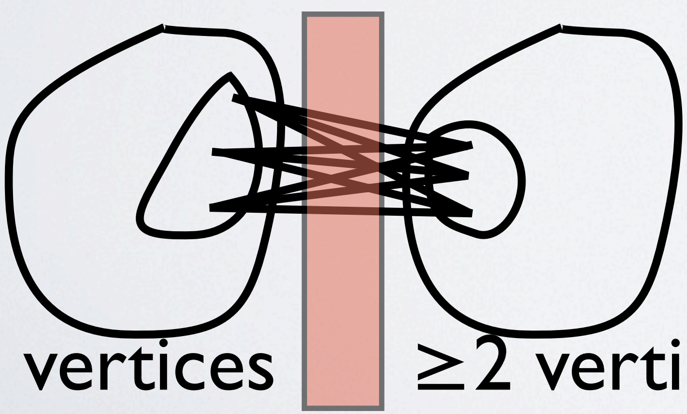
Our Theorem

∀n, ∃N s.t.

every **prime** graph on ≥N vertices
has C_n or $K_n \boxminus K_n$ as a **vertex-minor**

prime (with respect to the split decomposition)
= no splits (Cunningham 1982)

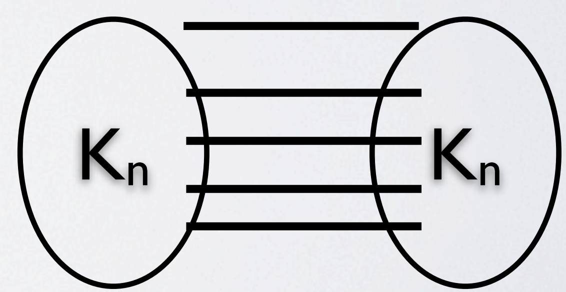
split=partition of the vertex-set s.t.



≥2 vertices ≥2 vertices

cf. 1-join of graphs

$K_n \boxminus K_n$



two cliques
joined by a matching

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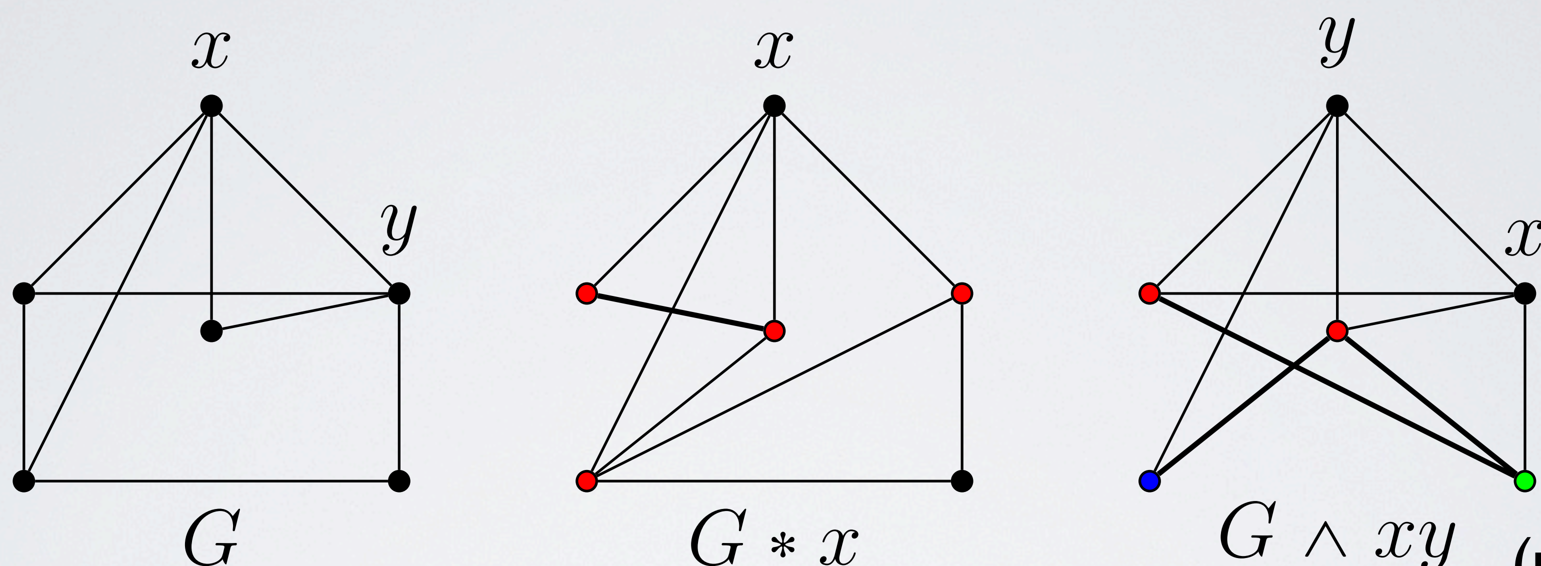
Local complementation and vertex-minors



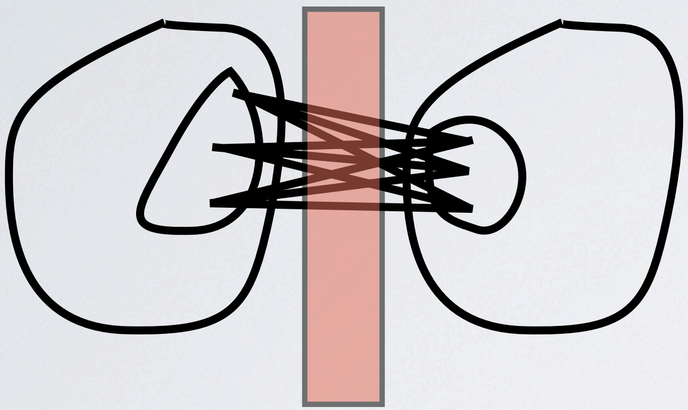
(local complementation at x)

(pivot xy)
= G*x*y*x

H is locally equivalent to G if $H=G*x_1*x_2*x_3$...

vertex-minor=graph obtained by applying a sequence of local complemention and vertex deletions

Why prime graphs & vertex-minors come together?



If (A,B) is a split of G, then it is also a split of G*v.

If G and H are locally equivalent, then G is prime iff H is prime

Bouchet (1987)
Every prime graph on ≥ 5 vertices has C_5 as a vertex-minor.

Our Theorem: $\forall n$, $\exists N$ s.t.
every **prime** graph on $\geq N$ vertices has C_n or $K_n \boxminus K_n$ as a **vertex-minor**

Why is this "best possible"?

- Both C_n and $K_n \boxminus K_n$ are prime!
- They can be arbitrary big.
- [Thm] Cn cannot have $K_m \boxminus K_m$ vertex-minor and $K_m \boxminus K_m$ cannot have C_n vertex-minor.

Our Theorem: $\forall n, \exists N$ s.t.
every **prime** graph on $\geq N$ vertices
has C_n or $K_n \boxminus K_n$ as a **vertex-minor**

This is an exact characterization thm!

Let I be a set of graphs closed under taking vertex-minors.
Prime graphs in I have bounded size
if and only if
$\{C_n : n \geq 3\} \not\subseteq I$ and $\{K_n \boxminus K_n : n \geq 3\} \not\subseteq I$

Our Theorem: $\forall n, \exists N$ s.t.
every **prime** graph on $\geq N$ vertices
has C_n or $K_n \boxminus K_n$ as a **vertex-minor**

Overview of the proof

Proposition 1:
$\forall n, \exists N$ s.t.

$N \sim 6.75n^7$

if a prime graph has a induced path of length N, then it has C_n as a vertex-minor.

Proposition 2:
$\forall n, \exists N$ s.t.

N~2^2^2^......^2

every **prime** graph on $\geq N$ vertices has C_n or $K_n \boxminus K_n$ as a **vertex-minor**

Our Theorem: $\forall n, \exists N$ s.t.

every **prime** graph on $\geq N$ vertices has C_n or $K_n \boxminus K_n$ as a **vertex-minor**

Part 1: Making a cycle from a long path

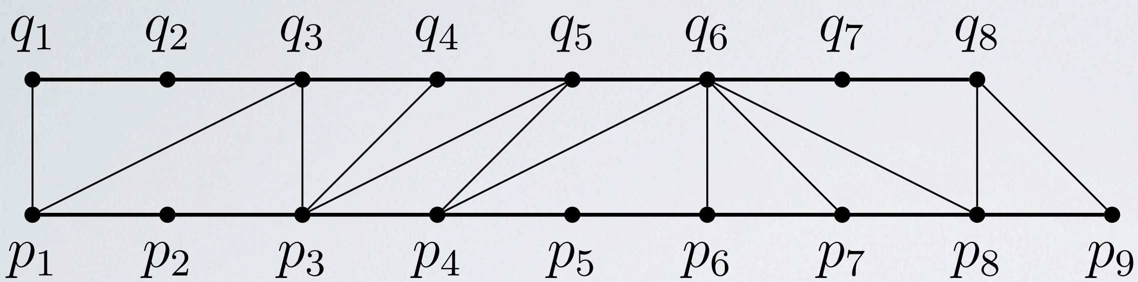
Proof: > 10 pages in the paper
"Blocking Sequences"

Proposition 1:
$\forall n, \exists N$ s.t.

$N \sim 6.75n^7$

if a prime graph has a induced path of length N,
then it has C_n as a vertex-minor.

Generalized ladder



Two induced paths
+ non-crossing chords

Lemma 1:
∀n, ∃N s.t.

$N \sim 4.5n^5$

every generalized ladder on ≥N verticess
has C_n as a vertex-minor.

Proposition 1:
∀n, ∃N s.t.

$N \sim 6.75n^7$

if a prime graph has a induced path of length N,
then it has C_n as a vertex-minor.

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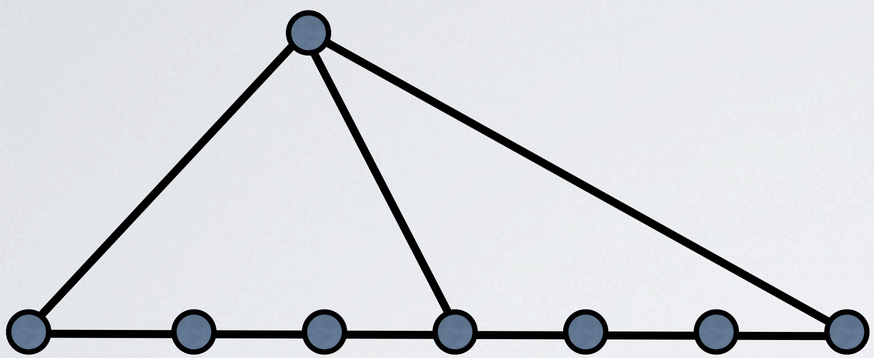
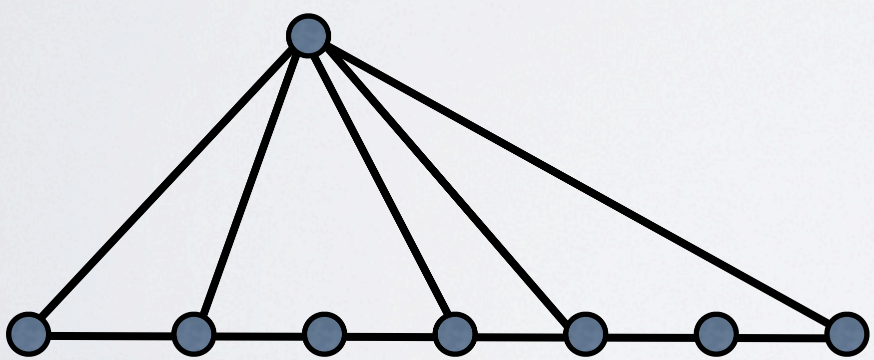
Example (simplest case)
Fan



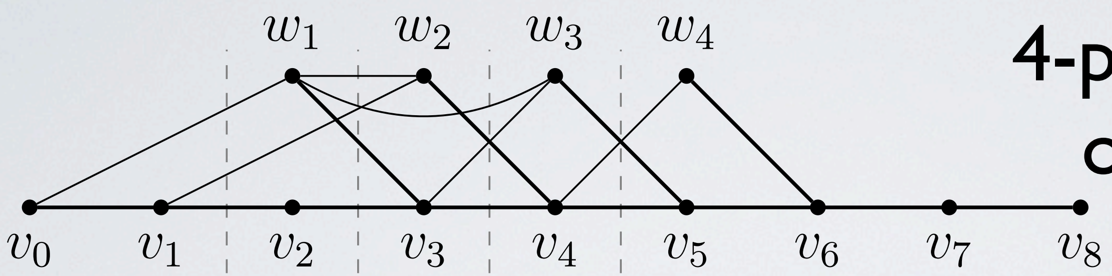
small number of chords

Every big fan has C_n as a vertex-minor.



large number of chords

Finding a gen. ladder



4-patched path
of length 8

Very very very long induced path
$\Longrightarrow$ very very long "k-patched" path
$\Longrightarrow$ very long "fully patched" path
$\Longrightarrow$ big generalized ladder

Use the technique "blocking sequences" by J. Geelen (1995)

Proposition 1:
$\forall n, \exists N$ s.t.
if a prime graph has a induced path of length N,
then it has C_n as a vertex-minor.

$N \sim 6.75 n^7$

Part 2: Making a bigger broom

proof: ~11 pages

"Ramsey"

Proposition 2:

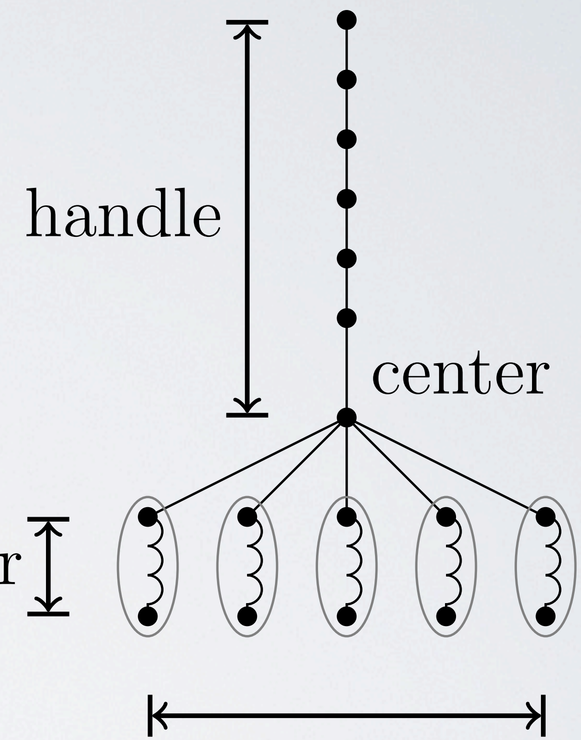
$\forall n, \exists N$ s.t.

N~2^2^2^......^2

every **prime** graph on $\geq N$ vertices

has C_n or $K_n \boxminus K_n$ as a **vertex-minor**

(h,w,l)-broom

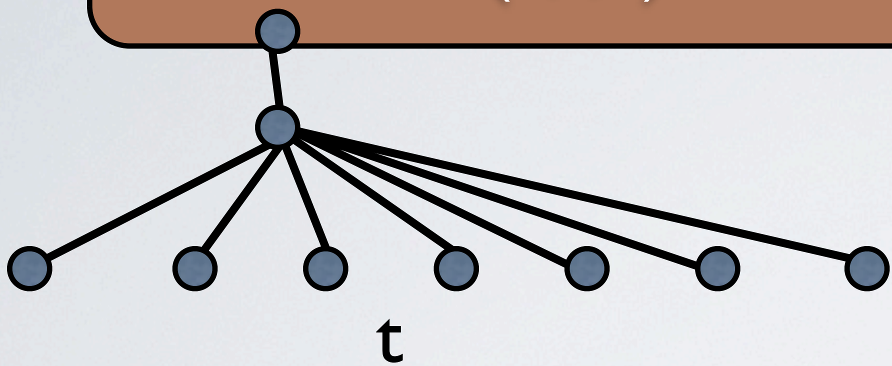


Handle = induced path

Fibers= connected components of BROOM–HANDLE

Suppose a prime graph G has no vertex-minor isomorphic to P_c or $K_c \boxminus K_c$.

If G is big, then
G has a (1,t,1)-broom for huge t as a vertex-minor.

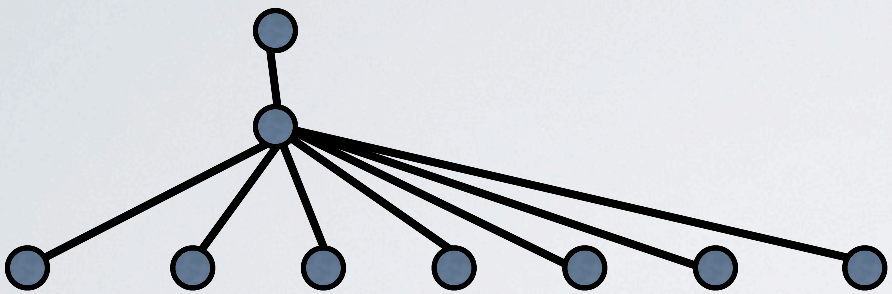


Large degree: Apply Ramsey!
Long path: P_c

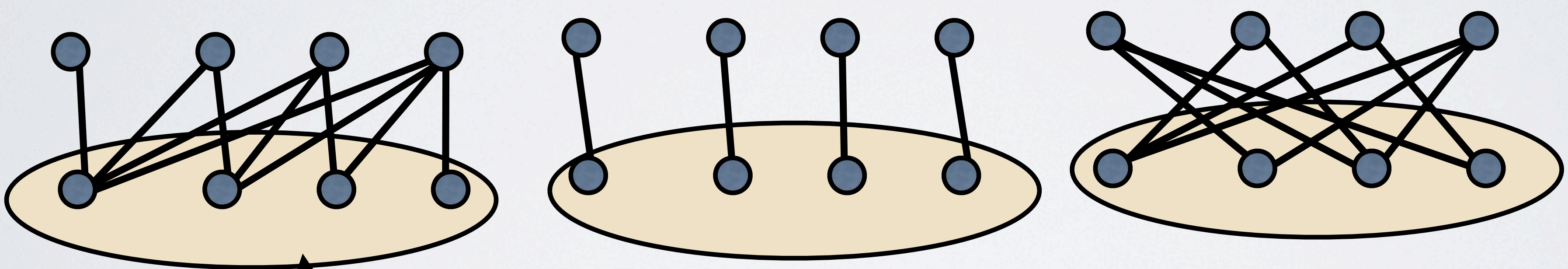
If G has a (h,N,1)-broom for very big N, then
G has a (h,t,2)-broom as a vertex-minor.

If G has a (h,N,k)-broom for very big N, then
G has a (h,t,k+1)-broom as a vertex-minor.

If G has a (h,N,1)-broom for very big N, then
G has a (h,t,2)-broom as a vertex-minor.



Leaves have distinct neighbors;
we can find...



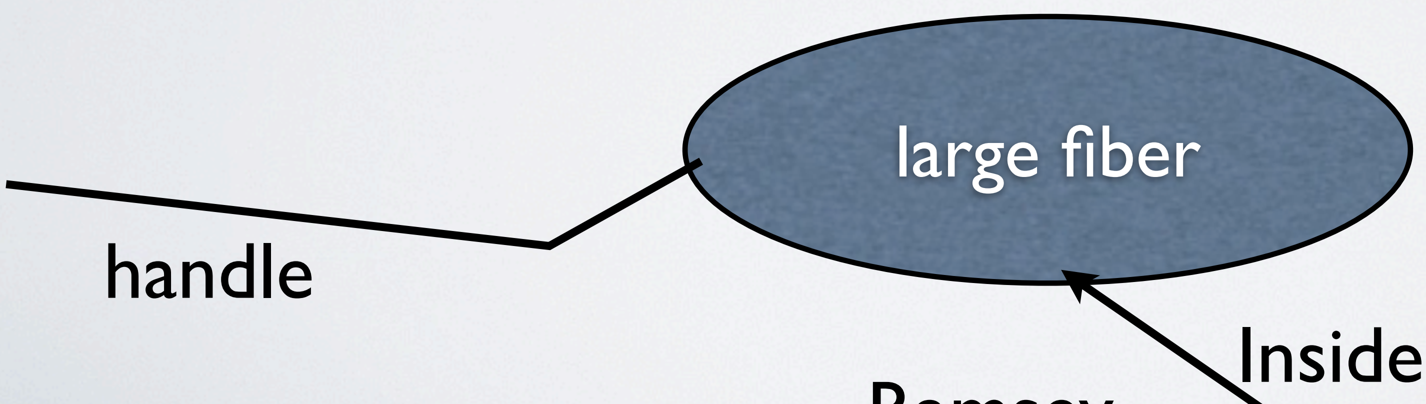
a clique or a stable set

Most cases reduce to P_c or $K_c \boxminus K_c$

If G has a (h,N,1)-broom for very big N, then
G has a (h,t,2)-broom as a vertex-minor.

If G has a (h,N,k)-broom for very big N, then
G has a (h,t,k+1)-broom as a vertex-minor.

If G has a (h,1,k)-broom for very big k, then
G has a (h+1,t,1)-broom as a vertex-minor.



If G has a (h,N,1)-broom for very big N, then
G has a (h,t,2)-broom as a vertex-minor.

If G has a (h,N,k)-broom for very big N, then
G has a (h,t,k+1)-broom as a vertex-minor.

If G has a (h,1,k)-broom for very big k, then
G has a (h+1,t,1)-broom as a vertex-minor.

Starting from a (1,N,1)-broom for very large N,
we can get a broom with very tall handle!

Hidden details

- Blocking sequences
  - how to find a short blocking sequence
  - how to get a (k+1)-patched path from a k-patched path by sacrificing a bounded number of edges in the long path
- how to get a long cycle from a gen. ladder
  - max degree 3 case
  - max degree 4 to 3
  - general to max degree 4

Thank you / Questions?

Our Theorem: $\forall n$, $\exists N$ s.t.
every **prime** graph on $\geq N$ vertices
has C_n or $K_n \boxminus K_n$ as a **vertex-minor**

Corollary: $\forall n$, $\exists N$ s.t.
every graph
without C_n or $K_n \boxminus K_n$ as a **vertex-minor**
is either a graph on $\leq N$ vertices
or the 1-join of two such graphs.

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