석사 학위논문
Master's Thesis

Pivot-minor들을 이용한 rank-width와 tree-width의 관계에 대한 연구

Connecting rank-width and tree-width via pivot-minors

권 오 정 (權 五 政 Kwon, O-joung)
수리과학과
Department of Mathematical Sciences

KAIST

2012

Pivot-minor들을 이용한 rank-width와 tree-width의 관계에 대한 연구

Connecting rank-width and tree-width via pivot-minors

Connecting rank-width and tree-width via pivot-minors

Advisor : Professor Oum, Sang-il

by

Kwon, O-joung

Department of Mathematical Sciences

KAIST

A thesis submitted to the faculty of KAIST in partial fulfillment of the requirements for the degree of Master of Science in the Department of Mathematical Sciences . The study was conducted in accordance with Code of Research Ethics[1].

2012. 5. 10.

Approved by

Professor Oum, Sang-il

[Advisor]

---

[1]Declaration of Ethical Conduct in Research: I, as a graduate student of KAIST, hereby declare that I have not committed any acts that may damage the credibility of my research. These include, but are not limited to: falsification, thesis written by someone else, distortion of research findings or plagiarism. I affirm that my thesis contains honest conclusions based on my own careful research under the guidance of my thesis advisor.

Pivot-minor들을 이용한 rank-width와 tree-width의 관계에 대한 연구

권 오 정

위 논문은 한국과학기술원 석사학위논문으로
학위논문심사위원회에서 심사 통과하였음.

2012년 5월 16일

심사위원장	엄 상 일	(인)
심사위원	김 동 수	(인)
심사위원	Andreas Holmsen	(인)

MMAS
20104251

권 오 정. Kwon, O-joung. Connecting rank-width and tree-width via pivot-minors. Pivot-minor들을 이용한 rank-width와 tree-width의 관계에 대한 연구. Department of Mathematical Sciences . 2012. 27p. Advisor Prof. Oum, Sang-il. Text in English.

ABSTRACT

We prove that every graph of rank-width k is a pivot-minor of a graph of tree-width at most $2k$ and every graph of linear rank-width k is a pivot-minor of a graph of path-width at most $k+1$. We also prove that graphs of rank-width at most 1 are exactly vertex-minors of trees, and graphs of linear rank-width at most 1 are precisely vertex-minors of paths. In addition, we show that bipartite graphs of rank-width at most 1 are exactly pivot-minors of trees and bipartite graphs of linear rank-width at most 1 are precisely pivot-minors of paths.

Also, we calculate the linear rank-width of complete binary trees. And we show that if a tree has linear rank-width at least k, then it has the complete binary tree of height k as a vertex-minor. Finally, we describe an interesting question that a class of graphs with unbounded linear rank-width contains all trees as vertex-minors or pivot-minors. We prove that this question for pivot-minor is false and we suggest another version of the question.

Contents

List of Tables

List of Figures

Chapter 1. Introduction

Rank-width is a width parameter of graphs, introduced by Oum and Seymour [11], measuring how easy it is to decompose a graph into a tree-like structure where the "easiness" is measured in terms of the matrix rank function derived from edges formed by vertex partitions. Rank-width is a generalization of another, more well-known width parameter called tree-width, introduced by Robertson and Seymour [15]. *Pivot-minor* and *vertex-minor* relations are graph containment relations such that rank-width cannot increase when taking pivot-minors or vertex-minors of a graph [11].

First, we focus on the graphs of small rank-width. It is well known that every graph of small tree-width also has small rank-width; Oum [12] showed that if a graph has tree-width k, then its rank-width is at most $k+1$. The converse does not hold in general, as complete graphs have rank-width 1 and arbitrary large tree-width. In Chapter 3, we prove that for every graph G with rank-width at most k and $|V(G)| \geq 3$, there exists a graph H having G as a pivot-minor such that H has tree-width at most $2k$ and $|V(H)| \leq (2k+1)|V(G)| - 6k$. Furthermore, we prove that for every graph G with linear rank-width at most k and $|V(G)| \geq 3$, there exists a graph H having G as a pivot-minor such that H has path-width at most $k+1$ and $|V(H)| \leq (2k+1)|V(G)| - 6k$. To prove these theorems, we construct a graph having G as a pivot-minor, called a *rank-expansion.* And we prove that a rank-expansion has small tree-width.

As a corollary, we give new characterizations of two graph classes: graphs with rank-width at most 1 and graphs with linear rank-width at most 1. We show that a graph has rank-width at most 1 if and only if it is a vertex-minor of a tree. We also prove that a graph has linear rank-width at most 1 if and only if it is a vertex-minor of a path. Moreover, if the graph is bipartite, we prove that a vertex-minor relation can be replaced with a pivot-minor relation in both theorems. Table 1 summarizes our theorems in Chapter 3.

The remaining chapters are related to the following questions. Let C be a class of graphs with unbounded linear rank-width. One question is that all trees are appeared as vertex-minors of graphs in the class C, and the other is that all trees are appeared as pivot-minors of graphs in C. Since a pivot-minor of a graph is also a vertex-minor of the graph, the latter question is stronger than the former. In Chapter 4, we prove that for a positive integer n, the linear rank-width of the complete binary tree of height n is $\lceil \frac{n}{2} \rceil$ and the linear rank-width of the incidence graph of the complete binary tree of height n is $\lfloor \frac{n}{2} \rfloor + 1$. Moreover, we prove that for a non-negative integer k, a tree has linear rank-width at least k, then it has the complete binary tree of height k as a vertex-minor.

But in Chapter 5, we prove that a complete binary tree of height at least 5 cannot be a pivot-minor of the incidence graph of a binary tree. It implies that the question for pivot-minor is false because the set of all the incidence graphs of binary trees have unbounded linear rank-width, by the result in Chapter 4. So we suggest a question that if C is a class of graphs with unbounded linear rank-width then all incidence graphs of trees are appeared as pivot-minors of graphs in C.

In Chapter 2, we introduce most of the terminology used in this thesis. Chapter 3 is a joint work with Oum [9].

G has rank-width $\leq k$	$\Rightarrow$	G is a pivot-minor of a graph of tree-width $\leq 2k$
G has linear rank-width $\leq k$	$\Rightarrow$	G is a pivot-minor of a graph of path-width $\leq k+1$
G has rank-width ≤ 1	$\Leftrightarrow$	G is a vertex-minor of a tree
G has linear rank-width ≤ 1	$\Leftrightarrow$	G is a vertex-minor of a path
G is bipartite and has rank-width ≤ 1	$\Leftrightarrow$	G is a pivot-minor of a tree
G is bipartite and has linear rank-width ≤ 1	$\Leftrightarrow$	G is a pivot-minor of a path

Table 1.1: Summary of theorems in Chapter 3

Chapter 2. Preliminaries

In the thesis, all graphs are simple and undirected. Let $G = (V, E)$ be a graph. For $v \in V$, let $N(v)$ be the set of vertices adjacent to v and $\deg(v) := |N(v)|$. And let $\delta(v)$ be the set of edges incident with v. For $S \subseteq V$, $G[S]$ denotes the subgraph of G induced on S. For two sets A and B, $A\Delta B = (A \cup B) \setminus (A \cap B)$.

A *vertex partition* of a graph G is a pair (A, B) of subsets of V such that $A \cup B = V$ and $A \cap B = \emptyset$. A vertex $v \in V$ is a *leaf* if $\deg(v) = 1$; Otherwise we call it an *inner vertex*. An edge $e \in E$ is an *inner edge* if e does not have a leaf as an end. Let $V_I(G)$ and $E_I(G)$ be the set of inner vertices of G and inner edges of G, respectively.

For an $X \times Y$ matrix M and subsets $A \subseteq X$ and $B \subseteq Y$, $M[A, B]$ denotes the $A \times B$ submatrix $(m_{i,j})_{i \in A, j \in B}$ of M. If $A = B$, then $M[A] = M[A, A]$ is called a *principal submatrix* of M. The adjacency matrix of a graph G, which is a $(0, 1)$-matrix over the binary field, will be denoted by $A(G)$.

A *rooted binary tree* is a tree with a root vertex such that the root has degree 2 and all other inner vertices have degree 3. For a positive integer n, the *complete rooted binary tree* of height n, denoted by T_n, is a rooted binary tree such that the length from the root to a leaf is n. The *incidence graph* $I(G)$ of a graph $G = (V, E)$ is a graph on vertices $V \cup E$ such that for $x, y \in V \cup E$, x is adjacent to y in $I(G)$ if and only if one of x, y is a vertex of G, the other is an edge of G and x is incident with y in G. The incidence graph of T_n is denoted by H_n. The graphs T_2 and H_2 are shown in Figure 2.1.

Pivoting matrices.

Let $M = \begin{array}{c} \\ X \\ V\setminus X \end{array} \!\!\begin{array}{c} \begin{array}{cc} X & V\setminus X \end{array} \\ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array}$ be a symmetric or skew-symmetric $V \times V$ matrix over a field F.

If $A = M[X]$ is nonsingular, then we define

$$M * X = \begin{array}{c} \\ X \\ V\setminus X \end{array} \!\!\begin{array}{c} \begin{array}{cc} X & V\setminus X \end{array} \\ \begin{pmatrix} A^{-1} & A^{-1}B \\ -CA^{-1} & D - CA^{-1}B \end{pmatrix} \end{array}.$$

This operation is called a *pivot*. Tucker showed the following theorem.

Theorem 2.1 (Tucker [17]). *Let $M[X]$ be a nonsingular principal submatrix of a square matrix M. Then $M * X[Y]$ is nonsingular if and only if $M[X\Delta Y]$ is nonsingular.*

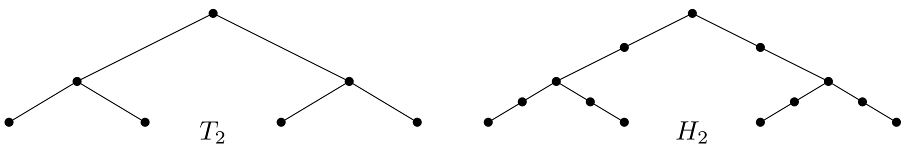

Figure 2.1: The complete binary tree T_2 and its incidence graph H_2.

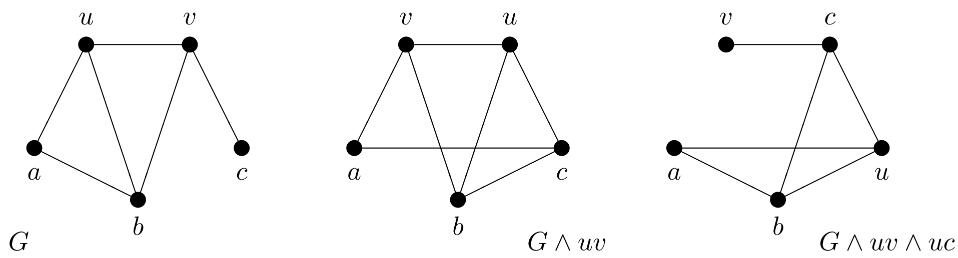

Figure 2.2: Pivoting an edge uv. Note that $G \wedge uv \wedge uc = G \wedge vc$.

Vertex-minors and pivot-minors.

The graph obtained from $G = (V, E)$ by applying *local complementation* at a vertex v is $G * v = (V, E\Delta\{xy : xv, yv \in E, x \neq y\})$. The graph obtained from G by *pivoting* an edge uv is defined by $G \wedge uv = G * u * v * u$.

To see how we obtain the resulting graph by pivoting an edge uv, let $V_1 = N(u) \cap N(v)$, $V_2 = N(u) \backslash N(v) \backslash \{v\}$ and $V_3 = N(v) \backslash N(u) \backslash \{u\}$. One can easily verify that $G \wedge uv$ is identical to the graph obtained from G by complementing adjacency of vertices between distinct sets V_i and V_j and swapping the vertices u and v [11]. See Figure 2.2 for example.

In fact, if $uv \in E$, then $A(G \wedge uv) = A(G) * \{u, v\}$. Since $\det(A(G)[\{u, v\}]) = A(G)(u, v)$, Theorem 2.1 is useful for dealing with a sequence of pivoting. In Figure 2.2, we can easily check that $G \wedge uv \wedge uc = G \wedge vc$. For $X \subseteq V$, if $A(G)[X]$ is nonsingular, then we denote $G \wedge X$ as the graph having the adjacency matrix $A(G) * X$.

A graph H is a *vertex-minor* of G if H can be obtained from G by applying a sequence of vertex deletions and local complementations. A graph H is a *pivot-minor* of G if H can be obtained from G by applying a sequence of vertex deletions and pivoting edges. From the definition, every pivot-minor of a graph is a vertex-minor of the graph. Note that every pivot-minor of a bipartite graph is bipartite.

Rank-width and linear rank-width.

The *cut-rank* function $\text{cutrk}_G : 2^V \to \mathbb{Z}$ of a graph G is defined by

$$\text{cutrk}_G(X) = \text{rank}(A(G)[X, V \backslash X]).$$

A tree is *subcubic* if it has at least two vertices and every inner vertex has degree 3. A *rank-decomposition* of a graph G is a pair (T, L), where T is a subcubic tree and L is a bijection from the vertices of G to the leaves of T. For an edge e in T, $T \backslash e$ induces a partition (X_e, Y_e) of the leaves of T. The *width* of an edge e is defined as $\text{cutrk}_G(L^{-1}(X_e))$. The *width* of a rank-decomposition (T, L) is the maximum width over all edges of T. The *rank-width* of G, denoted by $\text{rw}(G)$, is the minimum width of all rank-decompositions of G. If $|V| \leq 1$, then G admits no rank-decomposition and $\text{rw}(G) = 0$.

A subcubic tree is a *caterpillar* if it contains a path P such that every vertex of a tree has distance at most 1 to some vertex of P. A *linear rank-decomposition* of a graph G is a rank-decomposition (T, L) of G, where T is a caterpillar. The *linear rank-width* of G is defined as the minimum width of all linear rank-decompositions of G. If $|V| \leq 1$, then G admits no linear rank-decomposition and $\text{lrw}(G) = 0$.

A *linear layout* of G is a sequence $S = (v_1, v_2, \ldots, v_{|V(G)|})$ of $V(G)$ and for that sequence S, let $S(v_i) = i$. The *linear rank-width* of G with respect to S, denoted by $\text{lrw}_S(G)$, is defined as the maximum over all $\text{cutrk}\{v_k : 1 \leq k \leq i\}$ for $1 \leq i \leq |V(G)|$. Clearly, the set of all linear rank-decompositions of

G has one to one correspondence to the set of all linear layouts of G. In a linear rank-decomposition (T, L) of G, if G has rank-width at least 1, then the width of a leaf edge of T does not contribute to the width of the linear rank-decomposition. Thus, we easily verify that the width of the linear rank-decomposition with respect to S is the same as $\text{lrw}_S(T)$. Therefore the linear rank-width of G is the same as $\min\{\text{lrw}_S(G) : S \text{ is a linear layout of } G\}$. For two linear layouts $S_1 = (v_1, v_2, \ldots, v_n)$ and $S_2 = (w_1, w_2, \ldots, w_m)$, we define $S_1 \oplus S_2 = (v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_m)$.

Note that if a graph H is a vertex-minor or a pivot-minor of a graph G, then $\text{rw}(H) \leq \text{rw}(G)$ and $\text{lrw}(H) \leq \text{lrw}(G)$ [11]. Trivially, $\text{rw}(G) \leq \text{lrw}(G)$.

Tree-width and path-width.

Let T be a tree, and let $B = \{B_t\}_{t \in V(T)}$ be a family of vertex sets $B_t \subseteq V$ indexed by the vertices $t \in V(T)$, called *bags*. The pair (T, B) is called a *tree-decomposition* of G if it satisfies the following three axioms.

(T1) $V = \bigcup_{v \in V(T)} B_t$.

(T2) For every edge $uv \in E$, there exists a vertex t of T such that u, $v \in B_t$.

(T3) For t_1, t_2 and $t_3 \in V(T)$, $B_{t_1} \cap B_{t_3} \subseteq B_{t_2}$ whenever t_2 is on the path from t_1 to t_3.

The *width* of a tree-decomposition (T, B) is $\max\{|B_t| - 1 : t \in V(T)\}$. The *tree-width* of G, denoted by $\text{tw}(G)$, is the minimum width of all tree-decompositions of G. A *path-decomposition* of a graph G is a tree-decomposition (T, B) where T is a path. The *path-width* of G, denoted by $\text{pw}(G)$, is the minimum width of all path-decompositions of G.

Matroid.

Let E be a set and $\mathcal{I}$ be a set of subsets of E. A pair $\mathcal{M} = (E, \mathcal{I})$ is a *matroid* on the ground set E if it satisfies the following three axioms.

(T1) $\mathcal{I} \neq \emptyset$.

(T2) If $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$.

(T3) If $X, Y \in \mathcal{I}$ and $|X| = |Y| + 1$, then there exists $x \in X \setminus Y$ such that $Y \cup \{x\} \in \mathcal{I}$.

We call $X \in \mathcal{I}$ an *independent set* of the matroid $\mathcal{M}$. An independent set X of $\mathcal{M}$ is a *base* of $\mathcal{M}$ if it is a maximal independent set. We call $X \in \mathcal{I}$ an *cobase* of $\mathcal{M}$ if $E \setminus X$ is a base of $\mathcal{M}$. The *dual matroid* of $\mathcal{M}$ is $\mathcal{M}^* = (E, \{X \subseteq E : X \text{ is a subset of a cobase in } \mathcal{M}\})$. For $e \in E$, let $\mathcal{M} \setminus e = (E \setminus \{e\}, \mathcal{J})$ where $\mathcal{J} = \{X \subseteq E \setminus \{e\} : X \in \mathcal{I}\}$. We call it the *deletion* of e from $\mathcal{M}$. For $e \in E$, let $\mathcal{M}/e = (\mathcal{M}^* \setminus e)^*$ and call it the *contraction* of e from $\mathcal{M}$. A matroid $\mathcal{N}$ is a *minor* of a matroid $\mathcal{M}$ if $\mathcal{N}$ is obtained from $\mathcal{M}$ by a sequence of deletions and contractions.

Let A be a $V \times W$ matrix over a field $\mathbb{F}$. Let $\mathcal{M}(A) = (W, \mathcal{I})$ where $\mathcal{I}$ is the set of subsets of W which represent a linearly independent set over $\mathbb{F}$. Then $\mathcal{M}(A)$ is a matroid, and we call it a $\mathbb{F}$*-representable matroid*. In particular, a matroid representable over $GF(2)$ is called a *binary matroid*.

Let G be a graph. The *graphic matroid* $\mathcal{M}(G)$ of G is defined as $\mathcal{M}(G) = (E(G), \mathcal{I})$ where $\mathcal{I} = \{U \subseteq E(G) : U \text{ forms a forest in } G\}$. By the definition, a base of $\mathcal{M}(G)$ forms a spanning tree of G if G is connected. The next theorem is well known.

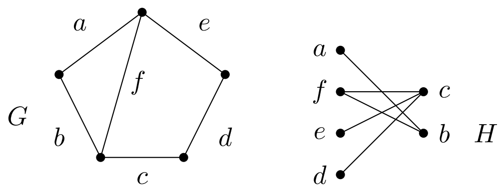

Figure 2.3: A graph G and the fundamental graph of the graphic matroid $\mathcal{M}(G)$ with respect to a base $\{a, f, e, d\}$.

Theorem 2.2 (See Oxley [13])**.** *Let G be a graph and I_G be the vertex-edge incidence matrix of G. Then the binary matroid $\mathcal{M}(I_G)$ is the same as $\mathcal{M}(G)$. Therefore every graphic matroid is a binary.*

For a base B in $\mathcal{M}$, a bipartite graph G with bipartition $B \cup (E \setminus B)$ is the *fundamental graph* of $\mathcal{M}$ with respect to B if for $v \in B$ and $w \in E \setminus B$, v and w are adjancet in G if and only if $B \setminus \{v\} \cup \{w\}$ is a base of $\mathcal{M}$. See Figure 2.3 for example.

Chapter 3. Graphs of small rank-width

3.1 Graphs of small rank-width are pivot-minor of graphs of small tree-width

In this section, for a graph G with rank-width k, we construct a graph having tree-width at most $2k$ such that it has G as a pivot-minor.

Theorem 3.1.1. *Let k be a non-negative integer. Let G be a graph of rank-width at most k and $|V(G)| \geq 3$. Then there exists a graph H having a pivot-minor isomorphic to G such that tree-width of H is at most $2k$ and $|V(H)| \leq (2k+1)|V(G)| - 6k$.*

Theorem 3.1.2. *Let k be a non-negative integer. Let G be a graph of linear rank-width at most k and $|V(G)| \geq 3$. Then there exists a graph H having a pivot-minor isomorphic to G such that path-width of H is at most $k+1$ and $|V(H)| \leq (2k+1)|V(G)| - 6k$.*

We need the following lemma.

Lemma 3.1.3. *Let G be a graph and (A_1, B_1), (A_2, B_2) be two vertex partitions such that $A_2 \subseteq A_1$. Let $S \subseteq A_1$ be a set corresponding to a basis of row vectors in $A(G)[A_1, B_1]$. Then there exists a subset of A_2 representing a basis of row vectors in $A(G)[A_2, B_2]$ containing $S \cap A_2$.*

Proof. Because $A_2 \subseteq A_1$, rows in $A(G)[S \cap A_2, B_2]$ are independent. Therefore we can extend $S \cap A_2$ to a basis of rows in $A(G)[A_2, B_2]$. □

To prove Theorems 3.1.1 and 3.1.2, we construct a *rank-expansion* of a graph. Let G be a connected graph and (T, L) be a rank-decomposition of G. We fix a leaf $x \in V(T)$. For $e \in E(T)$, let T_e be the component of $T \setminus e$ which does not contain x, and let $A_e = L^{-1}(V(T_e))$, $B_e = V(G) \setminus A_e$ and $M_e = A(G)[A_e, B_e]$. For each $a \in A_e$, let $R^e_a = M_e[\{a\}, B_e]$ the row vector of M_e.

First, for each edge $e = uv \in E(T)$, we orient the edge towards v if $v \in V(T_e)$. We choose a vertex set $U_e \subseteq A_e$ such that $\{R^e_w\}_{w \in U_e}$ forms a basis of row vectors in M_e and $(U_e \cap A_f) \subseteq U_f$ if the tail of an edge f is the head of e. Since R^e_a can be uniquely expressed as a linear combination of vectors of $\{R^e_w\}_{w \in U_e}$ for each $a \in A_e$, there exists a unique $A_e \times U_e$ matrix P_e such that $P_e A(G)[U_e, B_e] = A(G)[A_e, B_e]$. If the tail of an edge f is the head of an edge e, then let $C_f = P_e[U_f, U_e]$.

Let H be a *rank-expansion* $\boldsymbol{R}(G, T, L, x, \{U_f\}_{f \in E(T)})$ of a graph G such that

$$V(H) = \bigcup_{v \in V_I(T)} \bigcup_{e \in \delta(v)} (U_e \times \{e\} \times \{v\})$$

$$\begin{aligned} E(H) = \{\{(a,e,v),(a,e,w)\} &: e = vw \in E_I(T), a \in U_e\} \\ \cup \{\{(a,e,v),(b,f,v)\} &: v \in V_I(T), e, f \in E(T), v \text{ is the head of } e \text{ and the tail of } f, \\ &\qquad a \in U_f, b \in U_e \text{ and } C_f(a,b) \neq 0\} \\ \cup \{\{(a,f_1,v),(b,f_2,v)\} &: v \text{ is the tail of both } f_1 \text{ and } f_2 \in E(T), \\ &\qquad a \in U_{f_1}, b \in U_{f_2} \text{ and } ab \in E(G)\}. \end{aligned}$$

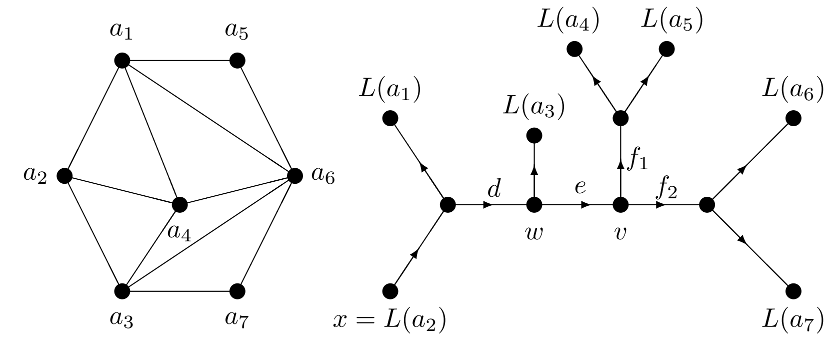

Figure 3.1: A graph G and a rank-decomposition (T, L) of G with a fixed leaf $x \in V(T)$. Note that the edge $e \in E(T)$ has width 3 and e is directed from w to v.

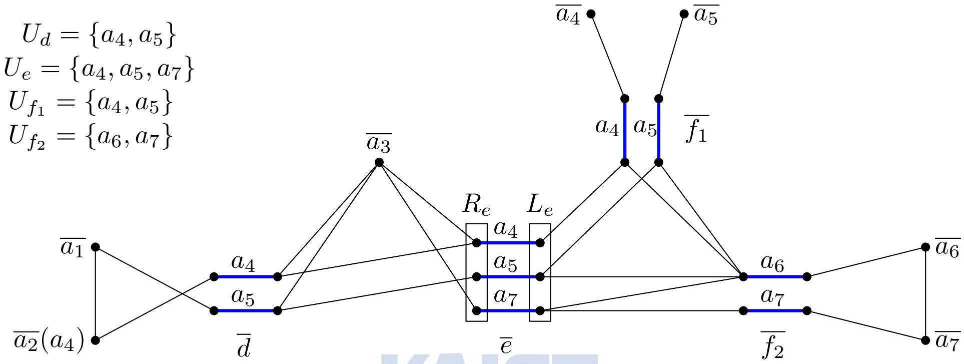

Figure 3.2: A rank-expansion of the graph G in Figure 3.1.

For $v \in V_I(T)$, let $S_v = \bigcup_{e \in \delta(v)} U_e \times \{e\} \times \{v\} \subseteq V(H)$. For $e = vw \in E_I(T)$, let $\overline{e} = \{(a, e, v), (a, e, w) : a \in U_e\} \subseteq V(H)$ and for $W \subseteq E_I(T)$, let $\overline{W} = \bigcup_{f \in W} \overline{f} \subseteq V(H)$. If $e \in E_I(T)$ is directed from w to v, let $L_e = S_v \cap \overline{e}$ and $R_e = S_w \cap \overline{e}$. For a vertex a in $V(G)$ and $e = \{L(a), v\} \in E(T)$, let $\overline{a}$ be the unique vertex in $U_e \times \{e\} \times \{v\}$ and let $\overline{e} = \overline{a}$.

We discuss the number of vertices in the rank-expansion H. We easily observe that $|E_I(T)| = |V(G)| - 3$. So if $\text{rw}(G) \leq k$, then $|\overline{e}| \leq 2k$ for each $e \in E_I(T)$, and we deduce that $|V(H)| \leq 2k|E_I(T)| + |V(G)| = 2k(|V(G)| - 3) + |V(G)| = (2k+1)|V(G)| - 6k$.

First, we prove that every rank-expansion of a graph has the given graph as a pivot-minor. To obtain G as a pivot-minor of H, we will pivot $\bigcup_{e \in E_I(T)} \overline{e}$ to H.

Lemma 3.1.4. *Let G be a graph and $uv \in E(G)$. If $\deg(u) = 1$, then $G \wedge uv \setminus \{u, v\} = G \setminus \{u, v\}$.*

Proof. It is clear from the definition. □

For convenience, let $\det(A(H)[\emptyset]) = 1$.

Lemma 3.1.5. *Let $W \subseteq E_I(T)$. Then $A(H)[\overline{W}]$ is nonsingular.*

Proof. We proceed by induction on $|W|$. If W is empty, then it is trivial. If $|W| \geq 1$, then W induces a forest in T, and therefore there must be an edge $f \in W$ which has a leaf in $T[W]$. By induction hypothesis, $A(H)[\overline{W \setminus \{f\}}]$ is nonsingular. Since every edge in $H[\overline{f}]$ is incident with a leaf in $H[\overline{W}]$, by Lemma 3.1.4, pivoting all edges in $\overline{f}$ does not change the graph $H[\overline{W \setminus \{f\}}]$. So, $A(H[\overline{W}] \wedge \overline{f})[\overline{W \setminus \{f\}}] = A(H)[\overline{W \setminus \{f\}}]$ and therefore, by Theorem 2.1, $A(H)[\overline{f} \Delta \overline{W \setminus \{f\}}] = A(H)[\overline{W}]$ is nonsingular. □

Lemma 3.1.6. *Let $a, b \in V(G)$ and let P be a path from $L(a)$ to $L(b)$ in T. Then for $E(P) \cap E_I(T) \subseteq W \subseteq E_I(T)$, $A(H)[\overline{W} \cup \{\overline{a}, \overline{b}\}]$ is nonsingular if and only if $A(H)[\overline{E(P)}]$ is nonsingular.*

Proof. We use induction on $|W|$. If $W = E(P) \cap E_I(T)$, then it is trivial, because $\overline{W} \cup \{\overline{a}, \overline{b}\} = \overline{E(P)}$. So we may assume that $|W| > |E(P) \cap E_I(T)|$. Since P is a maximal path in T, the subgraph of T having the edge set $W \cup E(P)$ must have at least 3 leaves. Thus there is an edge f in $W \setminus E(P)$ incident with a leaf in $T[W \cup E(P)]$ other than $L(a)$ and $L(b)$. Since every edge in $\overline{f}$ is incident with a leaf in $H[\overline{W}]$, by Lemma 3.1.4, $A(H[\overline{W} \cup \{\overline{a}, \overline{b}\}] \wedge \overline{f})[\overline{W \setminus \{f\}} \cup \{\overline{a}, \overline{b}\}] = A(H)[\overline{W \setminus \{f\}} \cup \{\overline{a}, \overline{b}\}]$. By induction hypothesis and Theorem 2.1, we deduce that

$$\begin{aligned} A(H)[\overline{E(P)}] \text{ is nonsingular} &\Leftrightarrow A(H)[\overline{W \setminus \{f\}} \cup \{\overline{a}, \overline{b}\}] \text{ is nonsingular} \\ &\Leftrightarrow A(H[\overline{W} \cup \{\overline{a}, \overline{b}\}] \wedge \overline{f})[\overline{W \setminus \{f\}} \cup \{\overline{a}, \overline{b}\}] \text{ is nonsingular} \\ &\Leftrightarrow A(H)[\overline{W} \cup \{\overline{a}, \overline{b}\}] \text{ is nonsingular.} \end{aligned}$$

□

Lemma 3.1.7. *Let $P = (e_{n+1}, e_n, \ldots, e_1)$ be the directed path from w to v in T.*
Then $C_{e_1} C_{e_2} \ldots C_{e_n} A(G)[U_{e_{n+1}}, B_{e_{n+1}}] = A(G)[U_{e_1}, B_{e_{n+1}}]$.

Proof. We proceed by induction on n. If $n = 1$, then by definition,

$$C_{e_1} A(G)[U_{e_2}, B_{e_2}] = P_{e_2}[U_{e_1}, U_{e_2}] A(G)[U_{e_2}, B_{e_2}] = A(G)[U_{e_1}, B_{e_2}].$$

We may assume that $n \geq 2$. By induction hypothesis,

$$C_{e_2} C_{e_3} \ldots C_{e_n} A(G)[U_{e_{n+1}}, B_{e_{n+1}}] = A(G)[U_{e_2}, B_{e_{n+1}}].$$

Since $C_{e_1} A(G)[U_{e_2}, B_{e_2}] = A(G)[U_{e_1}, B_{e_2}]$ and $B_{e_{n+1}} \subseteq B_{e_2}$,

$$C_{e_1} A(G)[U_{e_2}, B_{e_{n+1}}] = A(G)[U_{e_1}, B_{e_{n+1}}].$$

Therefore, we conclude that $C_{e_1} C_{e_2} \ldots C_{e_n} A(G)[U_{e_{n+1}}, B_{e_{n+1}}] = C_{e_1} A(G)[U_{e_2}, B_{e_{n+1}}] = A(G)[U_{e_1}, B_{e_{n+1}}]$.

□

Lemma 3.1.8.

$$\det \left(\begin{array}{c|cccccc} 0 & C_1 & 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & I & C_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & C_3 & & 0 & 0 \\ 0 & 0 & 0 & I & & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I & C_n \\ C_{n+1} & 0 & 0 & 0 & \cdots & 0 & I \end{array}\right) = (-1)^n \det(C_1 C_2 \ldots C_{n+1}).$$

Proof. By elementary row operation,

$$\det \left(\begin{array}{c|cccccc} 0 & C_1 & 0 & 0 & \cdots & 0 & 0 \\ \hline 0 & I & C_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & C_3 & & 0 & 0 \\ 0 & 0 & 0 & I & & 0 & 0 \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I & C_n \\ C_{n+1} & 0 & 0 & 0 & \cdots & 0 & I \end{array}\right)$$

$$
= \det \left(\begin{array}{c|cccccc}
0 & 0 & -C_1C_2 & 0 & \cdots & 0 & 0 \\ \hline
0 & I & C_2 & 0 & \cdots & 0 & 0 \\
0 & 0 & I & C_3 & & 0 & 0 \\
0 & 0 & 0 & I & & 0 & 0 \\
\vdots & & & & \ddots & & \vdots \\
0 & 0 & 0 & 0 & \cdots & I & C_n \\
C_{n+1} & 0 & 0 & 0 & \cdots & 0 & I
\end{array} \right)
$$

$$
= \det \left(\begin{array}{c|cccccc}
0 & 0 & 0 & (-1)^2 C_1C_2C_3 & \cdots & 0 & 0 \\ \hline
0 & I & C_2 & 0 & \cdots & 0 & 0 \\
0 & 0 & I & C_3 & & 0 & 0 \\
0 & 0 & 0 & I & & 0 & 0 \\
\vdots & & & & \ddots & & \vdots \\
0 & 0 & 0 & 0 & \cdots & I & C_n \\
C_{n+1} & 0 & 0 & 0 & \cdots & 0 & I
\end{array} \right)
$$

$$
= \det \left(\begin{array}{c|cccccc}
(-1)^n C_1C_2 \dots C_{n+1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \hline
0 & I & C_2 & 0 & \cdots & 0 & 0 \\
0 & 0 & I & C_3 & & 0 & 0 \\
0 & 0 & 0 & I & & 0 & 0 \\
\vdots & & & & \ddots & & \vdots \\
0 & 0 & 0 & 0 & \cdots & I & C_n \\
C_{n+1} & 0 & 0 & 0 & \cdots & 0 & I
\end{array} \right)
$$

$$
= (-1)^n \det(C_1C_2 \dots C_{n+1}). \qquad \square
$$

Proposition 3.1.9. *Let $k \geq 1$. Let G be a connected graph with rank-width k and $|V(G)| \geq 3$. Then a rank-expansion of G has a pivot-minor isomorphic to G.*

Proof. Let (T, L) be a rank decompostion of a graph G and let x be a leaf in T. We orient each edge f away from x. For each $f \in E(T)$, if m is the width of f, we choose a basis $U_f = \{u_1^f, u_2^f, \dots, u_m^f\} \subseteq A_f$ of rows in the matrix $A(G)[A_f, B_f]$ such that $(U_e \cap A_f) \subseteq U_f$ if the head of an edge e is the tail of f. Since G is connected, $|U_f| \geq 1$. Let H be a rank-expansion $\boldsymbol{R}(G, T, L, x, \{U_f\}_{f \in E(T)})$ of a graph G. By Lemma 3.1.4, for every $W \subseteq E_I(T)$, $A(H)[\overline{W}]$ is nonsingular. We will prove that for a, $b \in V(G)$, $\overline{a}\overline{b} \in E(H \wedge \overline{E_I(T)})$ if and only if $ab \in E(G)$.

Let a, b be distinct vertices in $V(G)$. We consider the path P from $L(a)$ to $L(b)$ in T. By Lemma 3.1.6, $\overline{a}$ is adjacent to $\overline{b}$ in $H \wedge \overline{E_I(T)}$ if and only if $\overline{a}$ is adjacent to $\overline{b}$ in $H[\overline{E(P)}] \wedge (\overline{E(P) \cap E_I(T)})$. Therefore, by Theorem 2.1,

$$
\begin{aligned}
\overline{a}\overline{b} \in E(H \wedge \overline{E_I(T)}) &\Leftrightarrow \overline{a}\overline{b} \in E(H[\overline{E(P)}] \wedge (\overline{E(P) \cap E_I(T)})) \\
&\Leftrightarrow A\left(H[\overline{E(P)}] \wedge (\overline{E(P) \cap E_I(T)})\right)[\{\overline{a}, \overline{b}\}] \text{ is nonsingular} \\
&\Leftrightarrow A\left(H[\overline{E(P)}]\right)[(\overline{E(P) \cap E_I(T)}) \Delta \{\overline{a}, \overline{b}\}] \text{ is nonsingular} \\
&\Leftrightarrow A(H[\overline{E(P)}]) \text{ is nonsingular.}
\end{aligned}
$$

Thus, it is enough to show that $\det(A(H[\overline{E(P)}])) = A(G)(a, b)$.

If $L(b) = x$, then $P = (e_{n+1}, e_n, \dots, e_1, e_0)$ is a directed path from $L(b)$ to $L(a)$. The submatrix of $A(H)$ induced by $\overline{E(P)}$ is

$$
\begin{array}{c|c|ccccc|c|ccccc}
 & \bar{b} & L_{e_1} & L_{e_2} & \cdots & L_{e_{n-1}} & L_{e_n} & \bar{a} & R_{e_1} & R_{e_2} & \cdots & R_{e_{n-1}} & R_{e_n} \\
\bar{a} & 0 & C_{e_0} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\hline
R_{e_1} & 0 & I & C_{e_1} & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
R_{e_2} & 0 & 0 & I & & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & & & \ddots & & \vdots & 0 & & & \ddots & & \vdots \\
R_{e_{n-1}} & 0 & 0 & 0 & \cdots & I & C_{e_{n-1}} & 0 & 0 & 0 & \cdots & 0 & 0 \\
R_{e_n} & C_{e_n} & 0 & 0 & \cdots & 0 & I & 0 & 0 & 0 & \cdots & 0 & 0 \\
\hline
\bar{b} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & C^t_{e_n} \\
\hline
L_{e_1} & 0 & 0 & 0 & \cdots & 0 & 0 & C^t_{e_0} & I & 0 & \cdots & 0 & 0 \\
L_{e_2} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & C^t_{e_1} & I & & 0 & 0 \\
\vdots & \vdots & & & \ddots & & \vdots & 0 & & & \ddots & & \vdots \\
L_{e_{n-1}} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & I & 0 \\
L_{e_n} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & C^t_{e_{n-1}} & I
\end{array}
$$

$$= \left(\begin{array}{c|c} C & 0 \\ \hline 0 & C^t \end{array} \right).$$

Note that $\det(A(H)[\overline{E(P)}]) = \det(C)\det(C^t) = \det(C)^2$. By Lemma 3.1.8,

$$\det(C) = (-1)^n \det(C_{e_0} C_{e_1} \dots C_{e_n}).$$

Since $|U_{e_{n+1}}| = |B_{e_{n+1}}| = 1$ and $\operatorname{rank}(A(G)[U_e, B_e]) = |U_e|$ for all edges $e \in E(T)$,

$$A(G)[U_{e_{n+1}}, B_{e_{n+1}}] = (1).$$

By Lemma 5.1.3,

$$\begin{aligned}
C_{e_0} C_{e_1} \dots C_{e_n} &= C_{e_0} C_{e_1} \dots C_{e_n} A(G)[U_{e_{n+1}}, B_{e_{n+1}}] \\
&= A(G)[U_{e_0}, B_{e_{n+1}}] \\
&= A(G)(a, b).
\end{aligned}$$

Therefore $\det(A(H)[\overline{E(P)}]) = A(G)(a, b)$, as required.

Now we assume that $L(a) \neq x$ and $L(b) \neq x$. Then there exists a vertex y in $V(P)$ such that it has a shortest distance to x. Let $P_1 = (e_n, e_{n-1}, \dots, e_0)$ be the edges of P from y to $L(a)$ and $P_2 = (f_m, f_{m-1}, \dots, f_0)$ be the edges of P from y to $L(b)$.

Let $M = A(H)[R_{e_n}, R_{f_m}]$. By the construction of a rank-expansion, $M = A(G)[U_{e_n}, U_{f_m}]$. The submatrix of $A(H)$ induced by $\overline{E(P)}$ is

$$
\begin{array}{c|c|c}
 & \{\bar{b}\} \cup \bigcup_{i=1}^{n} L_{e_i} \cup \bigcup_{i=1}^{m} R_{f_i} & \{\bar{a}\} \cup \bigcup_{i=1}^{n} R_{e_i} \cup \bigcup_{i=1}^{m} L_{f_i} \\
\{\bar{a}\} \cup \bigcup_{i=1}^{n} R_{e_i} \cup \bigcup_{i=1}^{m} L_{f_i} & C & 0 \\
\hline
\{\bar{b}\} \cup \bigcup_{i=1}^{n} L_{e_i} \cup \bigcup_{i=1}^{m} R_{f_i} & 0 & C^t
\end{array}
$$

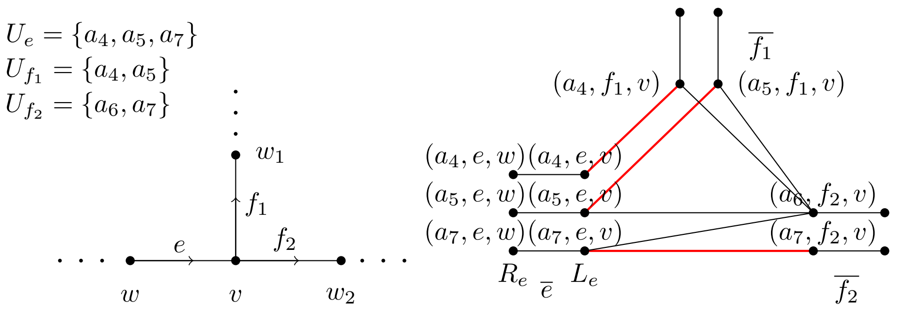

Figure 3.3: A rank-expansion of the graph G in Figure 3.1. By the construction of a rank-expansion, every vertex in L_e has exactly one neighbor in $R_{f_1} \cup R_{f_2} \setminus \{(a_6, f_2, v)\}$ in the subgraph $H[S_v]$.

where C is

$$
\begin{array}{c} \\ \bar{a} \\ R_{e_1} \\ R_{e_2} \\ \vdots \\ R_{e_{n-1}} \\ R_{e_n} \\ L_{f_m} \\ L_{f_{m-1}} \\ \vdots \\ L_{f_2} \\ L_{f_1} \end{array}
\begin{array}{c}
\begin{array}{cccccccccccc} \bar{b} & L_{e_1} & L_{e_2} & \cdots & L_{e_{n-1}} & L_{e_n} & R_{f_m} & R_{f_{m-1}} & \cdots & R_{f_2} & R_{f_1} \end{array} \\
\left(\begin{array}{c|ccccc|ccccc}
0 & C_{e_0} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\hline
0 & I & C_{e_1} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & I & & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & & & \ddots & & \vdots & & & \ddots & & \vdots \\
0 & 0 & 0 & \cdots & I & C_{e_{n-1}} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & I & M & 0 & \cdots & 0 & 0 \\
\hline
0 & 0 & 0 & \cdots & 0 & 0 & I & C^t_{f_{m-1}} & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & I & & 0 & 0 \\
\vdots & & & \ddots & & \vdots & & & \ddots & & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & I & C^t_{f_1} \\
C^t_{f_0} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & I
\end{array}\right)
\end{array}.
$$

It is enough to show that

$$C_{e_0} C_{e_1} \dots C_{e_{n-1}} M C^t_{f_{m-1}} C^t_{f_{m-2}} \dots C^t_{f_0} = A(G)(a, b).$$

Since $M = A(G)[U_{e_n}, U_{f_m}] \subseteq A(G)[U_{e_n}, B_{e_n}]$, by Lemma 5.1.3, we have

$$
\begin{aligned}
& C_{e_0} C_{e_1} \dots C_{e_{n-1}} M C^t_{f_{m-1}} C^t_{f_{m-2}} \dots C^t_{f_0} \\
& = C_{e_0} C_{e_1} \dots C_{e_{n-1}} A(G)[U_{e_n}, U_{f_m}] C^t_{f_{m-1}} C^t_{f_{m-2}} \dots C^t_{f_0} \\
& = A(G)[U_{e_0}, U_{f_m}] C^t_{f_{m-1}} C^t_{f_{m-2}} \dots C^t_{f_0} \\
& = (C_{f_0} C_{f_1} \dots C_{f_{m-1}} A(G)[U_{f_m}, U_{e_0}])^t \\
& = A(G)[U_{f_0}, U_{e_0}]^t = A(G)(a, b).
\end{aligned}
$$

So, $\det(A(H)[\overline{E(P)}]) = A(G)(a, b)$, as claimed. Therfore, $\bar{a}\bar{b} \in E(H \wedge \overline{E_I(T)})$ if and only if $ab \in E(G)$. We conclude that a rank-expansion of G has a pivot-minor isomorphic to G. $\square$

In the next proposition, we show that a rank-expansion has tree-width at most $2k$ when $\mathrm{rw}(G) \leq k$.

Proposition 3.1.10. *Let $k \geq 1$. Let G be a connected graph with $|V(G)| \geq 3$. If G has rank-width k, Then G has a rank-expansion of tree-width at most $2k$. Moreover, if G has linear rank-width k, then G has a rank-expansion of path-width at most $k + 1$.*

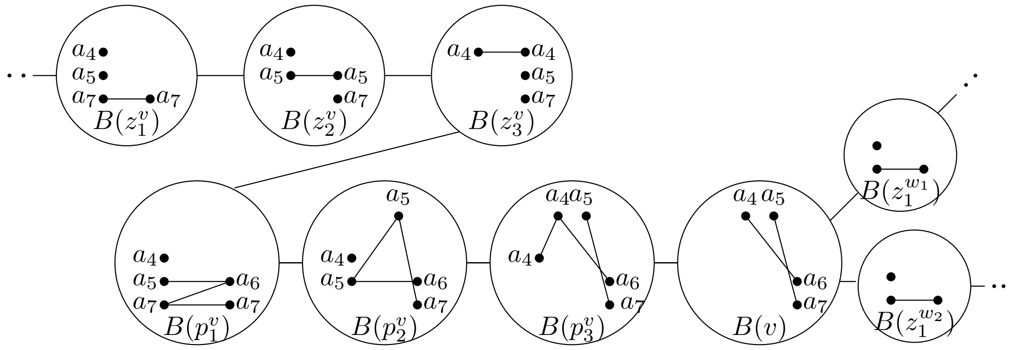

Figure 3.4: Tree-decomposition of a rank-expansion in Figure 3.3. The vertex sets $B(z_i^v)$ and $B(p_i^v)$, defined in Proposition 3.1.9, are bags which decompose $H[\overline{e}]$ and $H[S_v]$, respectively.

Proof. Let (T, L) be a rank-decomposition of G of width k. We fix a leaf $x \in V(T)$ and orient each edge f away from x. For each $f \in E(T)$, if m is the width of f, we choose a basis $U_f = \{u_1^f, u_2^f, \dots, u_m^f\} \subseteq A_f$ of rows in the matrix $A(G)[A_f, B_f]$ such that $(U_e \cap A_f) \subseteq U_f$ if the head of an edge e is the tail of f. Since G is connected, $|U_f| \geq 1$. Let H be a rank-expansion $\boldsymbol{R}(G, T, L, x, \{U_f\}_{f \in E(T)})$ of a graph G.

Let T' be a tree obtained from $T[V_I(T)]$ by replacing each edge from w to v with a path

$$wz_1^v z_2^v \dots z_{|U_e|}^v p_1^v p_2^v \dots p_{|U_e|}^v v.$$

Let y be the neighbor of x in T and let $B(y) = S_y$. For $v \in V_I(T) \setminus \{y\}$, let $e = vw$ be the edge incoming to v and f_1, f_2 be edges outgoing from v. Let $R^v = \{(a, f, v) \in R_{f_1} \cup R_{f_2} : a \notin U_e\}$. Since $(U_e \cap A_{f_i}) \subseteq U_{f_i}$ for each $i \in \{1, 2\}$, each vertex in L_e has exactly one neighbor in $R_{f_1} \cup R_{f_2} \setminus R^v$. Let $B(v) = R_{f_1} \cup R_{f_2}$ and $B(z_1^v) = R_e \cup \{(u_1^e, e, v)\}$, $B(p_1^v) = R^v \cup L_e \cup \{(a, f, v) \in R_{f_1} \cup R_{f_2} : a = u_1^e\}$. And for each $2 \leq i \leq |U_e|$, we define

$$\begin{aligned} B(z_i^v) &= B(z_{i-1}^v) \setminus \{(u_{i-1}^e, e, w)\} \cup \{(u_i^e, e, v)\} \\ B(p_i^v) &= B(p_{i-1}^v) \setminus \{(u_{i-1}^e, e, v)\} \cup \{(a, f, v) \in R_{f_1} \cup R_{f_2} : a = u_i^e\}. \end{aligned}$$

Now we show that the pair $(T', \{B(v)\}_{v \in V(T')})$ is a tree-decomposition of H. Note that for each $v \in V_I(T) \setminus \{y\}$ with the incoming edge e, $\bigcup_i E(H[B(z_i^v)]) = E(H[\overline{e}])$ and $\bigcup_i E(H[B(p_i^v)]) = E(H[S_v])$. Therefore all vertices and all edges in H are covered by $B(v)$ for some $v \in V(T')$. So the first and second axioms of a tree-decomposition are satisfied.

For the third axiom, it suffices to show that for every $t \in V(H)$, $T'[\{z : B(z) \ni t\}]$ is a subtree of T'. Let $t = (u_j^e, e, v) \in V(H)$ for some $e = vw \in E(T)$ and $1 \leq j \leq |U_e|$. If v is the head of e,

$$T'[\{z : B(z) \ni t\}] = T'[\{z_j^v, \dots, z_{|U_e|}^v, p_1^v, \dots, p_j^v\}],$$

and it forms a path. Suppose v is the tail of e. Let f be the edge incoming to v, and if $a \in U_f$, then let h be the integer such that $a = u_h^f$, if otherwise, let $h = 1$. Then

$$T'[\{z : B(z) \ni t\}] = T'[\{p_h^v, \dots, p_{|U_e|}^v, v, z_1^w, \dots, z_j^w\}].$$

It also forms a path, thus $(T', \{B(v)\}_{v \in V(T')})$ is a tree-decomposition of H.

Since $|B(y)| \leq 2k + 1$ and for each $v \in V_I(T) \setminus \{y\}$ with the incoming edge e, $|B(z_i^v)| = |B(z_1^v)| = |R_e| + 1 \leq k + 1$, $|B(p_i^v)| = |B(p_1^v)| = |R^v| + |L_e| + 1 \leq (2k - |U_e|) + |U_e| + 1 = 2k + 1$ and $|B(v)| \leq 2k$, the resulting tree-decomposition has width at most $2k$.

Suppose that G has linear rank-width at most k. Here, we choose $x \in V(T)$ such that x is an end of a longest path in T, and let y be the neighbor of x. For $v \in V_I(T)$ with outgoing edges f_1 and f_2, $|U_{f_1}| = 1$ or $|U_{f_2}| = 1$ because every inner vertex of T is incident with a leaf. Therefore, for each $v \in V_I(T) \setminus \{y\}$ and $1 \le i \le |U_e|$, $|B(p_i^v)| \le (k+1-|U_e|) + |U_e| + 1 = k+2$ and $|B(v)| \le k+1$, and $|B(y)| \le k+2$. Moreover, since $T[V_I(T)]$ is a path, T' is also a path. Therefore $(T', \{B(v)\}_{v \in V(T')})$ is a path-decomposition of H with path-width at most $k+1$. □

Proof of Theorem 3.1.1. If $k = 0$, then it is trivial. We assume that $k \ge 1$. We proceed by induction on the number of vertices.

Suppose G is connected. Since G has rank-width at most k and $|V(G)| \ge 3$, by Proposition 3.1.10, there is a rank-expansion H of G such that $\mathrm{tw}(H) \le 2k$, and $|V(H)| \le (2k+1)|V(G)| - 6k$. By Proposition 3.1.9, H has a pivot-minor isomorphic to G.

If G is disconnected, then we choose a largest component Y of G. Since $k \ge 1$, the component Y has at least 2 vertices. If $|V(Y)| = 2$, then G has rank-width 1 and tree-width 1, and $|V(G)| \le (2+1)|V(G)| - 6$ since $|V(G)| \ge 3$. We assume that $|V(Y)| \ge 3$. Then by induction hypothesis, there is a graph H_1 such that Y is isomorphic to a pivot-minor of H_1 and $\mathrm{tw}(H_1) \le 2k$ and $|V(H_1)| \le (2k+1)|V(Y)| - 6k$.

If $G \setminus V(Y)$ has tree-width at most 1, then G is isomorphic to a pivot-minor of the disjoint union of two graphs H_1 and $G \setminus V(Y)$, and the tree-width of it is equal to the tree-width of H_1. Since $|V(H_1)| + |V(G) \setminus V(Y)| \le (2k+1)|V(Y)| - 6k + |V(G) \setminus V(Y)| \le (2k+1)|V(G)| - 6k$, we obtain the result. If tree-width of $G \setminus V(Y)$ is at least 2, then $|V(G) \setminus V(Y)| \ge 3$. Therefore, by induction hypothesis, there is a graph H_2 such that $G \setminus V(Y)$ is isomorphic to a pivot-minor of H_2 and $\mathrm{tw}(H_2) \le 2k$ and $|V(H_2)| \le (2k+1)|V(G) \setminus V(Y)| - 6k$. So G is isomorphic to a pivot-minor of the disjoint union of two graphs H_1 and H_2, and the tree-width of it is at most $2k$, and $|V(H_1)| + |V(H_2)| \le (2k+1)|V(G)| - 6k$. Thus, we conclude the theorem. □

Proof of Theorem 3.1.2. We can easily obtain the proof of Theorem 3.1.2 from the proof of Theorem 3.1.1. □

3.2 Graphs with rank-width or linear rank-width at most 1

Distance-hereditary graphs are introduced by Bandelt and Mulder [2]. A graph G is *distance-hereditary* if for every connected induced subgraph H of G and vertices a, b in H, the distance between a and b in H is the same as in G. Oum [11] showed that distance-hereidtary graphs are exactly graphs of rank-width at most 1. Recently, Ganian [8] obtain a similar characterization of graphs of linear rank-width 1. In this section, we obtain another characterization for these classes in terms of vertex-minor relation.

Note that every tree has rank-width at most 1 and every path has linear rank-width at most 1.

Theorem 3.2.1. *Let G be a graph. The following are equivalent:*

1. *G has rank-width at most 1.*
2. *G is distance-hereditary.*
3. *G has no vertex-minor isomorphic to C_5.*
4. *G is a vertex-minor of a tree.*

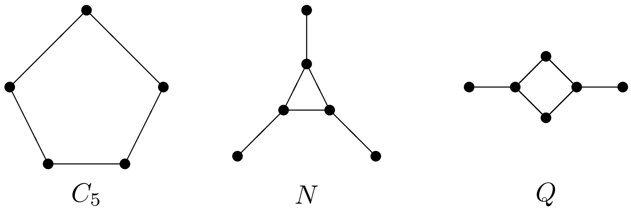

Figure 3.5: The graphs C_5, N and Q.

Proof. ((1) $\Leftrightarrow$ (2)) is proved by Oum [11], and ((2) $\Leftrightarrow$ (3)) follows from the Bouchet's theorem [4, 5]. Since every tree has rank-width at most 1, ((4) $\Rightarrow$ (1)) is trivial. We want to prove that (1) implies (4).

Let G be a graph of rank-width at most 1. We may assume that G is connected. If $|V(G)| \leq 2$, then G itself is a tree. So we may assume that $|V(G)| \geq 3$. Let (T, L) be a rank-decomposition of G of width 1. From Proposition 3.1.9, a rank-expansion H with the rank-decomposition (T, L) has G as a pivot-minor.

The width of each edge in T is 1. Thus for $v \in V_I(T)$, the subgraph $H[S_v]$ is a path of length 2 or a triangle because G is connected. Also for $e \in E_I(T)$, $H[\overline{e}]$ consists of an edge. Therefore H is connected and does not have cycles of length at least 4.

Let Q be a tree obtained from H by replacing each triangle abc with $K_{1,3}$ by adding a new vertex d, making d adjacent to a, b, c and deleting ab, bc, ca. Clearly H is a vertex-minor of the tree Q because we can obtain the graph H from Q by applying local complementation on those new vertices and deleting them. Therefore G is a vertex-minor of a tree, as required. □

We also obtain a characterization of graphs with linear rank-width at most 1. Obstructions sets for graphs of linear rank-width 1 are C_5, N and Q [1], depicted in Figure 3.5.

Lemma 3.2.2. *Every subcubic caterpillar is a pivot-minor of a path.*

Proof. Let H be a subcubic caterpillar. By the definition of a caterpillar, there is a path P in H such that every vertex in $V(H) \setminus V(P)$ is a leaf. We choose such path $P = p_1p_2 \ldots p_m$ in H with maximum length. We construct a path Q from P by replacing each edge p_ip_{i+1} with a path $p_ia_ib_ip_{i+1}$. We can obtain a pivot-minor of Q isomorphic to H by pivoting each edge a_ib_i and deleting all a_i and deleting b_i if p_i is not adjacent to a leaf in H. □

Theorem 3.2.3. *Let G be a graph. The following are equivalent:*

1. *G has linear rank-width at most* 1.
2. *G has no vertex-minor isomorphic to C_5, N or Q.*
3. *G is a vertex-minor of a path.*

Proof. ((1) $\Leftrightarrow$ (2)) is proved by Adler, Farley and Proskurowski [1]. Since every path has linear rank-width at most 1, ((3) $\Rightarrow$ (1)) is trivial. Let us prove that (1) implies (3).

Let G be a graph of linear rank-width at most 1. We may assume that G is connected and $|V(G)| \geq 3$. Let H be a rank-expansion of G with a linear rank-decompostion (T, L) of width 1. Note that T is a caterpillar.

Since (T, L) is a linear rank-decomposition of width 1, for each triangle in H, one of those vertices is of degree 2 in H. Let P be a caterpillar obtained from H by replacing each triangle with a path of length

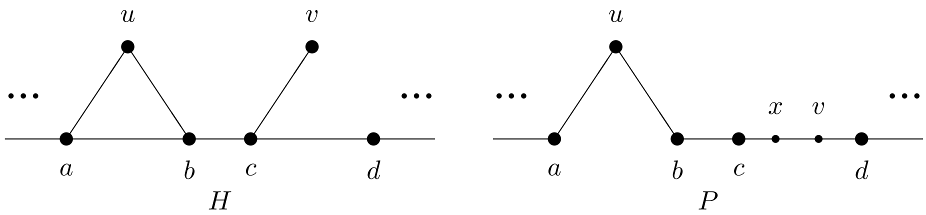

Figure 3.6: A rank-expansion H of a graph with linear rank-width 1. The graph H can be obtained from a path P by applying local complementation on u and pivoting xv and deleting x.

2 whose internal vertex has degree 2 in H. We can obtain H from P by applying local complementation on the inner vertex of those paths of length 2, H is a vertex-minor of P. And by Lemma 3.2.2, P is a pivot-minor of a path. Therefore G is a vertex-minor of a path. □

In Theorems 3.2.1 and 3.2.2, if a given graph is bipartite, we do not need to apply local complementation at some vertices. To prove it, we need the following lemma.

Lemma 3.2.4. *Let G be a connected bipartite graph with rank-width 1 and $|V(G)| \geq 3$. Let (T, L) be a rank-decomposition of width 1. Then a rank-expansion of G with respect to (T, L) is a tree.*

Proof. Let $x \in V(T)$ be a leaf and H be a rank-expansion $\boldsymbol{R}(G, T, L, x, \{U_f\}_{f \in E(T)})$ of G.

Suppose that H has a triangle. Then there exists a vertex $v \in V_I(T)$ such that $H[S_v]$ is the triangle. Let e_1, e_2 and e_3 be edges incident with v and assume that e_1 is the incoming edge. Let $U_{e_1} = \{a\}$, $U_{e_2} = \{b\}$ and $U_{e_3} = \{c\}$. By the construction of a rank-expansion, $bc \in E(G)$ and $R_a^{e_1} = R_b^{e_1} = R_c^{e_1}$. Since $R_a^{e_1}$ is a non-zero vector, there is a vertex $x \in V(G)$ such that x is adjacent to all of a, b and c. Therefore xbc is a triangle in G, contradiction. □

Theorem 3.2.5. *Let G be a graph. Then G is bipartite and has rank-width at most 1 if and only if G is a pivot-minor of a tree.*

Proof. We may assume that G is connected. Since every tree has rank-width at most 1, backward direction is trivial. If G is bipartite and has rank-width at most 1, then by Lemma 3.2.4, we have a rank-expansion of G which is a tree. Hence, G is a pivot-minor of a tree. □

Theorem 3.2.6. *Let G be a graph. Then G is bipartite and has linear rank-width 1 if and only if G is a pivot-minor of a path.*

Proof. We may assume that G is connected. Similarly, backward direction is trivial. Suppose G is bipartite and has linear rank-width 1. Let H be a rank-expansion of G with a linear rank-decomposition (T, L) of width 1. By Lemma 3.2.4, the graph H is a tree, and since T is a caterpillar, H is also a caterpillar. By Lemma 3.2.2, H is a pivot-minor of a path, and so is G. □

Chapter 4. Linear rank-width of trees

In this chapter, we prove that for a positive integer n, $\operatorname{lrw}(T_n) = \lceil \frac{n}{2} \rceil$ and $\operatorname{lrw}(H_n) = \lfloor \frac{n}{2} \rfloor + 1$. Also, we prove that if a tree has linear rank-width at least k, then it has T_k as a vertex-minor.

To obtain the linear rank-width of complete binary trees, we need the following lemma.

Lemma 4.1. *Let T be a tree and let $k \geq 1$. Then T has linear rank-width at most k if and only if for all vertices x in T at most two of the subtrees induced by x have linear rank-width k and all other subtrees have linear rank-width at most $k-1$.*

Proof. We first prove the backward direction. Suppose that for all $x \in V(T)$, there are at most two subtrees induced by x such that the linear rank-width of these subtrees is k and the linear rank-width of all remaining subtrees is at most $k-1$. We claim that there is a linear layout L of T such that $\operatorname{lrw}_L(T) \leq k$.

Let V_k be the set of vertices which induce exactly two subtrees with linear rank-width k. We assume that $|V_k| \neq 0$. For $a, b \in V_k$, if a vertex v is on the path from a to b, then $v \in V_k$ because v induces at least two subtrees containing a and b. Further, there exists a single path containing all vertices of V_k, if not, there exists a vertex x of V_k which induces at least 3 subtrees with linear rank-width k, contradiction. Let $V_k = x_1 x_2 \ldots x_p$.

We show that there exists a path P containing all the vertices of V_k such that for $x \in V(P)$, the subtrees induced by x and not containing vertices of P have linear rank-width at most $k-1$. We examine the first case for which V_k is not empty. Since x_1 is an end vertex of V_k, there is a unique subtree induced by x_1 having linear rank-width k and not containing a vertex in V_k. Let x_0 be the neighbor of x_1 in that subtree. Similarly, x_{p+1} be the neighbor of x_p in the subtree induced by x_p having linear rank-width k and not containing a vertex in V_k. Since x_0 and x_{p+1} are not in V_k, they induce exactly one subtree of linear rank-width k, which contains the path V_k. Let P be the path $x_0 x_1 x_2 \ldots x_p x_{p+1}$. It is evident that every subtree induced by a vertex in P and not containing a vertex in P has linear rank-width at most $k-1$.

Suppose V_k is empty. In this case, we take a vertex x_0 in T let $P = x_0 x_1 \ldots x_p$ be a maximal path in T such that for $2 \leq i \leq p$, x_i is the neighbor in the subtree induced by x_{i-1} having linear rank-width k and not containing a vertex in $\{x_0, x_1, \ldots, x_{i-1}\}$. Since P is maximal, if a subtree induced by x_p does not contain a vertex in P, then this subtree must have linear rank-width at most $k-1$. Let Q be a subtree induced by x_i for some $0 \leq i \leq p-1$ not containing a vertex in P. If Q has linear rank-width k, then since the subtree induced by x_i having a vertex x_{i+1} also has linear rank-width k, $x_i \in V_k$, contradiction.

Given a path $P = x_0 x_1 \ldots x_{p+1}$, now we explicit the linear layout of $V(G)$ with linear rank-width k. For $x_i \in V(P)$, if x_i induces subtrees $T_1^i, T_2^i, \ldots, T_l^i$ not containing a vertex in P, we define

$$L_i = (x_i) \oplus L_1^i \oplus L_2^i \oplus \ldots \oplus L_l^i.$$

And let

$$L = L_0 \oplus L_1 \oplus \ldots \oplus L_{p+1}.$$

Clearly L is a linear layout of T. We claim that $\operatorname{lrw}_L(T) \leq k$. Note that $\operatorname{cutrk}_T(\{x : L(x) \leq L(x_1)\}) = \operatorname{cutrk}_T(\{x_1\}) = 1$ and for $2 \leq i \leq p+1$, since $\{x : L(x) \leq L(x_i)\} = \{x_i\} \cup V(B)$ where B is the subtree induced by x_i containing x_{i-1}, $\mathrm{cutrk}_T(\{x : L(x) \leq L(x_i)\}) = \mathrm{rank}(A(T)[\{x_i\}, V(T) \setminus \{x : L(x) \leq L(x_i)\}]) = 1$. Let $y \in V(G) \setminus V(P)$. Then there exists a subtree T^i_j induced by x_i for some $0 \leq i \leq p+1$ such that $y \in V(T^i_j)$. Let $S_y = \{x \in V(T^i_j) : L(x) \leq L(y)\}$ and $T_y = \{x \in V(T) \setminus V(T^i_j) : L(x) \leq L(y)\}$. Then

$$A[S_y \cup T_y, V(T) \setminus (S_y \cup T_y)] = \begin{array}{c} \\ S_y \\ x_i \\ T_y \setminus \{x_i\} \end{array} \begin{array}{c|c} V(T^i_j) \setminus S_y & V(T) \setminus V(T^i_j) \setminus T_y \\ \leq k-1 & 0 \\ \leq 1 & \leq 1 \\ 0 & 0 \end{array}.$$

Therefore rank of the matrix $A[S_y \cup T_y, V(T) \setminus (S_y \cup T_y)]$ is at most k, so obtained linear rank-decomposition from the linear layout L has width at most k.

Now we prove the forward direction. Suppose that there is a linear layout L such that $\mathrm{lrw}_L(T)$ is at most k. Let a and b be the first and last vertices in the linear layout L such that $\mathrm{cutrk}_T(\{x : L(x) \leq L(a)\}) = \mathrm{cutrk}_T(\{x : L(x) \leq L(b)\}) = k$. Let x be a vertex in T and T_x be a subtree induced by x and not containing a and b. Let $y \in T_x$. We claim that $\mathrm{cutrk}_{T_x}(\{x \in V(T_x) : L(x) \leq L(y)\}) \leq k-1$. Since y is an arbitrary vertex in T_x, if it is true, then T_x has linear rank-width at most $k-1$.

If $L(y) < L(a)$ or $L(y) > L(b)$, then $\mathrm{cutrk}_{T_x}(\{x \in V(T_x) : L(x) \leq L(y)\}) \leq \mathrm{cutrk}_T(\{x \in V(T) : L(x) \leq L(y)\}) \leq k-1$. So, we may assume that $L(a) < L(y) < L(b)$. Let Q_x be the vertex set of one or two subtrees containing the vertices a and b induced by x. And let $T_y = \{u \in V(T_x) : L(u) \leq L(y)\}$, $Q_y = \{u \in V(Q_x) : L(u) \leq L(y)\}$ and $R_y = \{u \in V(T) \setminus V(T_x) \setminus V(Q_x) : L(u) \leq L(y)\}$. We observe that the rank of the matrix

$$\begin{array}{c} \\ T_y \\ Q_y \\ R_y \end{array} \begin{array}{c|c|c} V(T_x) \setminus T_y & V(Q_x) \setminus Q_y & V(T) \setminus V(T_x) \setminus V(Q_x) \setminus R_y \\ l & 0 & 0 \\ 0 & m & \leq 1 \\ \leq 1 & \leq 1 & n \end{array}.$$

is at most k. Since $L(a) < L(y) < L(b)$, m is greater than 0, therefore $l = \mathrm{cutrk}_{T_x}(T_y) \leq k-1$, as required. Clearly, the subtrees induced by x and containing a or b has linear rank-width at most k. Note that the same argument applies even if the vertex x is a and b. □

From Lemma 4.1, for a tree T, the linear rank-width of T is greater than k if there exists a vertex which induces at least 3 subtrees having linear rank-width at least k. Using this fact, we determind the linear rank-width of complete binary trees and the incidence graphs of complete binary trees.

Theorem 4.2. *For a positive integer n, $lrw(T_n) = \lceil \frac{n}{2} \rceil$.*

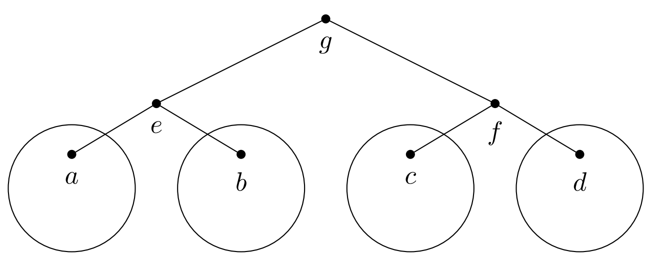

Figure 4.1: Binary tree.

Proof. Note that $\mathrm{lrw}(T_1) = 1$ and $\mathrm{lrw}(T_2) = 1$. We proceed by induction on n and assume that $n \geq 3$. Let g be the root of T_n and $N(g) = \{e, f\}$ and $N(e) = \{g, a, b\}$, $N(f) = \{g, c, d\}$. Clearly, $T_n \setminus e \setminus f \setminus g$ is the disjoint union of four copies of T_{n-2} with roots a, b, c, d. For $x \in \{a, b, c, d\}$, let $T_{n-2,x}$ be the component of $T_n \setminus e \setminus f \setminus g$ with the root x, in Figure 4.1. By induction hypothesis, the linear rank-width of T_{n-2} is $\lceil \frac{n-2}{2} \rceil = \lceil \frac{n}{2} \rceil - 1$. For $x \in \{a, b, c, d\}$, let $L(T_{n-2,x})$ be the linear layout of $T_{n-2,x}$ with linear rank-width $\lceil \frac{n}{2} \rceil - 1$.

Since the vertex e induces 3 subtrees which contains T_{n-2} as an induced subgraph, by Lemma 4.1, $\mathrm{lrw}(T_n) > \lceil \frac{n}{2} \rceil - 1$. And we define a linear layout of T_n as

$$(e) \oplus L(T_{n-2,a}) \oplus L(T_{n-2,b}) \oplus (g, f) \oplus L(T_{n-2,c}) \oplus L(T_{n-2,d}).$$

We easily check that the linear rank-decomposition with this linear layout has width at most $(\lceil \frac{n}{2} \rceil - 1) + 1 = \lceil \frac{n}{2} \rceil$. Therefore, $\mathrm{lrw}(T_n) = \lceil \frac{n}{2} \rceil$. □

Theorem 4.3. *For a positive integer n, lrw$(H_n) = \lfloor \frac{n}{2} \rfloor + 1$.*

Proof. It is clear that $\mathrm{lrw}(H_1) = 1$. And $\mathrm{lrw}(H_2) \leq 2$ is clear and since a vertex of degree 3 in H_2 induces 3 subtrees contain K_2 as a subgraph, therefore $\mathrm{lrw}(H_2) > 1$ by Lemma 4.1. So $\mathrm{lrw}(H_2) = 2$. By Lemma 4.1, we can prove $\mathrm{lrw}(H_n) > \lfloor \frac{n}{2} \rfloor$ similarly in Theorem 4.2. It is sufficient to show that $\mathrm{lrw}(H_n) \leq \lfloor \frac{n}{2} \rfloor + 1$. For a positive integer n, let J_n be the graph obtained from H_n by adding a leaf incident with the root and call that leaf vertex the root of J_n. In fact, we prove that $\mathrm{lrw}(J_n) \leq \lfloor \frac{n}{2} \rfloor + 1$ by induction on n. We can easily check that $\mathrm{lrw}(J_1) = 1$ and $\mathrm{lrw}(J_2) = 2$. We assume that $n \geq 3$.

Let j be the root of J_n and $N(j) = \{i\}$, $N(i) = \{j, g, h\}$, $N(g) = \{i, e\}$, $N(h) = \{i, f\}$, $N(e) = \{g, a, b\}$ and $N(f) = \{h, c, d\}$. Clearly, $J_n[V(J_n) \setminus \{i, j, g, h, e, f\}]$ is the disjoint union of four copies of J_{n-2} with roots a, b, c, d. For $x \in \{a, b, c, d\}$, let $J_{n-2,x}$ be the component of $J_n[V(J_n) \setminus \{i, j, g, h, e, f\}]$ with the root x and let $L(J_{n-2,x})$ be the linear layout of $J_{n-2,x}$ with linear rank-width $\lfloor \frac{n-2}{2} \rfloor + 1$. We define a linear layout of J_n as

$$(e) \oplus L(J_{n-2,a}) \oplus L(J_{n-2,b}) \oplus (g, i, j, h, f) \oplus L(J_{n-2,c}) \oplus L(J_{n-2,d}).$$

Also, we can see that the linear rank-decomposition with this linear layout has width at most $(\lfloor \frac{n-2}{2} \rfloor + 1) + 1 = \lfloor \frac{n}{2} \rfloor + 1$. So, $\mathrm{lrw}(H_n) \leq \mathrm{lrw}(J_n) \leq \lfloor \frac{n}{2} \rfloor + 1$. Therefore, $\mathrm{lrw}(H_n) = \lfloor \frac{n}{2} \rfloor + 1$. □

For a positive integer k, let M_K be the set of all graphs obtained from T_k by replacing some edges with paths of length 2. For $H \in M_K$, let v be the *root* of H if v is the root of the original complete binary tree of H. Note that a graph H in M_k has T_k as a vertex-minor because we can obtain T_k from H by applying local complementations on all vertices of degree 2 and deleting them.

We prove that if a tree has linear rank-width at least k, then it has T_k as a vertex-minor. In fact, if a tree has linear rank-width at least k, then it has a graph in M_k as a pivot-minor.

Proposition 4.4. *Let k be a non-negative integer and T be a tree of linear rank-width k. Then T has a graph in M_k as a pivot-minor.*

Proof. We show that for $v \in V(T)$, there is a graph $H \in M_k$ such that T has H with a root v as a pivot-minor or the graph $(V(H) \cup \{v\}, E(G) \cup \{av\})$ with the root a of H as a pivot-minor, without pivoting the vertex v. We use induction on k. If $k = 0$, then T and $H \in M_0$ are the graphs consist of one vertex, it is trivial. Suppose that $k \geq 1$.

Let $v \in V(T)$. By Lemma 4.1, there is a vertex $w \in V(T)$ such that at least three subtrees induced by w are of linear rank-width $k-1$. So, we may assume that two subtrees T_1 and T_2 do not have v. Let r_1 and r_2 be the neighbors of w in T_1 and T_2, respectively.

By induction hypothesis, for each i, there is a $H_i \in M_{k-1}$ such that T_i has H_i as a pivot-minor with r_i as a root and without pivoting r_i. Let $P = vv_1v_2 \ldots v_nw$ be the path from v to w in T. If n is even, then we can obtain an edge vw by pivoting $\{v_1, v_2, \ldots, v_n\}$ and deleting $\{v_1, v_2, \ldots, v_n\}$. Then we obtain a graph H such that $H = (V(J) \cup \{v\}, E(J) \cup \{vw\})$ with the root w of J and without pivoting v where $J \in M_k$. If n is odd, then we obtain a graph H in M_k with the root v by pivoting $\{w, v_1, v_2, \ldots, v_n\}$ and deleting $\{w, v_1, v_2, \ldots, v_n\}$.

Therefore T has a graph in M_k as a pivot-minor. □

We have the following corollary.

Corollary 4.5. *Let k be a non-negative integer. If a tree has linear rank-width at least k, then it has T_k as a vertex-minor.*

Proof. Let T be a tree of linear rank-width k. By Proposition 4.4, T has a graph in M_k as a pivot-minor. Since a graph in M_k is obtained from T_k by subdividing some edges, that graph has T_k as a vertex-minor. Therefore, T has T_k as a vertex-minor. □

Chapter 5. Incidence graphs of trees and binary matroids

It is an interesting problem to identify specific graphs so that if C is a class of graphs with unbounded rank-width or linear rank-width, then these specific graphs occur as pivot-minors of graphs in C. For rank-width, this problem for bipartite graphs, line graphs and circle graphs is solved by Oum [10, 11]. There is no known result for linear rank-width, so we concentrate on this problem for linear rank-width. For path-width, there is a theorem proved by Robertson and Seymour [14] that for a forest T, if a graph has path-width at least $|V(T)| - 1$ then G has T as a minor. We may guess that the specific graphs for linear rank-width are trees.

The main result of this chapter is that an incidence graph of a binary tree does not have B_6 as a pivot-minor where B_6 is the graph depected in Figure 5.1. It impliles that even if an incidence graph of a binary tree has large linear rank-width, it cannot have B_6 as a pivot-minor. Since for $n \geq 5$, T_n has B_6 as an induced subgraph, the graph also cannot have T_n as a pivot-minor.

Theorem 5.6. *The graph B_6 could not be a pivot-minor of the incidence graph of a binary tree.*

We give two proofs of the result. One proof is a more naive, and using a characterization of trees having nonsingular adjacency matrix. For the other proof, we use the property of the fundamental graphs of binary matroids.

5.1 Incidence graphs of trees

In this section, we prove Theorem 5.6. Lemma 5.1.1 is well known (See [6, 3]).

Lemma 5.1.1 ([6, 3]). *The adjacency matrix of a tree is nonsingular if and only if it has a perfect matching.*

Let T be a tree and $X \subseteq V(T)$ such that $T[X]$ is nonsingular. We verify that for two vertices x, y

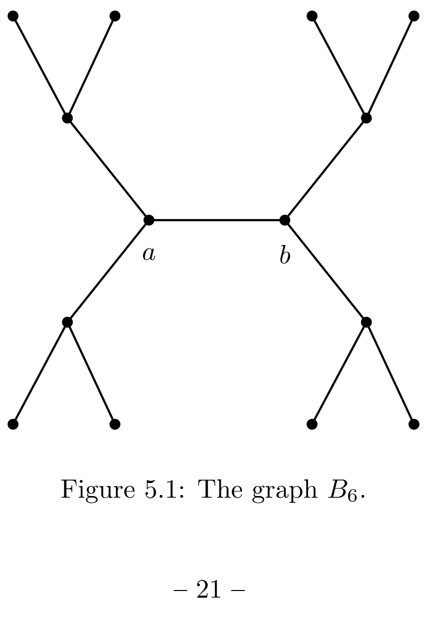

Figure 5.1: The graph B_6.

in T, $xy \in E(T \wedge X)$ if and only if $T[\{x,y\}\Delta X]$ has a perfect matching. By Theorem 2.1,

$$\begin{aligned} xy \in E(T \wedge X) &\Leftrightarrow A(T \wedge X)[\{x,y\}] \text{ is nonsingular} \\ &\Leftrightarrow A(T)[\{x,y\}\Delta X] \text{ is nonsingular} \\ &\Leftrightarrow T[\{x,y\}\Delta X] \text{ has a perfect matching.} \end{aligned}$$

Using this fact, we prove the following lemma.

Lemma 5.1.2. *Let n be an even integer. Let P be a path $v_0v_1v_2\ldots v_{n-1}v_nv_{n+1}$ and $H = V(P) \setminus \{v_0, v_{n+1}\}$. Then the following are satisfied.*

1. *$v_0v_i \in E(P \wedge H)$ if and only if $i = n+1$ or $i \neq 0$ and i is even.*
2. *$v_{n+1}v_i \in E(P \wedge H)$ if and only if $i = 0$ or $i \neq n+1$ and i is odd.*
3. *Suppose $0 < i < j < n+1$. Then $v_iv_j \in E(P \wedge H)$ if and only if i is odd and j is even.*

Proof. Since $P[H]$ is a path of even length, $P[H]$ has a perfect matching, and therefore $A(P)[H]$ is nonsingular. So, for $v, w \in V(P)$, $vw \in E(P \wedge H)$ if and only if $T[\{v,w\}\Delta H]$ has a perfect matching.

We claim that v_0 is adjacent to v_i for even i and v_{n+1} in $P \wedge H$. Since

$$\{v_0, v_{n+1}\}\Delta H = V(P)$$

and P has a perfect matching, $v_0v_{n+1} \in E(P \wedge H)$. If $i \neq n+1$, then

$$\{v_0, v_i\}\Delta H = \{v_0, v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n\},$$

and therefore $P[\{v_0, v_i\}\Delta H]$ has a perfect matching if and only if i is even, as claimed. Similarly, we can obtain that $v_iv_{n+1} \in E(P \wedge H)$ if and only if $i = 0$ or $i \neq n+1$ and i is odd. For $0 < i < j < n+1$, we can verify that $P[\{v_i, v_j\}\Delta H] = P[\{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{j-1}, v_{j+1}, \ldots v_n\}]$ has a perfect matching if and only if i is odd and j is even. □

Lemma 5.1.3. *Let n be an even non-negative integer. Let P be a path $v_0v_1v_2\ldots v_{n-1}v_nv_{n+1}$ and $H = V(P) \setminus \{v_0, v_{n+1}\}$ and T be a tree which is an induced subgraph of $P \wedge H$. If $v_0, v_{n+1} \in V(T)$, then $V(T) \subseteq H$ where $H = \{v_0, v_2, v_4, \ldots v_i\} \cup \{v_{i+1}, v_{i+3}, \ldots, v_{n-1}, v_{n+1}\}$ for some even integer $0 \leq i \leq n$. Moreover, if v_s, $v_t \in V(T)$ where s is odd and t is even and $s < t$, then $v_0, v_1, \ldots, v_{s-2} \notin V(T)$ or $v_{t+2}, v_{t+4}, \ldots, v_n, v_{n+1} \notin V(T)$.*

Proof. Suppose that $v_0, v_{n+1} \in V(T)$. Let $j = \max\{i : i \text{ is even}, v_i \in V(T)\}$ and $k = \min\{i : i \text{ is odd}, v_i \in V(T)\}$. If $j = 0$ or $k = n+1$, then $V(T) \subseteq \{v_0, v_2, v_4, \ldots v_i\} \cup \{v_{i+1}, v_{i+3}, \ldots, v_{n-1}, v_{n+1}\}$ where $i = 0$ or $i = n$, respectively. We may assume that $v_j, v_k \in V(P) \setminus \{v_0, v_{n+1}\}$. By Lemma 5.1.2, $j < k$. Also, if $v_i \in V(T)$ for some odd $i < j$, then $v_0v_jv_iv_{n+1}$ forms a cycle in T, which is a contradiction. So, for every odd $i < j$, v_i must be deleted. Similarly, for every even $i > k$, v_i must be deleted. Therefore, $V(T) \subseteq \{v_0, v_2, v_4, \ldots v_i\} \cup \{v_{i+1}, v_{i+3}, \ldots, v_{n-1}, v_{n+1}\}$ where $i = j$.

Second statement is also follows from Lemma 5.1.2. □

For $J \subseteq V(G)$, a vertex set H in G is a *closure* of J if $H = J \cup N(J)$, and for this H, let $B(H) = H \setminus J$.

Lemma 5.1.4. *Let T be a tree and $J \subseteq V(T)$ such that $A(T)[J]$ is nonsingular. Let $J_1, J_2, \ldots, J_m$ be the vertex set of components of $T[J]$ and H_i be the closure of J_i. If $A(T)(a,b) \neq A(T \wedge H)(a,b)$ then $\{a,b\} \subseteq H_i$ for some $1 \leq i \leq m$.*

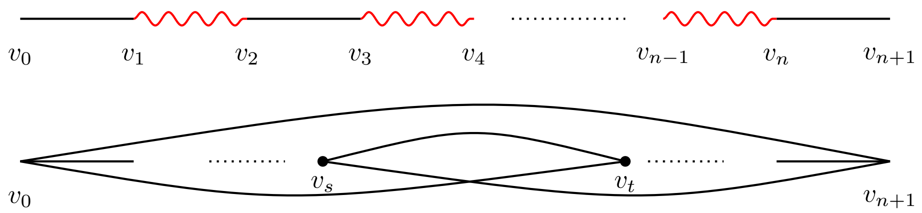

Figure 5.2: A path P in Lemma 5.1.2 and 5.1.3. The red edges denote the perfect matching on $P[H]$ where $H = V(P) \setminus \{v_0, v_{n+1}\}$. If s is odd and t is even, by Lemma 5.1.2, $v_0 v_t v_s v_{n+1}$ make a cycle in $P \wedge H$.

Proof. Let $a, b \in V(T)$. Suppose that $A(T)(a,b) \neq A(T \wedge J)(a,b)$. We first assume that $A(T)(a,b) = 1$. Then $A(T \wedge J)(a,b) = 0$, so there is no perfect matching in $T[\{a,b\}\Delta J]$. If $a \notin J$ and $b \notin J$, then since $ab \in E(G)$, there is a perfect matching in $T[\{a,b\}\Delta J] = T[\{a,b\} \cup J]$, contradiction. So, $a \in J_i$ or $b \in J_i$ for some $1 \leq i \leq m$, then $a, b \in H_i$.

Now we assume that $A(T)(a,b) = 0$. Then $A(T \wedge J)(a,b) = 1$, so there is a perfect matching in $T[\{a,b\}\Delta J]$. If $a \notin H_i$ for all $1 \leq i \leq m$, since $ab \notin E(T)$, $T[\{a,b\}\Delta J]$ could not have a perfect matching. So, $a \in H_i$ for some $1 \leq i \leq m$. If $b \notin H_i$, then since $ab \notin E(T)$, $T[\{a\}\Delta J_i]$ must have a perfect matching. But $|\{a\}\Delta J_i|$ is odd, contradiction. Therefore, $a, b \in H_i$ for some $1 \leq i \leq m$. □

Given a matching M in a graph G, a path P in G is an M*-alternating path* if the edges of P belong alternatively to the matching and not to the matching. Let H be the incidence graph of a graph G. We define $C(H)$ as the vertex set $V(H) \setminus V(G)$.

Lemma 5.1.5. *Let T be an indidence graph of a tree and $J \subseteq V(T)$ such that $A(T)[J]$ is nonsingular and $T[J]$ is connected. Let M be the perfect matching in $T[J]$ and H be a closure of J. If $v \in B(H) \cap C(T)$ or $v \in J \setminus C(T)$, then the following are satisfied.*

1. *If $v \in J \setminus C(T)$, then there is a unique maximal M-alternating path P in T starting from v such that the first edge of P is a matching edge. Also, if $v \in B(H) \cap C(T)$, then there is a unique maximal M-alternating path P in T starting from v.*

2. *$N_{T \wedge J}(v) \cap H \subseteq V(P)$.*

3. *If b, $c \in N_{T \wedge J}(v) \cap H$ such that b is on the path from v to c in T, then $N_{T \wedge J}(b) \subseteq N_{T \wedge J}(c)$.*

Proof. (1) Suppose $v \in J \setminus C(T)$. The path consists of the matching edge incident with v is a path satisfying the condition. Let P_1 and P_2 be distinct maximal paths starting from v with the matching edge. Let x be the last common vertex in the component of $P_1 \cap P_2$ containing v. Note that $\deg_T(x) \geq 3$. Since first different edges in P_1 and P_2 are not in M, the last edge of the component of $P_1 \cap P_2$ containing v is a matching edge, so the distance from v to x in T is odd and $x \in C(T)$. Then $\deg_T(x) = 2$, contradiction.

If $v \in B(H) \cap C(T)$, then v is adjacent to b in $J \setminus C(T)$. So if P is the unique maximal M-alternating path starting from b with the matching edge, then the path obtained from P by adding a vertex v and an edge vb is the unique maximal M-alternating path from v.

(2) Let $y \in H \setminus V(P)$. It is sufficient to show that $y \notin N_{T \wedge J}(v)$. Let z be the vertex in P such that the distance from y to z in T is minimum. Since $z \notin B(H)$, z is contained in some matching edge e in

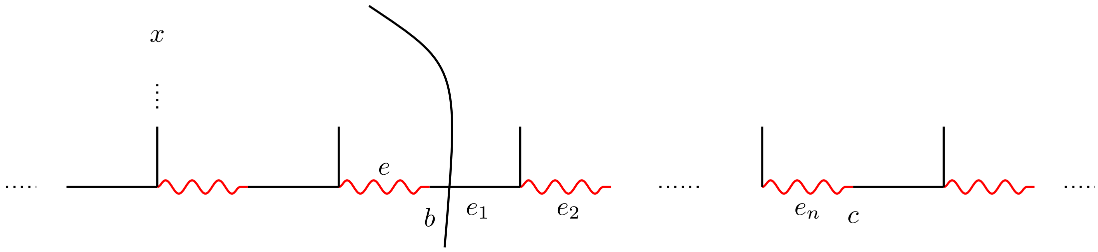

Figure 5.3: A tree T in Lemma 5.1.5 and $b \in J \cap C(T)$. The red edges denote the perfect matching on $T[J]$. If $T[\{b,x\}\Delta J]$ has a perfect matching, then x must be in the component of $T \setminus x$ connected by the matching edge e. So, if N is the perfect matching of $T[\{b,x\}\Delta J]$, then $\{e_2, e_4, \ldots, e_n\} \subseteq N$ and $N \setminus \{e_2, e_4, \ldots, e_n\} \cup \{e_1, e_3, \ldots, e_{n-1}\}$ is a perfect matching of $T[\{c,x\}\Delta J]$.

P. Since $\deg_T(z) = 3$, $z \notin C(T)$, and the edge e is not contained in the path from v to z. Therefore $T[\{a,y\}\Delta J]$ has no perfect matching, so $vy \notin E(T \wedge J)$. Thus $y \notin N_{T\wedge J}(v)$.

(3) Suppose that b, $c \in N_{T\wedge J}(v) \cap H$ such that b is on the path from v to c in T. To prove (3), it is enough to show that for $x \in V(T)$, if $T[\{b,x\}\Delta J]$ has a perfect matching, then $T[\{c,x\}\Delta J]$ also has a perfect matching. Suppose if $T[\{b,x\}\Delta J]$ has a perfect matching. Let e be the edge in M incident with b. Since $v \in B(H) \cap C(T)$ or $v \in J \setminus C(T)$, by Lemma 5.1.2, $b \in V(P) \cap C(T)$, and e is in the path from v to b. Therefore x has to be in the subtree induced by b containing v. Let N be the perfect matching in $T[\{b,x\}\Delta J]$ and $e_1e_2\ldots e_n$ be the path from b to c. Note that $\{e_2, e_4, \ldots, e_n\} \subseteq N$. Since $N \setminus \{e_2, e_4, \ldots, e_n\} \cup \{e_1, e_3, \ldots, e_{n-1}\}$ is a perfect matching in $T[\{c,x\}\Delta J]$, we obtain the result. □

We show that B_6 cannot be a pivot-minor of an incidence graph of binary tree.

Proof of Theorem 5.6. Let a, b be two vertices of B_6 in Figure 5.1. Suppose that B_6 is a pivot-minor of an incidence graph T of a binary tree. Then there exist $J, S \subseteq V(T)$ such that $T \wedge J \setminus S = B_6$. Let $J_1, J_2, \ldots, J_m$ be the vertex sets of components of $T[J]$ and H_i be the closure of J_i. Note that if $i \neq j$, then H_i and H_j can intersect with at most 1 vertex.

We first claim that $\{a,b\} \cap (B(H_j) \cap C(T)) = \emptyset$ for all $1 \leq j \leq m$, and $\{a,b\} \cap (J \setminus C(T)) = \emptyset$. If $a \in B(H_j) \cap C(T)$ for some $1 \leq j \leq m$, then since $\deg_T(a) = 2$, a may be contained in at most two distinct H_i. And since $\deg_{T\wedge J\setminus S}(a) = \deg_{B_6}(a) = 3$, we may assume that a has at least 2 neighbors of H_j in $T \wedge J \setminus S$. If $a \in J_j \setminus C(T)$ for some $1 \leq j \leq m$, then a has 3 neighbors of H_j in $T \wedge J \setminus S$. Let $c, d \in N_{T\wedge J\setminus S}(a)$. Then by Lemma 5.1.5, the vertices c, d and a are lying on a path in T and if c is on the path from a to d in T, then $N_{T\wedge J}(c) \subseteq N_{T\wedge J}(d)$. Since B_6 is a tree, c must be a leaf incident with a in $T \wedge J \setminus S$, which is a contradiction.

Now we prove the theorem. We first assume that $\{a,b\} \nsubseteq H_i$ for all $1 \leq i \leq m$. Because $\deg_{B_6}(a) = \deg_{B_6}(b) = 3$, at least one of a and b is in a H_j for some $1 \leq j \leq m$. By symmetry, we assume that $a \in H_j$. By Lemma 5.1.4, $A(T)(a,b) = A(T \wedge H)(a,b) = 1$, so $a \in B(H_j)$. By first claim, $a \notin C(T)$. So $b \in C(T)$, and since $\deg_T(b) = 2$, b must be contained in H_k for some $1 \leq k \leq m$ and $k \neq j$. Therefore $b \in B(H_k) \cap C(T)$, contradiction.

Therefore $\{a,b\} \subseteq H_i$ for some $1 \leq i \leq m$. We first assume that $\{a,b\} \subseteq J_i$, then by second claim, $\{a,b\} \subseteq J_i \cap C(T)$. Then the path from a to b has even length, by Lemma 5.1.2, $ab \notin T \wedge J$, contradiction. So, at least one of a and b is in $B(H_i)$. By symmetry, we assume that $a \in B(H_i)$. Then by first claim, $a \in B(H_i) \setminus C(T)$. If $b \in B(H_i)$, then by Lemma 5.1.2, the length of the path from a to b in T must be odd, $b \in C(T)$, contradiction. So $b \in J_i$. Then by second claim, $b \in J_i \setminus C(T)$. Then the path from a to b has odd length, by Lemma 5.1.2, $ab \notin T \wedge J$, we have a contradiction. $\square$

5.2 Fundamental graphs of binary matroids

In this section, we give another proof of Theorem 5.6. To prove the theorem, we will observe the relation between minors of binary matroids and pivot-minors of fundamental graphs of the binary matroids.

Let G be a bipartite graph with a bipartition $A \cup B$. Let $\mathrm{Bin}(G, A, B)$ be the binary matroid on V, represented by the $A \times V$ matrix $(I_A, A(G)[A, B])$, where I_A is the $A \times A$ identity matrix. By the construction, A is a base of the matroid $\mathrm{Bin}(G, A, B)$, and we can verify that G is the fundamental graph of $\mathrm{Bin}(G, A, B)$ with respect to the base A. The following is proved by Oum [11].

Corollary 5.2.1 ([11]). *1. Let N, M be binary matroids, and H, G be fundamental graph of N, M respectively. If N is a minor of M, then H is a pivot-minor of G.*

2. Let G be a bipartite graph with a bipartition $A \cup B = V(G)$. If H is a pivot-minor of G, then there is a bipartition $A' \cup B' = V(H)$ such that $\mathrm{Bin}(H, A', B')$ is a minor of $\mathrm{Bin}(G, A, B)$.

We need the following lemma.

Lemma 5.2.2. *Let G be a graph. Let B be a base of the matroid $\mathcal{M}(G)$ and H be the fundamental graph of the matroid $\mathcal{M}(G)$ with respect to B. Then $\mathcal{M}(G) = \mathrm{Bin}(H, B, E(G) \setminus B)$.*

Proof. By Theorem 2.2, $\mathcal{M}(G)$ is the same as the binary matroid represented by the incidence matrix of G. We can verify that for $U \subseteq E(G)$, U is independent in the binary matroid represented by the incidence matrix of G if and only if U is independent in $\mathrm{Bin}(H, B, E(G) \setminus B)$. Therefore $\mathcal{M}(G) = \mathrm{Bin}(H, B, E(G) \setminus B)$. $\square$

Let Q be a binary tree and T be the incidence graph of Q. Let P be the graph obtained from Q by adding a vertex v adjacent to all vertices in Q, call this vertex the *apex* vertex of P. Note that $\delta_P(v)$ forms a spanning tree in P, and T is the fundamental graph of the matroid $\mathcal{M}(P)$ with respect to a base $\delta_P(v)$. Therefore, by Lemma 5.2.2, $\mathcal{M}(P) = \mathrm{Bin}(T, V(T) \setminus C(T), C(T))$.

Let $A \cup B = V(B_6)$ be a bipartition of B_6. Since B_6 is connected, B_6 has exactly two types of bipartitions, but they induce same binary matroid since they are symmetric. So if B_6 is a pivot-minor of T, then $\mathrm{Bin}(B_6, A, B)$ must be a minor of $\mathrm{Bin}(T, V(T) \setminus C(T), C(T))$. Let C be the set of graphs such

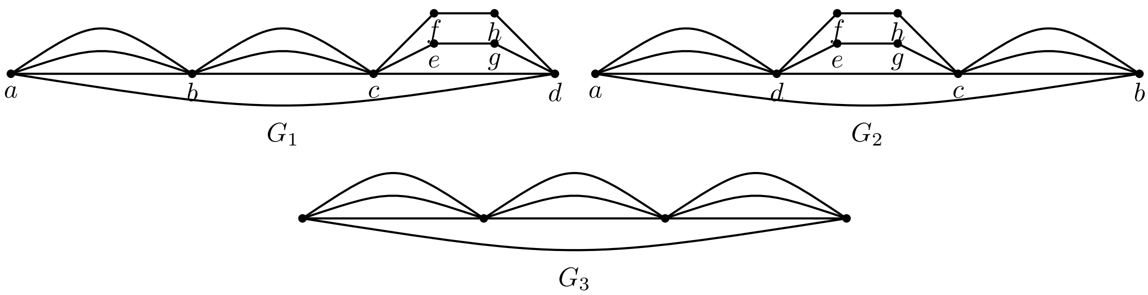

Figure 5.4: The graphs G_1, G_2 and G_3. Note that G_2 can be obtained from G_1 by a Whitney twisting about the cut {b, d}.

that the graphic matroid of graphs in C are $\mathrm{Bin}(B_6, A, B)$. If a graph G in C is a minor of the graph P, then $\mathcal{M}(G) = \mathrm{Bin}(B_6, A, B)$ is a minor of $\mathcal{M}(P) = \mathrm{Bin}(T, V(T) \setminus C(T), C(T))$. Then by Corollary 5.2.1, B_6 is a pivot-minor of T. So to prove main theorem, it is sufficient to show that for a graph G in C, P does not have G as a minor.

The next theorem characterize all graphs in C. The graph obtained from G by applying a *Whitney twisting* is defined by taking a vertex cut of size two and switching the role of each vertex in one of the components. A graph G is 2-*connected* if $|V(G)| \geq 3$ and $G \setminus v$ is connected for all $v \in V(G)$.

Theorem 5.2.3 ([18, 13]). *Let G be a 2-connected graph and H be a graph without isolated vertex. Then $\mathcal{M}(G) = \mathcal{M}(H)$ if and only if H can be obtained from G by a sequence of twisting operations.*

Now we show that $C = \{G_1, G_2\}$. Using Lemma 5.2.2, we can verify that $\mathrm{Bin}(B_6, A, B) = \mathcal{M}(G_1)$ where G_1 is the graph depicted in Figure 5.4. Since G_1 is 2-connected, by Theorem 5.2.3, every graph in C is obtained from G_1 by a sequence of twisting operations. If a vertex cut of size 2 in G_1 contains a vertex in $\{e, f, g, h\}$, then the other vertex in the cut is one of c and d. So the cut induces two components and a component induced by the cut has one vertex. Therefore, the graph obtained from G by twisting with that cut is isomorphic to G_1. The other cuts are $\{a, c\}, \{b, d\}$. The graph obtained from G by twisting with a cut $\{a, c\}$ is isomorphic to G_1 since both ab and ac has 3 multiple edges. The graph obtained from G by twisting with a cut $\{b, d\}$ is G_2, depected in Figure 5.4. Therefore, $C = \{G_1, G_2\}$.

Note that G_1 and G_2 have G_3 as a minor. So if we prove that P does not have G_3 as a minor, then P does not have G_1 or G_2 as a minor. We prove the following.

Lemma 5.2.4. *Let G be a minor of P. Then there is a vertex w in G such that $G \setminus w$ does not have multiple edges.*

Proof. Let v be the apex vertex of P. Since G is a minor of P, there exists $U, V \subseteq E(G)$ and $S \subseteq V(G)$ such that $P \setminus S \setminus U/V$. If $v \in S$ then G is a minor of a tree, so G does not have multiple edges. Thus $v \notin S$. Suppose that e is a multiple edge in G. Then e is contained in a cycle in $P \setminus S$. Since every cycle in $P \setminus S$ contains the vertex v, thus one end of e came from v. Let w be the vertex in G came from v. Then $G \setminus w$ has no multiple edges, as required. □

But G_3 has a vertex such that multiple edges remains after deleting that vertex. Thus P does not have G_3 as a minor. It implies the main theorem.

Proof of Theorem 5.6. By Lemma 5.2.4, P does not have G_3 as a minor. So P does not have G_1 or G_2 as a minor. Thus T does not have B_6 as a pivot-minor. Since T is the incidence graph of an arbitrary binary tree, we obtain the result. □

Therefore, now we have a following question.

Question. *If C is a class of graphs with unbounded linear rank-width, then every incidence graph of a binary tree occurs as a pivot-minor of some graph in C.*

References

[1] I. Adler, A. M. Farley, and A. Proskurowski. Obstructions for linear rankwidth at most 1. *CoRR*, abs/1106.2533, 2011.

[2] H.-J. Bandelt and H. M. Mulder. Distance-hereditary graphs. *J. Combin. Theory Ser. B*, 41(2):182–208, 1986.

[3] R. B. Bapat. *Graphs and matrices*. Universitext. Springer, London, 2010.

[4] A. Bouchet. Isotropic systems. *European J. Combin.*, 8(3):231–244, 1987.

[5] A. Bouchet. Transforming trees by successive local complementations. *J. Graph Theory*, 12(2):195–207, 1988.

[6] D. M. Cvetković, M. Doob, and H. Sachs. *Spectra of graphs*, volume 87 of *Pure and Applied Mathematics*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1980. Theory and application.

[7] J. A. Ellis, I. H. Sudborough, and J. S. Turner. The vertex separation and search number of a graph. *Inform. and Comput.*, 113(1):50–79, 1994.

[8] R. Ganian. Thread graphs, linear rank-width and their algorithmic applications. In *Combinatorial Algorithms*, volume 6460 of *Lecture Notes in Comput. Sci.*, pages 38–42. Springer, 2011.

[9] O. Kwon and S. Oum. Graphs of small rank-width are pivot-minors of graphs of small tree-width. arXiv:1203.3606v1., 2012.

[10] S. Oum. Excluding a bipartite circle graph from line graphs. *J. Graph Theory*, 60(3):183–203, 2009.

[11] S. Oum. Rank-width and vertex-minors. *J. Combin. Theory Ser. B*, 95(1):79–100, 2005.

[12] S. Oum. Rank-width is less than or equal to branch-width. *J. Graph Theory*, 57(3):239–244, 2008.

[13] J. Oxley. *Matroid theory*, volume 21 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, second edition, 2011.

[14] N. Robertson and P. D. Seymour. Graph minors. I. Excluding a forest. *J. Combin. Theory Ser. B*, 35(1):39–61, 1983.

[15] N. Robertson and P. D. Seymour. Graph minors. II. Algorithmic aspects of tree-width. *J. Algorithms*, 7(3):309–322, 1986.

[16] R. Thomas. Tree decompositions of graphs. Lecture notes for the Catlin Memorial Workshop. 1996.

[17] A. W. Tucker. A combinatorial equivalence of matrices. In R. Bellman and M. Hall, Jr., editors, *Combinatorial Analysis*, pages 129–140. American Mathematical Society, Providence, R.I., 1960.

[18] H. Whitney. 2-Isomorphic Graphs. *Amer. J. Math.*, 55(1-4):245–254, 1933.

Summary

Connecting rank-width and tree-width via pivot-minors

Tree-width가 k인 그래프는 rank-width가 $k+1$ 이하임이 알려져 있었지만, rank-width가 작은 그래프에 대해서는 tree-width와 관련된 결과가 없었다. 3장에서 우리는 처음으로 rank-width가 k 이하인 그래프는 tree-width가 $2k$인 그래프의 pivot-minor로 얻을 수 있음을 증명하였고, 또한 linear rank-width가 k 이하인 그래프는 path-width가 $k+1$인 그래프의 pivot-minor로서 얻을 수 있음을 증명하였다. 이 증명의 보조정리로서 rank-width가 1인 그래프는 정확히 트리의 vertex-minor들이고 linear rank-width가 1인 그래프는 정확히 패쓰의 vertex-minor들임을 증명하였다. 그리고 그래프들이 이분 그래프이면, 이 보조정리들의 vertex-minor를 pivot-minor로 대체할 수 있음을 보였다.

4장에서는, 높이가 k인 완전 이진 트리들과 그들의 결합 그래프들의 linear rank-width를 계산하였다. 또한 트리가 linear rank-width가 k보다 크면 높이가 k인 완전 이진 트리들을 vertex-minor로서 가짐을 증명하였다.

5장에서는, 우리는 그래프가 linear rank-width가 어떤 수 이상이면 반드시 어떤 그래프를 vertex-minor 혹은 pivot-minor로서 가진다는 문제에 대해서 생각해보았다. 처음에는 유명한 Robertson과 Seymour의 path-width 정리처럼 문제의 어떤 그래프가 트리가 될 것이라고 추측을 하였다. 하지만, pivot-minor에 대해서는 이 추측이 잘못 되었다는 사실을 증명하였고, 트리를 대신하여 트리의 결합 그래프가 어떤 그래프에 들어갈 것이라고 문제를 제시하였다.