A Polynomial Kernel for Linear Rank-width One Vertex Deletion

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Joint work with
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Fixed parameter tractability

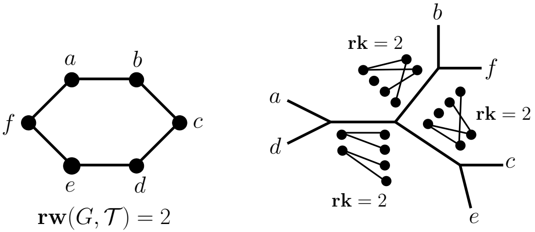
Graph modification into a graph class $\mathcal{C}$

Input : A graph G, an integer k
Parameter : k
Question : Is there a vertex subset S of G with $|S| \leq k$ such that $G - S$ is in $\mathcal{C}$?

- **FPT algorithm (fixed parameter tractable algorithm)** :
  an algorithm that runs in time $f(k) \cdot |V(G)|^{\mathcal{O}(1)}$.
- **Kernel of size $g(k)$** :
  A parameterized problem has a kernel of size $g(k)$ if there is a polynomial-time algorithm to transform every instance (G, k) into an instance (G', k') such that
  (1) (G, k) is a Yes-instance if and only if (G', k') is a Yes-instance,
  (2) $|V(G')| \leq g(k)$ and $k' \leq k$.

The generic problem

RANK-WIDTH-c VERTEX DELETION ($c \in \mathbb{N}^+$)
Input : Graph $G = (V, E)$, integer k
Question : $\exists ?\ S \subseteq V$ such that $|S| \leqslant k$ and $\mathbf{rw}(G - S) \leqslant c$



$-$Graphs of bounded rank-width are well-quasi-ordered under the **vertex-minor** relation.

Lemma: [Oum'05] If $G = (V, E)$ is a minimal vertex-minor such that $\mathbf{rw}(G) > c$, then $|V| \leqslant \frac{6^{c+1}-1}{5}$.

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Theorem [Oum'06]: For fixed c, there is an $O(n^3)$ algorithm that either finds a rank-decomposition of width $3k-1$ or confirms that $\mathbf{rw}(G) > c$.

Observation: Let $G = (V, E)$ be a graph and $S \subseteq V$. Then

$$\mathbf{rw}(G - S) \leqslant c \Rightarrow \mathbf{rw}(G) \leqslant c + k$$

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By the meta-theorem on graphs of bounded rank-width (or clique-width)[Courcelle, Makowsky, Rotics'00], we obtain:

Theorem: Rank-width-c Vertex Deletion parameterized by solution size is FPT.

Questions and results

- Can Rank-width-c Vertex Deletion be solved in single exponential FPT-time ?
  - Tree-width-c Vertex Deletion admits a single exponential FPT-time algorithm. (Fomin et al. 2012 ; Kim et al 2013)
- Can Rank-width-One Vertex Deletion be solved in single exponential FPT-time ?
- Do these problems admit a polynomial-size kernel ?

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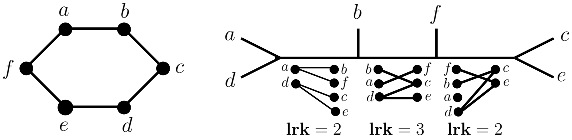
- Can Rank-width-One Vertex Deletion be solved in single exponential FPT-time ?
- Do these problems admit a polynomial-size kernel ?

$\rightsquigarrow$ Open but

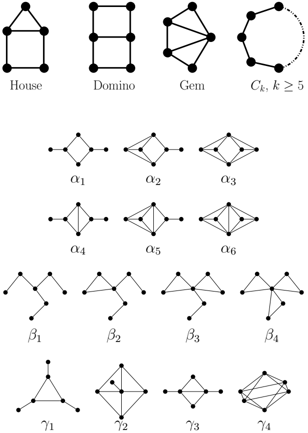
Linear Rank-width-One Vertex Deletion (LRw1VD) can be solved in $O(8^k \cdot n^8)$ and has a polynomial size kernel



$\mathbf{lrw}(G, \mathcal{T}) = 3$

Obstructions for graphs of linear rank-width 1

- Class of graphs characterized by an infinite list of forbidden induced subgraphs. (Adler, Farley, Proskurowski 2014)



Basic approach

The FPT algorithm for LRw1VD:

- We kill the small obstructions by branching
- and then show that the problem can be solved in polynomial time on instances without small obstructions

Theorem: The LRw1VD problem can be solved in $O(8^k \cdot n^8)$-time.

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The polynomial kernel for LRw1VD:

- We identify a small set of vertices S containing every minimal hitting set of the small obstructions
- We show how to reduce $G - S$ in polynomial time

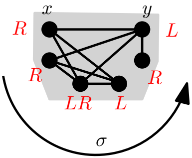
(Same approach used for Pathwidth-One Vertex Deletion)

Theorem: The LRw1VD problem admits a kernel of size $O(k^{33})$.

Thread blocks and thread graphs

(G, x, y, σ, ℓ) with $x, y \in V(G)$, σ is a (linear) ordering on V and $\ell : V(G) \to \{\{L\}, \{R\}, \{L, R\}\}$, is a thread block if:

- $\ell(x) = \{R\}$ and $\ell(y) = \{L\}$,
- for $v, w \in V(G)$, $vw \in E(G)$ iff $v <_\sigma w$, $R \in \ell(v)$, $L \in \ell(w)$.



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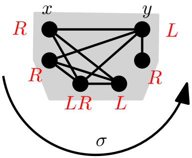
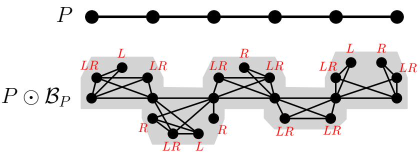


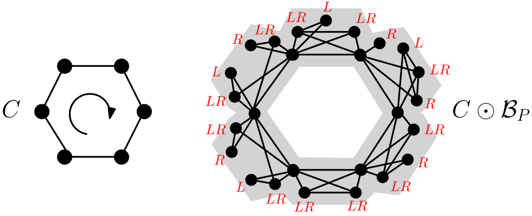
A connected thread graph is a graph that can be obtained by substituting a thread block to every arc of a directed path P.



Theorem [Ganian'09] A graph G is a thread graph iff $\mathbf{lrw}(G) \leq 1$.

Necklace graphs

A connected necklace graph is a graph that can be obtained by substituting a thread block to every arc of a circuit C.



Lemma: Removing one vertex to a connected necklace graph is enough to obtain a thread graph.

An FPT algorithm for LRw1VD

Let Ω_N be the subset of forbidden induced subgraphs of thread graphs that have size at most 8.

Lemma: A connected Ω_N-free graph is either a thread graph or a necklace graph built on a circuit of length at least 9.

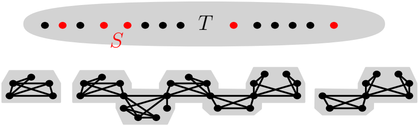
Algorithm

- As long as, G has a forbidden induced subgraph H of size at most 8, branch on the vertices of H
- Remove one vertex per connected component of G that is not a thread graph

Kernel for LRw1VD – Hitting small obstructions

Lemma: There is a polynomial time algorithm that finds a non-empty set $T \subseteq V$ such that

1. $G - T$ is a thread graph,
2. $\forall S \subseteq V$, $|S| \leqslant k$, S is a minimal hitting set for Ω_N in G iff it is a minimal hitting set for Ω_N contained in $G[T]$, and
3. $|T| \leqslant 8 \cdot 8!(k+1)^8 + k$.



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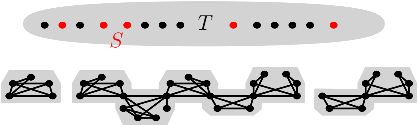

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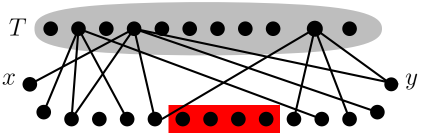
Crucial part : bounding the size of each thread block of $G - T$

Kernel for LRw1VD – Irrelevant vertex

We set $\mu(k) = 8 \cdot 8!(k+1)^8 + k$

Lemma: If $G - T$ contains a thread block $B(x, y)$ of size at least $(k+2)(\mu(k)+2)^2 + 1$, then we can find an irrelevant vertex in $B(x, y)$ in polynomial time.



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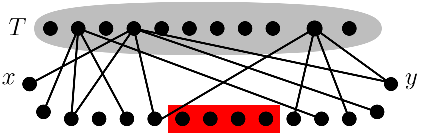

Proof sketch

$\rightsquigarrow$ For every $v \in T$ mark the $k+2$ first vertices with label R and $k+2$ last vertices with label L

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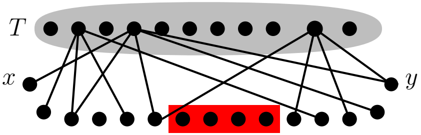

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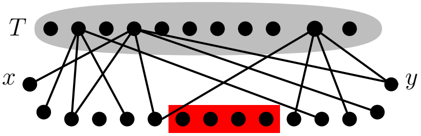

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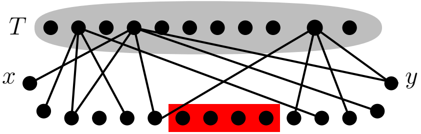

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$\rightsquigarrow$ unmarked vertices in the middle are irrelevant

Claim that a vertex w in middle is irrelavent.
Let S be the minimum deleting set of $G - w$, but not deleting set for G.
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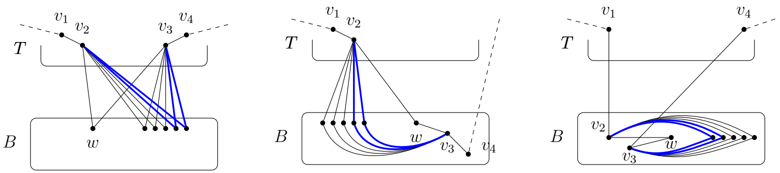

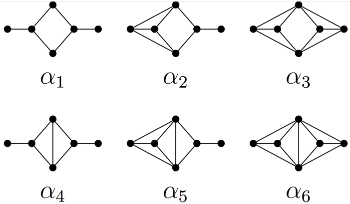

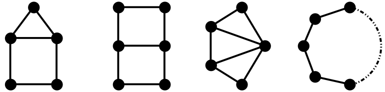

House Domino Gem C_k, $k \geq 5$

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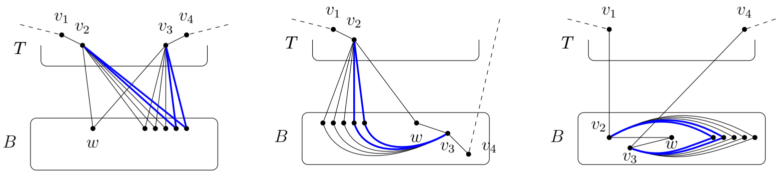

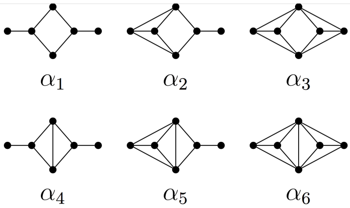

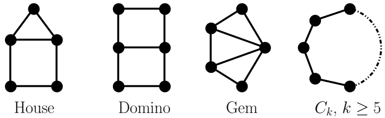

Kernel for LRw1VD – Remaining part

1. a connected component C with at least $19(6\mu(k)+1)$ thread blocks
   $\rightarrow \exists$ a block which can be shrunk into a vertex.
2. at least $2\mu(k)+1$ connected components containing at least two vertices
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Theorem: The Linear Rank-width-One Vertex Deletion problem admits a kernel with $O(k \cdot \mu(k)^4) = O(k^{33})$ vertices.

Open problems

- Improve the time complexity of the FPT algorithm and the kernel size (Is $O^*(c^k)$ reasonable for $c < 4$?).
- Does RANK-WIDTH-c VERTEX DELETION has a single exponential FPT algorithm? Does RANK-WIDTH-c VERTEX DELETION has a polynomial kernel? Even for $c = 1$.

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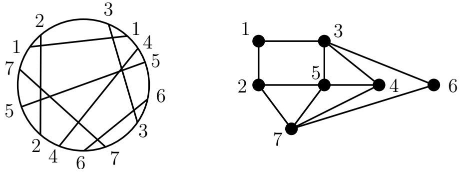
- Does RANK-WIDTH-c VERTEX DELETION has a single exponential FPT algorithm? Does RANK-WIDTH-c VERTEX DELETION has a polynomial kernel? Even for $c = 1$.
- What about the CIRCLE GRAPH VERTEX DELETION problem?



- **Circle-$\mathcal{F}$-vertex-minor deletion admits a polynomial kernel? (Like a planar-$\mathcal{F}$-minor deletion) If $\mathcal{F}$ contains a cycle or a path?**