Coloring graphs without fan vertex-minors and cycle pivot-minors

Ilkyoo Choi, O-JOUNG KWON, and Sang-il Oum

TU Berlin

Korea Advanced Institute of Science and Technology (KAIST)

Southern Italian Workshop on Algorithms and Graphs 2016

September 26, 2016

A clique is a set of pairwise adjacent vertices in a graph.
The clique number $\omega(G)$ is the size of the largest clique in a graph G.

A graph G is k-colorable if the following is possible:

- each vertex receives a color from $\{1, \ldots, k\}$
- adjacent vertices receive different colors

The chromatic number $\chi(G)$ is the minimum such k.

$$\omega(G) \leq \chi(G)$$

A clique is a set of pairwise adjacent vertices in a graph.
The clique number $\omega(G)$ is the size of the largest clique in a graph G.

A graph G is k-colorable if the following is possible:

- each vertex receives a color from $\{1, \ldots, k\}$
- adjacent vertices receive different colors

The chromatic number $\chi(G)$ is the minimum such k.

$$\omega(G) \leq \chi(G)$$

G is perfect if every induced subgraph H satisfies $\omega(H) = \chi(H)$.

A clique is a set of pairwise adjacent vertices in a graph.
The clique number $\omega(G)$ is the size of the largest clique in a graph G.

A graph G is k-colorable if the following is possible:

- each vertex receives a color from $\{1, \ldots, k\}$
- adjacent vertices receive different colors

The chromatic number $\chi(G)$ is the minimum such k.

$$\omega(G) \leq \chi(G)$$

G is perfect if every induced subgraph H satisfies $\omega(H) = \chi(H)$.

Strong Perfect Graph Conjecture (1961 Berge)

Given a graph G, every induced subgraph H satisfies $\omega(H) = \chi(H)$ iff G contains no C_k and no $\overline{C_k}$ as induced subgraphs for any odd $k \geq 5$.

Theorem (2006 Chudnovsky–Robertson–Seymour–Thomas)

The Strong Perfect Graph Conjecture is true.

Is there a function f such that $\chi(G) \leq f(\omega(G))$ for all graphs G?

Is there a function f such that $\chi(G) \leq f(\omega(G))$ for all graphs G?

Theorem (1955 Mycielski, 1954 Blanche Descartes, 1959 Erdős)

For any k, there exists a graph G with no triangle and $\chi(G) \geq k$.
For any k, there exists a graph G with girth at least 6 and $\chi(G) \geq k$.
For any k, g, there exists a graph G with girth at least g and $\chi(G) \geq k$.

Is there a function f such that $\chi(G) \leq f(\omega(G))$ for all graphs G? *NO!*

Theorem (1955 Mycielski, 1954 Blanche Descartes, 1959 Erdős)

For any k, there exists a graph G with no triangle and $\chi(G) \geq k$.
For any k, there exists a graph G with girth at least 6 and $\chi(G) \geq k$.
For any k, g, there exists a graph G with girth at least g and $\chi(G) \geq k$.

Is there a function f such that $\chi(G) \leq f(\omega(G))$ for all graphs G? *NO!*

Theorem (1955 Mycielski, 1954 Blanche Descartes, 1959 Erdős)

For any k, there exists a graph G with no triangle and $\chi(G) \geq k$.
For any k, there exists a graph G with girth at least 6 and $\chi(G) \geq k$.
For any k, g, there exists a graph G with girth at least g and $\chi(G) \geq k$.

Definition

A class $\mathcal{C}$ of graphs is χ-*bounded* if there is a function f where $\chi(G) \leq f(\omega(G))$ for all $G \in \mathcal{C}$.

Some questions about light versions of SPGT

Conjecture (1985 Gyárfás)

The following classes are χ-bounded:

1. *The class of graphs with no induced cycles of odd length ≥ 5*
2. *Given k, the class of graphs with no induced cycles of length $\geq k$*
3. *Given k, the class of graphs with no induced cycles of odd length $\geq k$*

Some questions about light versions of SPGT

Conjecture (1985 Gyárfás)

The following classes are χ-bounded:

1. *The class of graphs with no induced cycles of odd length ≥ 5*
2. *Given k, the class of graphs with no induced cycles of length $\geq k$*
3. *Given k, the class of graphs with no induced cycles of odd length $\geq k$*

Partial cases............. no induced cycles of.......

Some questions about light versions of SPGT

Conjecture (1985 Gyárfás)

The following classes are χ-bounded:

1. The class of graphs with no induced cycles of odd length ≥ 5

2. Given k, the class of graphs with no induced cycles of length $\geq k$

3. Given k, the class of graphs with no induced cycles of odd length $\geq k$

Partial cases............ no induced cycles of.......

2008 Addario-Berry–Chudnovsky–Havet–Reed–Seymour: even length

2013 Bonamy–Charbit–Thomassé: length divisible by 3

2015+ Lagoutte: length 3 and even length ≥ 6

2015+ Chudnovsky–Scott–Seymour: length 3 and odd length ≥ 7

length 5 and length 3 and odd length $\geq k$

length 5 and length $\geq k$

Some questions about light versions of SPGT

Conjecture (1985 Gyárfás)

The following classes are χ-bounded:

1. *The class of graphs with no induced cycles of odd length ≥ 5*
2. *Given k, the class of graphs with no induced cycles of length $\geq k$*
3. *Given k, the class of graphs with no induced cycles of odd length $\geq k$*

Partial cases............. no induced cycles of.......

2008 Addario-Berry–Chudnovsky–Havet–Reed–Seymour: even length
2013 Bonamy–Charbit–Thomassé: length divisible by 3
2015+ Lagoutte: length 3 and even length ≥ 6
2015+ Chudnovsky–Scott–Seymour: length 3 and odd length ≥ 7
length 5 and length 3 and odd length $\geq k$
length 5 and length $\geq k$
2015+ Scott–Seymour: length 3 and odd length $\geq k$

Some questions about light versions of SPGT

Conjecture (1985 Gyárfás)

The following classes are χ-bounded:

1. *The class of graphs with no induced cycles of odd length ≥ 5*
2. *Given k, the class of graphs with no induced cycles of length $\geq k$*
3. *Given k, the class of graphs with no induced cycles of odd length $\geq k$*

Partial cases............. no induced cycles of.......

2008 Addario-Berry–Chudnovsky–Havet–Reed–Seymour: even length

2013 Bonamy–Charbit–Thomassé: length divisible by 3

2015+ Lagoutte: length 3 and even length ≥ 6

2015+ Chudnovsky–Scott–Seymour: length 3 and odd length ≥ 7

length 5 and length 3 and odd length $\geq k$

length 5 and length $\geq k$

2015+ Scott–Seymour: length 3 and odd length $\geq k$

2015+ Scott–Seymour: *CLAIM TO SOLVE 1.!*

2015+ Chudnovsky–Scott–Seymour: *CLAIM TO SOLVE 2.!*

Some questions about light versions of SPGT

Conjecture (1985 Gyárfás)

The following classes are χ-bounded:

1. *The class of graphs with no induced cycles of odd length ≥ 5*
2. *Given k, the class of graphs with no induced cycles of length $\geq k$*
3. *Given k, the class of graphs with no induced cycles of odd length $\geq k$*

Partial cases............. no induced cycles of.......

2008 Addario-Berry–Chudnovsky–Havet–Reed–Seymour: even length
2013 Bonamy–Charbit–Thomassé: length divisible by 3
2015+ Lagoutte: length 3 and even length ≥ 6
2015+ Chudnovsky–Scott–Seymour: length 3 and odd length ≥ 7
length 5 and length 3 and odd length $\geq k$
length 5 and length $\geq k$
2015+ Scott–Seymour: length 3 and odd length $\geq k$

2015+ Scott–Seymour: *CLAIM TO SOLVE 1.!*
2015+ Chudnovsky–Scott–Seymour: *CLAIM TO SOLVE 2.!*

3. implies both 1. and 2.!

H is a SOMETHING of G if H can be obtained from G by OPERATIONS

H is a SOMETHING of G if H can be obtained from G by OPERATIONS

[1] induced subgraph

– deleting vertices

H is a SOMETHING of G if H can be obtained from G by OPERATIONS

[1] induced subgraph
  – deleting vertices

[2] vertex-minor
  – deleting vertices
  – local complementations at vertices

H is a SOMETHING of G if H can be obtained from G by OPERATIONS

[1] induced subgraph
- deleting vertices

[2] vertex-minor
- deleting vertices
- local complementations at vertices

[3] pivot-minor
- deleting vertices
- pivoting edges (one operation = three local complementations)

H is a SOMETHING of G if H can be obtained from G by OPERATIONS

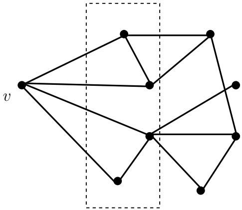
[1] induced subgraph
- deleting vertices

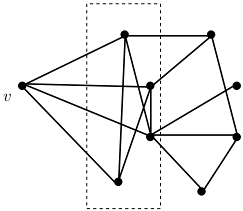
[2] vertex-minor
- deleting vertices
- local complementations at vertices

[3] pivot-minor
- deleting vertices
- pivoting edges (one operation = three local complementations)

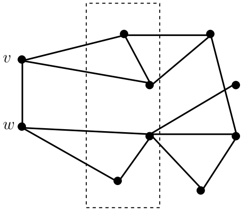
More operations imply easier to get the structure!
H-vertex-minor free implies H-pivot-minor free implies H-free.

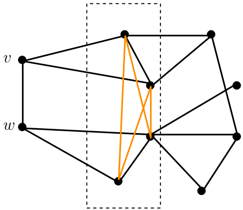
Local comlementation





Pivoting edge





H is a SOMETHING of G if H can be obtained from G by OPERATIONS

[1] induced subgraph

– deleting vertices

[2] vertex-minor

– deleting vertices

– local complementations at vertices

[3] pivot-minor

– deleting vertices

– pivoting edges (one operation = three local complementations)

More operations imply easier to get the structure!
H-vertex-minor free implies H-pivot-minor free implies H-free.

Bipartite graph

Distance-hereditary

Parity graph

Circle graph

Bipartite graph: vertex set has a partition into two independent sets

Distance-hereditary: distances are preserved in every induced subgraph

Parity graph: induced paths joining a pair of vertices have the same parity

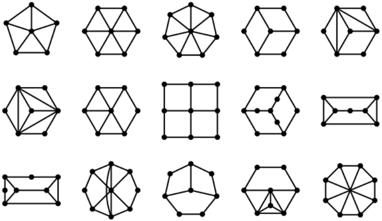
Circle graph: intersection graph of chords on a circle

Bipartite graph: vertex set has a partition into two independent sets
no C_3-pivot-minor

Distance-hereditary: distances are preserved in every induced subgraph
no C_5-vertex-minor (1987, 1988 Bouchet)
no C_5, C_6-pivot-minors (1986 Bandelt–Mulder)

Parity graph: induced paths joining a pair of vertices have the same parity
no C_5-pivot-minor (1984 Burlet–Uhry)

Circle graph: intersection graph of chords on a circle
three forbidden vertex-minors (1994 Bouchet)
fifteen forbidden pivot-minors (2009 Geelen–Oum)



Bipartite graph: vertex set has a partition into two independent sets

no C_3-pivot-minor

Distance-hereditary: distances are preserved in every induced subgraph

no C_5-vertex-minor (1987, 1988 Bouchet)

no C_5, C_6-pivot-minors (1986 Bandelt–Mulder)

Parity graph: induced paths joining a pair of vertices have the same parity

no C_5-pivot-minor (1984 Burlet–Uhry)

Circle graph: intersection graph of chords on a circle

three forbidden vertex-minors (1994 Bouchet)

fifteen forbidden pivot-minors (2009 Geelen–Oum)

Bipartite graph: vertex set has a partition into two independent sets
no C_3-pivot-minor

Distance-hereditary: distances are preserved in every induced subgraph
no C_5-vertex-minor (1987, 1988 Bouchet)
no C_5, C_6-pivot-minors (1986 Bandelt–Mulder)

Parity graph: induced paths joining a pair of vertices have the same parity
no C_5-pivot-minor (1984 Burlet–Uhry)

Circle graph: intersection graph of chords on a circle
three forbidden vertex-minors (1994 Bouchet)
fifteen forbidden pivot-minors (2009 Geelen–Oum)

Bipartite graph: vertex set has a partition into two independent sets
no C_3-pivot-minor

Distance-hereditary: distances are preserved in every induced subgraph
no C_5-vertex-minor (1987, 1988 Bouchet)
no C_5, C_6-pivot-minors (1986 Bandelt–Mulder)

Parity graph: induced paths joining a pair of vertices have the same parity
no C_5-pivot-minor (1984 Burlet–Uhry)

Circle graph: intersection graph of chords on a circle
three forbidden vertex-minors (1994 Bouchet)
fifteen forbidden pivot-minors (2009 Geelen–Oum)

Rank-width t graphs: well-decomposable
finite list of forbidden vertex-minors (2008 Oum)

Bipartite, distance-hereditary, parity graphs are perfect, thus χ-bounded.
Circle graphs are χ-bounded (1997 Kostochka–Kratochvíl)
Rank-width t graphs are χ-bounded (2012 Dvořák–Král)

Bipartite graph: vertex set has a partition into two independent sets
no C_3-pivot-minor

Distance-hereditary: distances are preserved in every induced subgraph
no C_5-vertex-minor (1987, 1988 Bouchet)
no C_5, C_6-pivot-minors (1986 Bandelt–Mulder)

Parity graph: induced paths joining a pair of vertices have the same parity
no C_5-pivot-minor (1984 Burlet–Uhry)

Circle graph: intersection graph of chords on a circle
three forbidden vertex-minors (1994 Bouchet)
fifteen forbidden pivot-minors (2009 Geelen–Oum)

Rank-width t graphs: well-decomposable
finite list of forbidden vertex-minors (2008 Oum)

Bipartite, distance-hereditary, parity graphs are perfect, thus χ-bounded.
Circle graphs are χ-bounded (1997 Kostochka–Kratochvíl)
Rank-width t graphs are χ-bounded (2012 Dvořák–Král)

Conjecture (1995 Geelen)

For any H, the class of graphs with no H-vertex-minor is χ-bounded.

Conjecture (1995 Geelen)

For any H, the class of graphs with no H-vertex-minor is χ-bounded.

Conjecture (1995 Geelen)

For any H, the class of graphs with no H-vertex-minor is χ-bounded.

2012 Dvořák–Král: H is W_5

Conjecture (1995 Geelen)

For any H, the class of graphs with no H-vertex-minor is χ-bounded.

2012 Dvořák–Král: H is W_5
2014 K–Oum: H is distance-hereditary
(1997 Scott: The class of graphs having no fixed tree as an induced subdivision is χ-bounded)

Conjecture (1995 Geelen)

For any H, the class of graphs with no H-vertex-minor is χ-bounded.

2012 Dvořák–Král: H is W_5
2014 K–Oum: H is distance-hereditary
(1997 Scott: The class of graphs having no fixed tree as an induced subdivision is χ-bounded)
2015+ Chudnovsky–Scott–Seymour: H is a cycle
(The class of graphs with no induced cycles of long length is χ-bounded)

Conjecture (1995 Geelen)

For any H, the class of graphs with no H-vertex-minor is χ-bounded.

2012 Dvořák–Král: H is W_5

2014 K–Oum: H is distance-hereditary

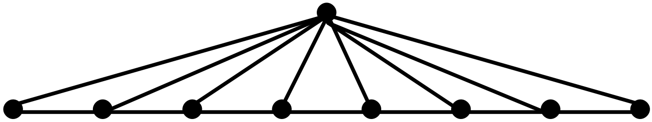
(1997 Scott: The class of graphs having no fixed tree as an induced subdivision is χ-bounded)

2015+ Chudnovsky–Scott–Seymour: H is a cycle

(The class of graphs with no induced cycles of long length is χ-bounded)

Theorem (2015 Choi–K.–Oum)

For any fan F, the class of graphs with no F-vertex-minor is χ-bounded.



Conjecture (Stronger version)

For any H, the class of graphs with no H-pivot-minor is χ-bounded.

Conjecture (Stronger version)

For any H, the class of graphs with no H-pivot-minor is χ-bounded.

Excluding a cycle of length k as a pivot-minor is interesting, as it does not exclude all induced cycles of length at least k.

For odd k and even ℓ, C_k has no pivot-minor C_ℓ and vice versa.

Conjecture (Stronger version)

For any H, the class of graphs with no H-pivot-minor is χ-bounded.

Excluding a cycle of length k as a pivot-minor is interesting, as it does not exclude all induced cycles of length at least k.

For odd k and even ℓ, C_k has no pivot-minor C_ℓ and vice versa.

Theorem (2015 Choi–K.–Oum)

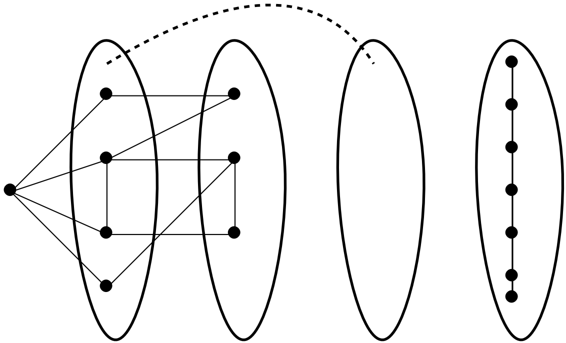
For any cycle C, the class of graphs with no C-pivot-minor is χ-bounded.

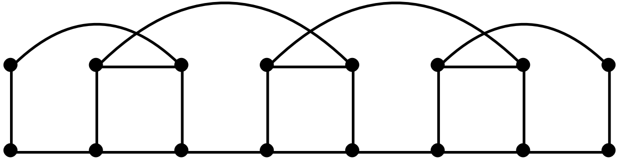
If Gyárfás's third conjecture (excluding odd cycles of length at least k) is true, then this theorem with odd cycle follows from the result.

Theorem (2015 Choi–K.–Oum)

For any fan F, the class of graphs with no F-vertex-minor is χ-bounded.

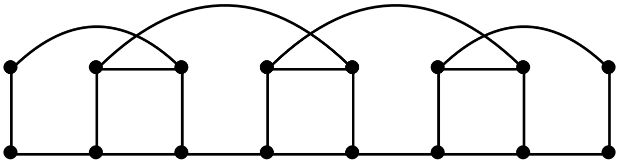
- We use a leveling technique; take a leveling from a vertex. We can say that there should be one level with large chromatic number.
- By Gyárfás's result, there exists a long induced path in this level.
- You cannot do contractions!





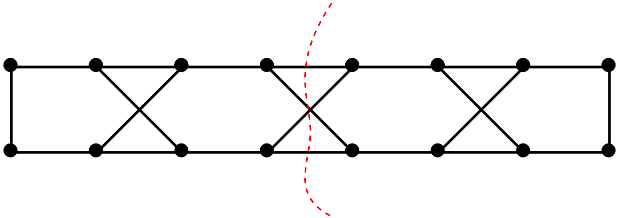
This graph does not contain fan F_6 as a vertex-minor.

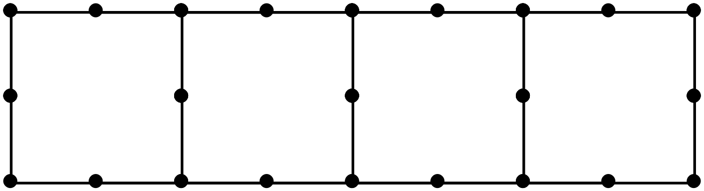
Why?



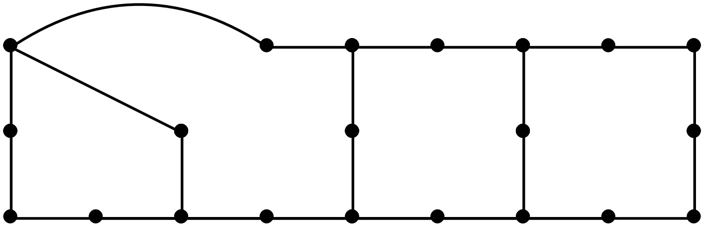
This graph does not contain fan F_6 as a vertex-minor.

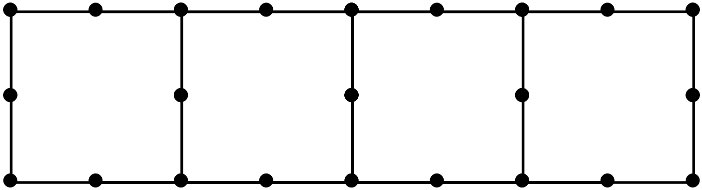
Why? Local complementaions do not destroy splits.



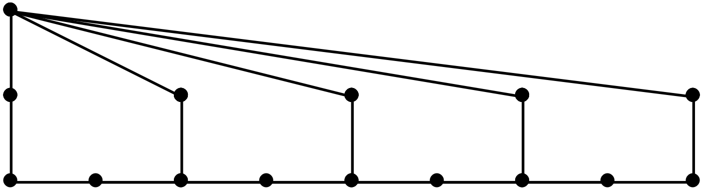


Pivoting edges





Pivoting edges



Theorem (2015 Choi–K.–Oum)

For any fan F, the class of graphs with no F-vertex-minor is χ-bounded.

1. Refining level L_t containing the path and the previous levels L_{t-1}, L_{t-2} to well-structured ladder
   - Iteratively use Ramsey's type argument
2. From v, going down to this well-structured ladder
   - Use the property that for every graph either it is connected or its complement is connected.
   - We push a connected graph on L_{t-2}.

Conjecture (1995 Geelen)

For any H, the class of graphs with no H-vertex-minor is χ-bounded.

Theorem (2015 Choi–K.–Oum)

For any fan F, the class of graphs with no F-vertex-minor is χ-bounded.

Question

For any wheel W, is the class of graphs with no W-vertex-minor χ-bounded?

Conjecture (1995 Geelen)

For any H, the class of graphs with no H-vertex-minor is χ-bounded.

Theorem (2015 Choi–K.–Oum)

For any fan F, the class of graphs with no F-vertex-minor is χ-bounded.

Question

For any wheel W, is the class of graphs with no W-vertex-minor χ-bounded?

Thank you for your attention!