Coloring graphs without fan vertex-minors and cycle pivot-minors

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A **clique** is a set of pairwise adjacent vertices in a graph. The **clique number** \( \omega(G) \) is the size of the largest clique in a graph \( G \).

A graph \( G \) is **\( k \)-colorable** if the following is possible:
- each vertex receives a **color** from \( \{1, \ldots, k\} \)
- adjacent vertices receive different colors

The **chromatic number** \( \chi(G) \) is the minimum such \( k \).

\[ \omega(G) \leq \chi(G) \]
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**Strong Perfect Graph Conjecture (1961 Berge)**

*Given a graph $G$, every induced subgraph $H$ satisfies $\omega(H) = \chi(H)$ iff $G$ contains no $C_k$ and no $\overline{C_k}$ as induced subgraphs for any odd $k \geq 5$.*


*The Strong Perfect Graph Conjecture is true.*
Is there a function $f$ such that $\chi(G) \leq f(\omega(G))$ for all graphs $G$?
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**Theorem (1955 Mycielski, 1954 Blanche Descartes, 1959 Erdős)**

- For any $k$, there exists a graph $G$ with no triangle and $\chi(G) \geq k$.
- For any $k$, there exists a graph $G$ with girth at least 6 and $\chi(G) \geq k$.
- For any $k$, $g$, there exists a graph $G$ with girth at least $g$ and $\chi(G) \geq k$. 


Is there a function $f$ such that $\chi(G) \leq f(\omega(G))$ for all graphs $G$? **NO!**

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Some questions about light versions of SPGT

Conjecture (1985 Gyárfás)

The following classes are $\chi$-bounded:

1. The class of graphs with no induced cycles of odd length $\geq 5$
2. Given $k$, the class of graphs with no induced cycles of length $\geq k$
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3. implies both 1. and 2.!


$H$ is a SOMETHING of $G$ if $H$ can be obtained from $G$ by OPERATIONS

\[\begin{align*}
1 & \text{ induced subgraph} \\
2 & \text{vertex-minor} \\
3 & \text{pivot-minor}
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More operations imply easier to get the structure!

$H$-vertex-minor free implies $H$-pivot-minor free implies $H$-free.
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Bipartite graph

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- no \( C_5 \)-vertex-minor \((1987, 1988 \text{ Bouchet})\)
- no \( C_5, C_6 \)-pivot-minors \((1986 \text{ Bandelt–Mulder})\)

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**Rank-width $t$** graphs: well-decomposable
- finite list of forbidden vertex-minors (2008 Oum)

Bipartite, distance-hereditary, parity graphs are **perfect**, thus **$\chi$-bounded**.
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Theorem (2015 Choi–K.–Oum)

For any fan $F$, the class of graphs with no $F$-vertex-minor is $\chi$-bounded.
Conjecture (Stronger version)

For any $H$, the class of graphs with no $H$-pivot-minor is $\chi$-bounded.
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Excluding a cycle of length $k$ as a pivot-minor is interesting, as it does not exclude all induced cycles of length at least $k$.

For odd $k$ and even $\ell$, $C_k$ has no pivot-minor $C_\ell$ and vice versa.
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Theorem (2015 Choi–K.–Oum)

For any cycle $C$, the class of graphs with no $C$-pivot-minor is $\chi$-bounded.

If Gyárfás’s third conjecture (excluding odd cycles of length at least $k$) is true, then this theorem with odd cycle follows from the result.
Theorem (2015 Choi–K.–Oum)

For any fan $F$, the class of graphs with no $F$-vertex-minor is $\chi$-bounded.

- We use a leveling technique; take a leveling from a vertex. We can say that there should be one level with large chromatic number.
- By Gyárfás’s result, there exists a long induced path in this level.
- You cannot do contractions!
This graph does not contain fan $F_6$ as a vertex-minor.

Why?
This graph does not contain fan $F_6$ as a vertex-minor.

Why? Local complementaions do not destroy splits.
Pivoting edges
Pivoting edges
Theorem (2015 Choi–K.–Oum)

For any fan $F$, the class of graphs with no $F$-vertex-minor is $\chi$-bounded.

1. Refining level $L_t$ containing the path and the previous levels $L_{t-1}, L_{t-2}$ to well-structured ladder
   - Iteratively use Ramsey’s type argument

2. From $v$, going down to this well-structured ladder
   - Use the property that for every graph either it is connected or its complement is connected.
   - We push a connected graph on $L_{t-2}$. 
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Theorem (2015 Choi–K.–Oum)
For any fan $F$, the class of graphs with no $F$-vertex-minor is $\chi$-bounded.

Question
For any wheel $W$, is the class of graphs with no $W$-vertex-minor $\chi$-bounded?
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Thank you for your attention!