Recognizing Small Pivot-Minors*

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Abstract

A graph G contains a graph H as a pivot-minor if H can be obtained from G by applying a sequence of vertex deletions and edge pivots. So far, pivot-minors have been considered mainly from a structural perspective and not much is known about whether graph classes that contain (or do not contain) a certain pivot-minor can be recognized in polynomial time. Our results provide a first step in this direction. We first prove that the Pivot-Minor problem, which is that of testing whether a given graph G contains a given graph H as a pivot-minor, is NP-complete. If H is fixed we denote this problem H-Pivot-Minor. Our second result is that H-Pivot-Minor is polynomial-time solvable for every connected graph H with $|V(H)| \leq 4$ except for one open case, namely $H = K_4$.

1 Introduction

Computing whether a graph H appears as a "pattern" inside some other graph G is a well-studied problem in the area of structural and algorithmic graph theory. The definition of a pattern depends on the set of graph operations that we are allowed to use. For instance, if we can obtain H from G via a sequence of vertex deletions, edge deletions and edge contractions, then G contains H as a minor. The Minor problem is that of testing whether a given graph G contains a given graph H as a minor. This problem is known to be NP-complete even if G and H are trees of small diameter [14]. Hence, it is natural to fix the graph H and let only G be part of the input. This leads to the H-Minor problem, and a celebrated result of Robertson and Seymour [22] states that the H-Minor problem can be solved in cubic time for every graph H. If we only allow vertex deletions and edge contractions, then we obtain the H-Induced Minor problem. In contrast, this problem can be NP-complete (see [6] for an example of a "hard" graph H on 68 vertices). Other well-known containment relations include containing a graph H as a contraction, an induced subgraph, a subdivision, a topological minor or an induced topological minor; see, for instance, [3, 10, 12, 13, 23] for a number of complexity results for these relations.

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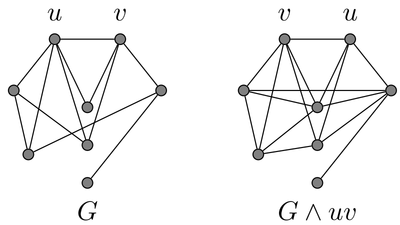

Figure 1 Graphs G and $G \wedge uv$.

In this paper we focus on the pivot-minor containment relation defined as follows. The *local complementation* at a vertex u in a graph G replaces every edge of the subgraph induced by the neighbours of u by a non-edge, and vice versa. We denote the resulting graph by $G * u$. The *edge pivot* is the operation that takes an edge uv, first applies a local complementation at u, then at v, and then at u again. We denote the resulting graph by $G \wedge uv = G * u * v * u$ and note that $G * u * v * u = G * v * u * v$. Alternatively, we can define the edge pivot operation as follows. Consider the set S_u of neighbours of u not adjacent to v, the set S_v of neighbours of v not adjacent to u and the set S_{uv} of common neighbours of u and v. Replace every edge between any two vertices of two distinct sets from $\{S_u, S_v, S_{uv}\}$ by a non-edge and vice versa. Then delete every edge between u and S_u and add every edge between u and S_v. Similarly, delete every edge between v and S_v and add every edge between v and S_u. See Figure 1 for an example of an edge pivot. A graph G contains a graph H as a *pivot-minor* if G can be modified into (an isomorphic copy of) H by a sequence of vertex deletions and edge pivots.

Pivot-minors were called *p-reductions* by Bouchet [1] and have been mainly studied from a structural perspective, as they form a very suitable tool for working with rank-width [16, 20]. Rank-width is a well-known width parameter (see [19] for a survey) and pivot-minors play a similar role for rank-width as minors do for treewidth. Oum [17] showed that for every positive constant k the class of graphs of rank-width at most k is well-quasi-ordered under the pivot-minor relation. Kwon and Oum [11] proved that every graph of rank-width at most k is a pivot-minor of a graph of treewidth at most $2k$, and that a graph of linear rank-width at most k is a pivot-minor of a graph of path-width at most $k + 1$.

Pivot-minors are closely related to so-called vertex-minors, introduced in the nineties as *ℓ-reductions* by Bouchet [1]. A graph G contains a graph H as a *vertex-minor* if G can be modified into (an isomorphic copy of) H by a sequence of vertex deletions and local complementations. Hence, if G contains H as a pivot-minor, then G contains H as a vertex-minor (but not necessarily vice versa). Bouchet [1] characterized circle graphs in terms of forbidden vertex-minors and by using this result, Geelen and Oum [9] were able to characterize circle graphs in terms of forbidden pivot-minors. Oum [18] conjectured that for each fixed bipartite circle graph H, every graph G of sufficiently large rank-width contains H as a pivot-minor. This conjecture is known to be true when G is a line graph, a bipartite graph, or a circle graph (see [18]).

We study pivot-minors from an algorithmic perspective, that is, we consider the following research question:

Can we decide if a graph H is a pivot-minor of a graph G in polynomial time?

If both G and H are part of the input, then we obtain the following decision problem:

Pivot-Minor	
Instance:	A pair of graphs G and H.
Question:	Does G have a pivot-minor isomorphic to H?

If H is not part of the input but fixed, then we obtain the H-Pivot-Minor problem. So far, it was only known that, for every graph H, the H-Pivot-Minor problem is solvable in cubic time for graphs G of bounded rank-width, as pivot-minor testing can be expressed in monadic second order logic with modulo-2 counting in a similar way to vertex-minor testing [5]. Hence, a systematic study into the computational complexity of H-Pivot-Minor for general graphs is currently lacking and this motivated our research.

Our Results. In Section 2 we prove that Pivot-Minor is NP-complete. Due to this, it is natural to study the computational complexity of H-Pivot-Minor, as proposed in [19]. To get a handle on this problem, we restrict ourselves to small graphs H. For every connected graph H on at most four vertices except for the complete graph K_4, we give an algorithm that solves H-Pivot-Minor in polynomial time.

To explain the idea behind our algorithms, we first observe that *H-pivot-minor-free* graphs, that is, graphs that do not contain H as a pivot-minor, are closed under vertex deletion. It is well known and readily seen that a class of graphs is closed under vertex deletion if and only if it can be characterized by a (possibly infinite) set of minimal forbidden induced subgraphs. For every connected graph $H \neq K_4$ on at most four vertices, we determine in Section 3 the set $\mathcal{F}_H$ of minimal forbidden induced subgraphs. Then, our algorithms simply test whether the input graph G contains an induced subgraph F in $\mathcal{F}_H$. If such an induced subgraph does not exist, then G is H-pivot-minor-free. Otherwise, G contains H as a pivot-minor. In fact, the graph F from $\mathcal{F}_H$ found by our algorithm contains H as a pivot-minor. Thus F can be seen as a certificate that can be used to verify H-pivot-minor containment (see [15] for a survey on certifying algorithms).

The only connected graph H on at most four vertices for which we do not know the complexity of H-Pivot-Minor is $H = K_4$. In Section 4 we prove that $\mathcal{F}_{K_4}$ must contain infinitely many graphs, and in the same section we discuss some other natural questions for future research.

2 When H Is a Part of the Input

In this section we prove that Pivot-Minor is NP-complete. We first need to introduce some terminology. A *matroid* is a pair $M = (E, \mathcal{I})$ of a finite set E, called the *ground set*, and a set $\mathcal{I}$ of subsets of E satisfying the following three properties:

- $\mathcal{I} \neq \emptyset$;
- if $Y \in \mathcal{I}$ and $X \subseteq Y$, then $X \in \mathcal{I}$ and
- if $X, Y \in \mathcal{I}$ with $|Y| = |X|+1$, then there exists an element $y \in Y \setminus X$ such that $X \cup \{y\} \in \mathcal{I}$.

A set $X \subseteq E$ is *independent* in $M = (E, \mathcal{I})$ if $X \in \mathcal{I}$. Otherwise X is *dependent*. The *rank* of a subset $X \subseteq E$ is the size of a largest independent subset of X. The *rank* of a matroid $M = (E, \mathcal{I})$ is the rank of E. A *base* of a matroid is a maximal independent set. A *circuit* of a matroid is a minimal dependent set. The *dual matroid* M^* of a matroid $M = (E, \mathcal{I})$ is a matroid on E such that X is a base of M^* if and only if $E \setminus X$ is a circuit in M. For a subset X of E, we define $M \setminus X$ to be the matroid $(E \setminus X, \mathcal{I}')$ such that $\mathcal{I}' = \{X' \subseteq E \setminus X \mid X' \in \mathcal{I}\}$. We define $M/X = (M^* \setminus X)^*$. A matroid N is a *minor* of a matroid M if $N = (M \setminus X)/Y$ for some disjoint sets X and Y.

A matroid $M = (E, \mathcal{I})$ is *binary* if there is a matrix over the binary field whose columns are indexed by E such that X is independent in M if and only if the corresponding columns are linearly independent. It is known that the dual matroid of a binary matroid is also binary.

A major example of binary matroids arises from graphs. For a graph $G = (V, E)$, let $\mathcal{I}$ be a set of subsets X of E such that the subgraph (V, X) has no cycles. Then $M(G) = (E, \mathcal{I})$ is a matroid, called the *cycle matroid* of G and such matroids are binary. It is known that circuits of $M(G)$ are precisely the edge set of cycles of G and if a graph H is a minor of G, then $M(H)$ is a minor of $M(G)$.

If G is connected and has n vertices and m edges, then $M(G)$ has rank $n-1$ because any spanning tree of G has $n-1$ edges, and $(M(G))^*$ has rank $m-n+1$. For more on matroid theory, we refer to the book of Oxley [21].

For a binary matroid $M=(E,\mathcal{I})$, the *fundamental graph* of M with respect to a base B is a bipartite graph on E with the bipartition $(B,E\setminus B)$ such that $x\in B$, $y\in E\setminus B$ are adjacent if and only if $(B\setminus\{x\})\cup\{y\}$ is a base of M. Conversely, for a bipartite graph G with a bipartition (A,B), we may define a binary matroid $\mathrm{Bin}(G,A,B)$ on $V(G)$ represented by the $A\times V(G)$ matrix

$$\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ A & \begin{pmatrix} I_A & M_{A,B} \end{pmatrix} \end{array}$$

over the binary field where I_A is the $A\times A$ identity matrix and $M_{A,B}$ is the $A\times B$ submatrix of the adjacency matrix of G whose (x,y)-entry is 1 if and only if x and y are adjacent. We use the following lemma to prove that Pivot-Minor is NP-complete.

▶ **Lemma 1** ([16, Corollary 3.6])**.** *The following statements hold:*

(1) *Let N, M be binary matroids, and H, G be fundamental graphs of N and M respectively. If N is a minor of M, then H is a pivot-minor of G.*
(2) *Let G be a bipartite graph with a bipartition $A\cup B=V(G)$. If H is a pivot-minor of G, then there is a bipartition $A'\cup B'=V(H)$ such that* $\mathrm{Bin}(H,A',B')$ *is a minor of* $\mathrm{Bin}(G,A,B)$.

We are now ready to prove our hardness result.

▶ **Theorem 2.** Pivot-Minor *is* NP-*complete.*

Proof. We reduce from Hamilton Cycle, which is the problem of testing whether a given graph has a Hamilton cycle. This problem is NP-complete even for 3-regular graphs [7]. Let $G=(V,E)$ be a 3-regular graph with n vertices and m edges. We may assume without loss of generality that $n\geq 5$ and that G is connected. Note that $2m=3n$. Consequently, $(M(G))^*$ has rank $m-n+1=\frac{1}{2}n+1$.

Let T be a spanning tree of G. Let G_T be the fundamental graph of $M(G)$ with respect to $E(T)$, which can be constructed in polynomial time. We claim that G has a Hamilton cycle if and only if $K_{1,n-1}$ is a pivot-minor of G_T.

For the forward direction, we use Lemma 1(1). If G has a Hamilton cycle C, then G contains C as a minor and thus $M(G)$ has $M(C)$ as a minor, and so G_T has every fundamental graph of $M(C)$ as a pivot-minor. This proves the forward direction, because every fundamental graph of $M(C)$ is isomorphic to $K_{1,n-1}$.

For the reverse direction, suppose that $K_{1,n-1}$ is a pivot-minor of G_T. Then by Lemma 1(2), $V(K_{1,n-1})$ has a bipartition (A',B') such that $\mathrm{Bin}(K_{1,n-1},A',B')$ is a minor of $M(G)=\mathrm{Bin}(G_T,A,B)$ for some partition (A,B) of $V(G_T)$. As $K_{1,n-1}$ is connected, it admits only two possible bipartitions. So $\mathrm{Bin}(K_{1,n-1},A',B')$ is either $M(C)$ or its dual $(M(C))^*$, where C is the cycle on n vertices. Therefore $M(C)$ or $(M(C))^*$ is a minor of $M(G)$. Equivalently, $M(C)$ is a minor of $M(G)$ or $(M(G))^*$. Because the rank of $M(C)$ is $n-1$ and the rank of $(M(G))^*$ is $\frac{1}{2}n+1<n-1$ (as $n\geq 5$) we find that $M(C)$ cannot be a minor of $(M(G))^*$. Thus, $M(C)$ is a minor of $M(G)$ and therefore $M(G)$ has a circuit of length at least n. This implies that G has a cycle of length n. ◀

3 When H Is Connected and Small But Not a K_4

In this section we consider every connected graph H on at most four vertices except for the case where $H=K_4$ (see Section 4 for some partial results on this case). For two vertex-disjoint

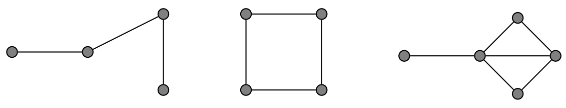

Figure 2 The list of forbidden induced subgraphs of graphs having no P_4 as a pivot-minor.

graphs G_1 and G_2, we let $G_1 + G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$ denote their *disjoint union*. We denote the path and cycle on n vertices by P_n and C_n, respectively. We let $\overline{G}$ denote the *complement* of a graph G, that is, the graph with vertex set $V(G)$ and an edge between two vertices u and v if and only if u and v are not adjacent in G. The *paw* is the graph $\overline{P_1 + P_3}$. The *diamond* is the graph $\overline{2P_1 + P_2}$. The *dart* is the graph $\overline{P_1 + \text{paw}}$. The *claw* is the graph $K_{1,3}$. In this section, we consider the cases where $H \in \{P_1, P_2, P_3, P_4, C_3, C_4, \text{paw}, \text{diamond}, \text{claw}\}$. For each graph H in this set, we determine a set $\mathcal{F}_H$ such that a graph G contains H as a pivot-minor if and only if G contains an induced subgraph in $\mathcal{F}_H$.

The case where H is the claw is more complicated than the other cases. As such, we first consider the other cases in Section 3.1 and discuss the claw in Section 3.2.

3.1 When H Is Not the Claw

In this section we focus on the cases where $H \in \{P_1, P_2, P_3, P_4, C_3, C_4, \text{paw}, \text{diamond}\}$. We say that a graph class is *pivot-minor-closed* if it is closed under vertex deletions and edge pivots. We say that a graph G is $(H_1, \ldots, H_p)$*-free* for some set $\mathcal{H} = \{H_1, \ldots, H_p\}$ of graphs if G contains no induced subgraph isomorphic to a graph in $\mathcal{H}$.

The cases where $H \in \{P_1, P_2\}$ are trivial. The cases $H = P_3$ and $H = C_3$ can also be readily seen, as follows.

▶ **Proposition 3.** *For a graph G, P_3 is a pivot-minor of G if and only if P_3 is an induced subgraph of G.*

Proof. If G contains P_3 as an induced subgraph, then G contains P_3 as a pivot-minor. If G does not contain P_3 as an induced subgraph, then G is the disjoint union of complete graphs. Any edge pivot does not change a complete graph. Hence, as complete graphs are closed under vertex deletion, complete graphs are pivot-minor-closed, and so G does not contain P_3 as a pivot-minor. ◀

▶ **Proposition 4.** *A graph G contains C_3 as a pivot-minor if and only if G contains an odd cycle.*

Proof. First observe that C_3 is a pivot-minor of C_n for every odd integer $n \geq 3$. If G contains an odd cycle, then a shortest odd cycle is induced and so G contains C_3 as a pivot-minor. Conversely, if G is bipartite, then every pivot-minor of G is bipartite, that is, the class of bipartite graphs is pivot-minor-closed (see also [16]). Hence G does not contain C_3 as a pivot-minor. ◀

▶ **Proposition 5.** *The following statements are equivalent for every graph G.*

(i) *P_4 is a pivot-minor of G.*
(ii) *C_4 is a pivot-minor of G.*
(iii) *P_4, C_4, or the dart is an induced subgraph of G* (see Figure 2).

Proof. P_4 and C_4 can be obtained from each other by pivoting one edge and so (i) and (ii) are equivalent. As the dart contains P_4 as a pivot-minor, (iii) implies (i) and (ii).

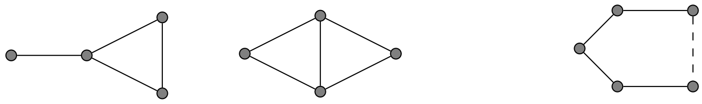

Figure 3 The list of forbidden induced subgraphs of graphs having no paw as a pivot-minor.

Now suppose that (iii) is false. It suffices to prove that (dart, P_4, C_4)-free graphs are pivot-minor-closed. We say that a graph is a *clique-star* if and only if it consists of mutually vertex-disjoint complete graphs K, $L_1, \ldots, L_p$ for some $p \geq 0$, such that every vertex of K is adjacent to every vertex of $L_1 \cup \cdots \cup L_p$ and there is no edge between any two distinct complete graphs L_i and L_j. We may also assume that either $p = 0$ or $p \geq 2$. We need the following claim.

Claim. Every connected component of G is a clique-star.

For contradiction, suppose that G has a connected component D that is not a clique-star. It is well known that if H is a P_4-free graph on at least two vertices, then either H or $\overline{H}$ is disconnected [4]. Hence we can partition $V(D)$ into two sets A and B, such that every vertex of A is adjacent to every vertex of B. Moreover, as D is not a complete graph, we may assume that B is not a clique. If A is not a clique either, then two non-adjacent vertices of A, together with two non-adjacent vertices of B, form an induced C_4, a contradiction. Hence A is a clique. We assume that A is chosen to be maximal subject to the condition that every vertex of A is adjacent to every vertex of B.

Suppose B induces a connected subgraph. Then we can partition B into two sets B_1 and B_2, such that every vertex of B_1 is adjacent to every vertex of B_2. As B is not a clique, this means that at least one of B_1 and B_2, say B_2, is not a clique. Then, by the same argument as before, B_1 must be a clique. This implies that every vertex of B_1 is adjacent to every other vertex of B_1 and to every vertex of B_2. However, this contradicts the maximality of A, as we could have chosen $A \cup B_1$ instead. Hence B does not induce a connected subgraph of D.

Let $J_1, \ldots, J_r$ denote the connected components of $D[B]$ for some $r \geq 2$. Suppose one of the graphs $J_1, \ldots, J_r$, say J_1, is not complete. Then J_1 contains an induced P_3, say on vertices u, v, w. Then u, v, w, together with a vertex of A and a vertex of J_2, form an induced dart, a contradiction. Hence we have proven the Claim.

It remains to prove that clique-stars are pivot-minor-closed. In order to see this, consider an edge uv of some clique-star. If u and v both belong to K or both belong to some L_i, then pivoting uv results in the same graph. In the remaining case, we may assume that u belongs to K and v belongs to L_1. After pivoting uv, we obtain a new clique-star with $K' = L_1$, $L_1' = K$ and $L_i' = L_i$ for $i = 2, \ldots, p$. Hence the class of (dart, P_4, C_4)-free graphs is pivot-minor-closed. ◀

▶ **Proposition 6.** *The following statements are equivalent for every graph G.*

- **(i)** *The paw is a pivot-minor of G.*
- **(ii)** *The diamond is a pivot-minor of G.*
- **(iii)** *The paw, the diamond, or an odd cycle of length at least 5 is an induced subgraph of G* (see Figure 3).
- **(iv)** *G has a connected component that is neither bipartite nor complete.*

Proof. By pivoting one edge, the diamond can be obtained from the paw and so (i) and (ii) are equivalent. Since every odd cycle on at least five vertices contains the paw as a pivot-minor, (iii) implies (i) and (ii).

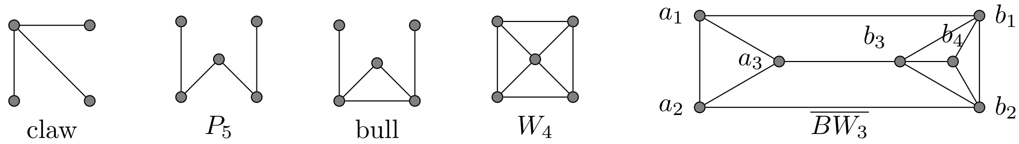

Figure 4 The list of forbidden induced subgraphs for graphs having no claw as a pivot-minor.

Now suppose that (iii) is false. Let D be a connected component of G. We claim that D is bipartite or complete. If not, C_3 is a proper subgraph of D. Let K be a maximal clique of D containing all three vertices of C_3. As D is not complete and K is chosen to be maximal, there exists a vertex $u \in V(D) \setminus K$ that has both a neighbour and a non-neighbour in K. Since K is a clique of size at least 3, D contains the paw or diamond as an induced subgraph, a contradiction. Thus (iv) implies (iii).

If (iv) is false, then as the classes of complete graphs and bipartite graphs are pivot-minor-closed, G does not contain the paw as a pivot-minor and so (i) is false. So (i) implies (iv). This completes the proof. ◀

Every condition for induced subgraphs in Propositions 3 and 5 can be checked in polynomial time, as there are only finitely many forbidden induced subgraphs to be checked in each of these cases. Proposition 4 can be checked in polynomial time because we can find an odd cycle, if one exists, in polynomial time by testing bipartiteness. In particular, checking these conditions will give us the graph F in $\mathcal{F}_H$ as an induced subgraph of G if one exists.

For Proposition 6, we do not know whether there exists a polynomial-time algorithm to decide if a given graph has an odd induced cycle of length at least 5 in general. However we can use condition (iv) in Proposition 6 to decide whether a graph has the paw as a pivot-minor and this structure allows us to find a forbidden induced subgraph F efficiently by using the argument in its proof.

In all cases above, it is trivial to find the edge pivots to obtain H from F. Hence we obtain the following result. For two graphs G and H we say that a sequence S of vertex deletions and edge pivots is an *H-pivot-minor-sequence* if H can be obtained from G after applying the operations of S.

▶ **Corollary 7.** *Let $H \in \{P_1, P_2, P_3, P_4, C_3, C_4, paw, diamond\}$. Then there is a polynomial-time algorithm for H-Pivot-Minor that gives an H-pivot-minor-sequence (if one exists).*

3.2 When H Is the Claw

The *bull* is the graph obtained from P_5 by adding an edge between the second vertex and the fourth vertex. The graph W_n is the graph obtained from C_n by adding one vertex adjacent to all vertices in the cycle. The graph BW_3 is the bipartite graph on seven vertices obtained from C_6 by adding one vertex adjacent to three pairwise non-adjacent vertices of the cycle. Figure 4 shows the complement of BW_3, denoted by $\overline{BW_3}$. In this section, we show that the claw is a pivot-minor of G if and only if the claw, P_5, the bull, W_4, or $\overline{BW_3}$ is an induced subgraph of G.

Lemma 9 is easy in the context of binary delta-matroids or matrix pivots (see [2, 18]). We provide a direct proof, which is inspired by the analogous proof for vertex-minors by Geelen and Oum [9, Lemma 3.2]. We write G/v to denote $(G \wedge zv) - v$ if a vertex v has a neighbour z, and $G - v$ if v is isolated. Two graphs are *pivot-equivalent* if one is obtained from the other by a sequence of edge pivots. For two neighbours x, y of v, because $(G \wedge xv) - v = (G \wedge yv \wedge xy) - v = (G \wedge yv - v) \wedge xy$, we find that $(G \wedge xv) - v$ is pivot-equivalent to $(G \wedge yv) - v$ and therefore the choice of z does not change the pivot-equivalence of graphs G/v. The proof of the following is in the appendix.

▶ **Lemma 8.** *Let v, x, y be distinct vertices of a graph G. If x, y are adjacent, then $(G \wedge xy) - v$ is pivot-equivalent to $G - v$ and $(G \wedge xy)/v$ is pivot-equivalent to G/v.*

▶ **Lemma 9.** *If a graph H is a pivot-minor of G and $v \in V(G) \setminus V(H)$, then H is a pivot-minor of $G - v$ or $(G \wedge vw) - v$ for some neighbour w of v in G.*

Proof. Let $G_0 = G$, $G_1 = G_0 \wedge x_1y_1$, $G_2 = G_1 \wedge x_2y_2$, $\ldots$, $G_m = G_{m-1} \wedge x_my_m$ and suppose H is an induced subgraph of $G_m - v$. For each i, if $x_i \neq v$ and $y_i \neq v$ then by Lemma 8, $G_i - v = (G_{i-1} \wedge x_iy_i) - v$ is pivot-equivalent to $G_{i-1} - v$ and $G_i/v = (G_{i-1} \wedge x_iy_i)/v$ is pivot-equivalent to G_{i-1}/v. If $x_i = v$, then $G_i - v = (G_{i-1} \wedge x_iy_i) - v = G_{i-1}/v$ and $G_i/v = (G_{i-1} \wedge y_iv \wedge y_iv) - v = G_{i-1} - v$. Thus we deduce that G/v or $G - v$ is pivot-equivalent to $G_m - v$. ◀

The next lemma will allow us to focus on the *connected* case. Indeed, it will be sufficient to show that if G is a connected graph, then the claw is a pivot-minor of G if and only if $3P_1$, W_4, or $\overline{BW_3}$ is an induced subgraph of G.

▶ **Lemma 10.** *A graph G is (bull, claw, P_5)-free if and only if every component of G is $3P_1$-free.*

Proof. Note that the bull, the claw and P_5 are all connected graphs that contain an induced subgraph isomorphic to $3P_1$. Therefore, if every component of G is $3P_1$-free, then every component of G is (bull, claw, P_5)-free and so G is (bull, claw, P_5)-free.

Now suppose that G contains a component containing an induced subgraph isomorphic to $3P_1$, say on the vertex set $\{x, y, z\}$. We will show that G contains an induced subgraph isomorphic to the bull, the claw or P_5. Let $P = p_1p_2 \cdots p_m$ be a shortest path from $x = p_1$ to $y = p_m$. If $m \geq 5$, then G contains P_5 as an induced subgraph. Thus, we may assume $m \leq 4$, which in particular implies that P does not contain z. Let $Q = q_1q_2 \cdots q_n$ be a shortest path from $z = q_1$ to a vertex q_n on P. If z is adjacent to a vertex on P, then G must contain the claw or the bull because z is non-adjacent to x and y. So we may assume $n \geq 3$.

If q_{n-1} has only q_n as a neighbour on P, then $G[V(P) \cup V(Q)]$ contains P_5 or the claw as an induced subgraph. If q_{n-1} has exactly two neighbours on P and these neighbours are adjacent, then $G[V(P) \cup V(Q)]$ contains the bull as an induced subgraph. In the remaining case, q_{n-1} has two non-adjacent neighbours on P, which means that $G[V(P) \cup V(Q)]$ contains the claw as an induced subgraph. This completes the proof. ◀

Our proof will focus on showing that if a connected graph G contains the claw as a pivot-minor then it contains a graph from $\{3P_1, W_4, \overline{BW_3}\}$ as an induced subgraph. We will do this by induction on $|V(G)|$. It is easy to see that pivoting any edge of a claw again yields a claw. Therefore, the above claim holds in the case when $|V(G)| \leq 4$, and so we may assume that $|V(G)| \geq 5$. If G has a pivot-minor isomorphic to the claw, then by Lemma 9, there is a vertex $v \in V(G)$ such that $G - v$ or G/v contains a pivot-minor isomorphic to the claw for some neighbour w of v. By the induction hypothesis, we may assume that G/v contains the claw as a pivot-minor. In Lemmas 12, 13 and 14, we will show that if G/v contains an induced subgraph isomorphic to $3P_1$, W_4 or $\overline{BW_3}$, then G contains an induced subgraph in $\{3P_1, W_4, \overline{BW_3}\}$; these lemmas will form the main steps in our induction. We start by proving Lemma 11, which deals with some special cases, but first we define some notation that we will use throughout the next four lemmas.

Given a graph G containing an edge vw, we define the following sets: $S_1 := N_G(v) \setminus \{w\}$, $S_2 := N_G(v) \cap N_G(w)$, $S_3 := N_G(w) \setminus \{v\}$, and $S_4 := V(G) \setminus (N_G(v) \cup N_G(w))$ (see Figure 5 for an illustration). Note that the adjacency relations between $S_1 \cup S_2 \cup S_3$ and S_4 are not changed when pivoting on vw, and that the adjacency relations between two vertices in the same set S_i in G are not changed when pivoting on vw.

▶ **Lemma 11.** *Let vw be an edge of a graph G. If $G \wedge vw$ satisfies at least one of the following conditions, then G contains an induced subgraph isomorphic to $3P_1$ or W_4.*

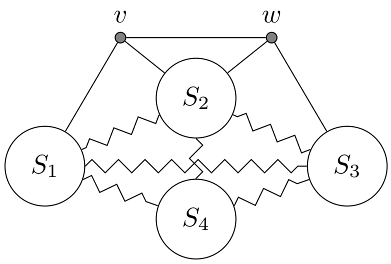

Figure 5 The sets S_1, S_2, S_3, S_4 in G.

(i) *There exist $v_1 \in S_1$, $v_2 \in S_2$ and $v_3 \in S_3$ such that $\{v_1, v_2, v_3\}$ is a clique or an independent set in $G \wedge vw$.*

(ii) *There exist distinct $v_1, v_2 \in S_2$ and $v_3 \in S_1 \cup S_3$ such that v_1 and v_2 are non-adjacent and v_3 is complete or anti-complete to $\{v_1, v_2\}$ in $G \wedge vw$.*

(iii) *There exists an induced path $v_1v_2v_3$ in $G \wedge vw$ and a vertex $v_4 \in S_4$ such that v_4 is adjacent to v_1, v_2, and v_3 in $G \wedge vw$ and $\{v_1, v_2, v_3\} \subseteq S_1 \cup S_2$ or $\{v_1, v_2, v_3\} \subseteq S_2 \cup S_3$.*

(iv) *$S_1 \cup S_4$ or $S_3 \cup S_4$ is not a clique in $G \wedge vw$.*

(v) *$G \wedge vw$ has an induced cycle C of length 4 such that $V(C) \subseteq S_1 \cup S_2 \cup \{w\}$ or $V(C) \subseteq S_2 \cup S_3 \cup \{w\}$.*

Proof. (i), (ii), (iii) are trivial. For (iv), by symmetry, assume that $S_1 \cup S_4$ is not a clique in $G \wedge vw$. Note that $(G \wedge vw)[S_1 \cup S_4] = G[S_1 \cup S_4]$. Two non-adjacent vertices in $S_1 \cup S_4$ and w form an independent set of size 3 in G.

For (v), we may assume that both S_1 and S_3 are cliques by (iv) and so $|V(C) \cap S_1| \leq 2$ and $|V(C) \cap S_3| \leq 2$. We may assume that $|V(C) \cap S_1|$, $|V(C) \cap S_3| \neq 1$ by (ii). We may assume that $|V(C) \cap S_1|$, $|V(C) \cap S_3| \neq 2$ because otherwise $V(C)$ induces C_4 dominated by v or w in G, inducing W_4. Therefore $V(C) \subseteq S_2 \cup \{w\}$ and so $V(C) \cup \{v\}$ induces W_4 in G. ◀

▶ **Lemma 12.** *Let vw be an edge of a graph G. If $(G \wedge vw) - v$ contains $3P_1$ as an induced subgraph, then G contains $3P_1$ or W_4 as an induced subgraph.*

Proof. Let T be an independent set of size 3 in $(G \wedge vw) - v$. By Lemma 11(iv), we may assume that $S_1 \cup S_4$ and $S_3 \cup S_4$ are cliques in $G \wedge vw$. Then $w \notin T$ because w is complete to $S_1 \cup S_2$ in $(G \wedge vw) - v$ and so $T \cap S_2 \neq \emptyset$.

If $|T \cap S_2| = 1$, then $|T \cap S_1| = |T \cap S_3| = 1$ and by Lemma 11(i), we are done in this case. If $T \subseteq S_2 \cup S_4$, then T is independent in G. So we may assume that $|T \cap S_2| = 2$ and $|T \cap (S_1 \cup S_3)| = 1$ and the proof is completed by Lemma 11(ii). ◀

▶ **Lemma 13.** *Let vw be an edge of a graph G. If $G \wedge vw$ contains W_4 as an induced subgraph, then G contains $3P_1$, W_4 or $\overline{BW_3}$ as an induced subgraph.*

Proof. We proceed by the induction on $|V(G)|$. By the induction hypothesis, we may assume that $G \wedge vw = W_4$ or $(G \wedge vw) - v = W_4$ or $(G \wedge vw) - v - w = W_4$. Let z be the centre of W_4 and let $v_1v_2v_3v_4$ be the cycle in W_4.

Suppose that G has no induced subgraph isomorphic to $3P_1$ or W_4. It is trivial to see that $G \wedge vw \neq W_4$. By Lemma 11(iv), both $S_1 \cup S_4$ and $S_3 \cup S_4$ are cliques in $G \wedge vw$.

If $(G \wedge vw) - v = W_4$, then $z \neq w$ by Lemma 11(v). We may assume $v_1 = w$. Then v_4zv_2 is an induced path in $S_1 \cup S_2$ in $G \wedge vw$ and so $v_3 \notin S_4$ by Lemma 11(iii). Thus, $v_3 \in S_3$. By Lemma 11(ii), $v_4 \in S_1$ or $v_2 \in S_1$. As S_1 is a clique, we may assume that $v_4 \in S_1$ and $v_2 \in S_2$, contradicting to Lemma 11(i) because $z \in S_1 \cup S_2$.

Therefore $(G \wedge vw) - v - w = W_4$. We may assume that $G \wedge vw$ is non-isomorphic to G and therefore $S_1 \cup S_3 \neq \emptyset$.

If $v_1, v_2, v_3, v_4 \in S_2 \cup S_4$, then by the above assumption, we may assume that $z \in S_1$. By Lemma 11(ii), S_2 is a clique. So we may assume that $v_1, v_2 \in S_2$ and $v_3, v_4 \in S_4$ because S_4 is also a clique. In this case G is isomorphic to $\overline{BW_3}$. Therefore we may assume that $\{v_1, v_2, v_3, v_4\} \cap S_1 \neq \emptyset$ by the symmetry of S_1 and S_3.

If $z \in S_4$, then by Lemma 11(iii), $|S_1| + |S_2| \leq 2$ and $|S_2| + |S_3| \leq 2$. Then $|S_2| \leq 1$ because $S_1 \neq \emptyset$. If $|S_1 \cup S_2 \cup S_3| = 4$, then $|S_1| + |S_2| + |S_3| = 4$ and so $|S_1| = |S_3| = 2$ and $S_2 = \emptyset$. As S_1 and S_3 are cliques, we may assume $v_1, v_2 \in S_1$ and $v_3, v_4 \in S_3$. Then $G - v - w$ is isomorphic to W_4, a contradiction. If $|S_1 \cup S_2 \cup S_3| = 3$, then we may assume that $v_1 \in S_4$. Since $S_1 \cup S_4$ and $S_3 \cup S_4$ are cliques, $|S_1| \leq 1$ and $|S_3| \leq 1$. Therefore $|S_1| = |S_2| = |S_3| = 1$. Then G isomorphic to $\overline{BW_3}$. So we may assume that $z \notin S_4$.

If $S_4 \neq \emptyset$, then we may assume that $v_1 \in S_4$. Since $S_1 \cup S_4$ and $S_3 \cup S_4$ are cliques, $v_3 \in S_2$. By symmetry between v_2 and v_4, we may assume that $v_2 \in S_1$. Then $z \in S_2$ by Lemma 11(i) considering $\{v_1, v_2, z\}$ and $\{v_2, v_3, z\}$. Since $S_1 \cup S_4$ is a clique, $v_4 \in S_2 \cup S_3$. By Lemma 11(iii), $v_4 \notin S_2$ and so $v_4 \in S_3$. Then G is isomorphic to $\overline{BW_3}$. So we may assume that $S_4 = \emptyset$.

By Lemma 11(v), $\{v_1, v_2, v_3, v_4\} \cap S_3 \neq \emptyset$.

If $z \in S_2$, then we may assume $v_1 \in S_1$. Then by Lemma 11(i), $v_2, v_4 \notin S_3$ and $v_3 \in S_3$. By Lemma 11(i), $v_2, v_4 \notin S_1$ and so $v_2, v_4 \in S_2$, contradicting Lemma 11(ii). So we may assume that $z \in S_1 \cup S_3$. By symmetry between S_1 and S_3, we may assume that $z \in S_1$.

If $v_1 \in S_2$, then by Lemma 11(i), $v_2, v_4 \notin S_3$ and so $v_3 \in S_3$. By Lemma 11(i) again, $v_2, v_4 \notin S_2$ and therefore $v_2, v_4 \in S_1$, contradicting the assumption that S_1 is a clique. So we may assume that $v_1, v_2, v_3, v_4 \notin S_2$. Since S_1 and S_3 are cliques, we may assume that $v_1, v_2 \in S_1$ and $v_3, v_4 \in S_3$. Then G is isomorphic to $\overline{BW_3}$. ◀

The proof of the following lemma is presented in the appendix.

▶ **Lemma 14.** *Let G be a graph containing an edge vw. If $G \wedge vw$ contains $\overline{BW_3}$ as an induced subgraph, then G contains $3P_1$, W_4 or $\overline{BW_3}$ as an induced subgraph.*

We are now ready to prove the main result of this section.

▶ **Theorem 15.** *A graph G contains the claw as a pivot-minor if and only if G contains a graph from $\{\text{claw}, P_5, \text{bull}, W_4, \overline{BW_3}\}$ as an induced subgraph.*

Proof. We first prove the "if" part. Suppose G contains a graph $H \in \{\text{claw}, P_5, \text{bull}, W_4, \overline{BW_3}\}$ as an induced subgraph. If H is the claw, then trivially G contains the claw as a pivot-minor. If $H = P_5$, then by pivoting the edge between the second and third vertex we obtain a graph which contains the claw as an induced subgraph. If H is the bull, then by pivoting an edge incident to the vertex of degree 2 we obtain a graph which contains the claw as an induced subgraph. If $H = W_4$, then by pivoting an edge incident to the vertex of degree 4 we obtain a graph which contains the claw as an induced subgraph. If $H = \overline{BW_3}$, then let $U_1 = \{a_1, a_2, a_3\}$ and $U_2 = \{b_1, b_2, b_3, b_4\}$ be the two cliques of H and $a_i b_i \in E(H)$ for $i = 1, 2, 3$. By pivoting an edge $a_1 b_1$, we obtain a subgraph induced by $\{a_2, a_3, b_2, b_3, b_4\}$ that is isomorphic to W_4.

Next, we prove the "only if" part. Suppose G contains the claw as a pivot-minor. We use induction on $|V(G)| = n$ to prove that G contains a graph from $\{\text{claw}, P_5, \text{bull}, W_4, \overline{BW_3}\}$ as an induced subgraph. We may assume that $n \geq 5$.

We may assume that G is connected because the claw is connected and the edge pivot preserves connectedness of each component of a graph. As $n \geq 5 > |V(\text{claw})|$, Lemma 9 implies that there is a vertex $v \in V(G)$ such that $G - v$ or $(G \wedge vw) - v$, for some neighbour w of v, contains the claw as a pivot-minor.

If $G - v$ contains the claw as a pivot-minor, then by the induction hypothesis, $G - v$ contains an induced subgraph in $\{\text{claw}, P_5, \text{bull}, W_4, \overline{BW_3}\}$, hence so does G. Now we assume that $(G \wedge vw) - v$,

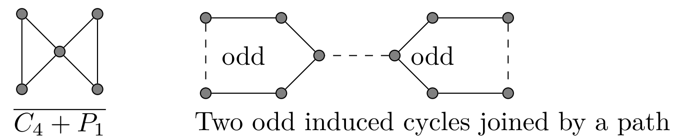

Figure 6 An infinite family of minimal forbidden induced subgraphs for the class of K_4-pivot-minor-free graphs.

for some neighbour w of v, contains the claw as a pivot-minor. By the induction hypothesis, $(G \wedge vw) - v$ contains the claw, P_5, the bull, W_4, or $\overline{BW_3}$ as an induced subgraph, which implies that $(G \wedge vw) - v$ contains $3P_1$, W_4 or $\overline{BW_3}$ as an induced subgraph. Applying Lemmas 12, 13, and 14, we find that G contains an induced graph in $\{3P_1, W_4, \overline{BW_3}\}$. If G contains $3P_1$ as an induced subgraph, then Lemma 10 implies that G contains a graph from $\{\text{claw}, P_5, \text{bull}\}$ as an induced subgraph. This completes the proof. ◀

In the same way as for Corollary 7 we can deduce the following result from Theorem 15.

▶ **Corollary 16.** *There is a polynomial-time algorithm for* claw-Pivot-Minor *that gives a* claw-*pivot-minor-sequence (if one exists).*

4 Conclusions

Question 7 in [19] asked whether the problems of deciding if a given graph G contains a fixed graph H as a pivot-minor or as a vertex-minor, respectively, can be solved in polynomial time. In our paper we showed some partial progress towards this question. We proved that Pivot-Minor is NP-complete and provided polynomial-time algorithms for H-Pivot-Minor for every connected graph H on at most four vertices except for $H = K_4$.

As future work we aim to determine the complexity of H-Pivot-Minor for every (not necessarily connected) graph H on at most four vertices including $H = K_4$. We also do not know if there exists a graph H for which H-Pivot-Minor is NP-complete.

At the moment our only technique for proving polynomial-time solvability is to find the structural characterizations in terms of forbidden induced subgraphs. For $H = K_4$, this turns out to be non-trivial. The set $\mathcal{F}_{K_4}$ has infinite size, as it contains the following infinite class of graphs. Let $\mathcal{H}$ be the class of graphs obtained from taking two odd cycles and adding a path of length 0 or more connecting one vertex of one odd cycle with one vertex of the other odd cycle (see Figure 6). Note that no graph of $\mathcal{H}$ is an induced subgraph of another graph of $\mathcal{H}$. Moreover, every graph in $\mathcal{H}$ contains the graph $\overline{C_4 + P_1}$ as a pivot-minor, while $\overline{C_4 + P_1}$ contains K_4 as a pivot-minor. Finally, the graphs in $\mathcal{H}$ can be shown to be minimal, as any graph obtained from identifying a vertex of some odd cycle with an end-vertex of a path that ends in the centre of a claw does not contain K_4 as a pivot-minor.

We note that a proof for the Minor Recognition conjecture [8] for binary matroids would yield another technique to obtain computational complexity results for pivot-minors. In particular, if this conjecture is true, then for every graph H the H-Pivot-Minor problem is polynomial-time solvable for bipartite graphs. This follows from Lemma 1, which implies that a bipartite connected graph H is a pivot-minor of a bipartite graph G if and only if for binary matroids M and N that have G and H as fundamental graphs, respectively, N or the dual of N is a minor of M (if H is not connected, then we try all possible ways of making duals per component of H).

It would also be interesting to perform a similar complexity study with respect to vertex-minors. As far as we are aware the complexity of deciding whether a graph G contains a graph H as a vertex-minor is not known even if both G and H are part of the input.

References

1 André Bouchet. Circle graph obstructions. *J. Combin. Theory Ser. B*, 60(1):107–144, 1994.
2 André Bouchet and Alain Duchamp. Representability of Δ-matroids over GF(2). *Linear Algebra Appl.*, 146:67–78, 1991.
3 A. E. Brouwer and H. J. Veldman. Contractibility and NP-completeness. *J. Graph Theory*, 11(1):71–79, 1987.
4 D. G. Corneil, H. Lerchs, and L. Stewart Burlingham. Complement reducible graphs. *Discrete Appl. Math.*, 3(3):163–174, 1981.
5 Bruno Courcelle and Sang-il Oum. Vertex-minors, monadic second-order logic, and a conjecture by Seese. *J. Combin. Theory Ser. B*, 97(1):91–126, 2007.
6 M. R. Fellows, J. Kratochví l, M. Middendorf, and F. Pfeiffer. The complexity of induced minors and related problems. *Algorithmica*, 13(3):266–282, 1995.
7 M. R. Garey, D. S. Johnson, and R. Endre Tarjan. The planar Hamiltonian circuit problem is NP-complete. *SIAM J. Comput.*, 5(4):704–714, 1976.
8 Jim Geelen, Bert Gerards, and Geoff Whittle. Towards a structure theory for matrices and matroids. In *International Congress of Mathematicians. Vol. III*, pages 827–842. Eur. Math. Soc., Zürich, 2006.
9 Jim Geelen and Sang-il Oum. Circle graph obstructions under pivoting. *J. Graph Theory*, 61(1):1–11, 2009.
10 Martin Grohe, Ken-ichi Kawarabayashi, Dániel Marx, and Paul Wollan. Finding topological subgraphs is fixed-parameter tractable. In *STOC'11—Proceedings of the 43rd ACM Symposium on Theory of Computing*, pages 479–488. ACM, New York, 2011.
11 O-joung Kwon and Sang-il Oum. Graphs of small rank-width are pivot-minors of graphs of small tree-width. *Discrete Appl. Math.*, 168:108–118, 2014.
12 Benjamin Lévêque, David Y. Lin, Frédéric Maffray, and Nicolas Trotignon. Detecting induced subgraphs. *Discrete Appl. Math.*, 157(17):3540–3551, 2009.
13 Benjamin Lévêque, Frédéric Maffray, and Nicolas Trotignon. On graphs with no induced subdivision of K_4. *J. Combin. Theory Ser. B*, 102(4):924–947, 2012.
14 Jiří Matoušek and Robin Thomas. On the complexity of finding iso- and other morphisms for partial k-trees. *Discrete Math.*, 108(1-3):343–364, 1992.
15 Ross M. Mcconnell, Kurt Mehlhorn, Stefan Näher, and Pascal Schweitzer. Survey: Certifying algorithms. *Comput. Sci. Rev.*, 5(2):119–161, 2011.
16 Sang-il Oum. Rank-width and vertex-minors. *J. Combin. Theory Ser. B*, 95(1):79–100, 2005.
17 Sang-il Oum. Rank-width and well-quasi-ordering. *SIAM J. Discrete Math.*, 22(2):666–682, 2008.
18 Sang-il Oum. Excluding a bipartite circle graph from line graphs. *J. Graph Theory*, 60(3):183–203, 2009.
19 Sang-il Oum. Rank-width: Algorithmic and structural results. *Discrete Appl. Math.*, (in press).
20 Sang-il Oum and Paul Seymour. Approximating clique-width and branch-width. *J. Combin. Theory Ser. B*, 96(4):514–528, 2006.
21 James Oxley. *Matroid theory*, volume 21 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, second edition, 2011.
22 Neil Robertson and P. D. Seymour. Graph minors. XIII. The disjoint paths problem. *J. Combin. Theory Ser. B*, 63(1):65–110, 1995.
23 Pim van 't Hof, Marcin Kamiński, Daniël Paulusma, Stefan Szeider, and Dimitrios M. Thilikos. On graph contractions and induced minors. *Discrete Appl. Math.*, 160(6):799–809, 2012.

A Appendix

We prove Lemma 8.

▶ **Lemma 8 (restated).** *Let v, x, y be distinct vertices of a graph G. If x, y are adjacent, then $(G \wedge xy) - v$ is pivot-equivalent to $G - v$ and $(G \wedge xy)/v$ is pivot-equivalent to G/v.*

Proof. The first statement is trivial. We prove the second statement. As a pivot operation does not change the fact that v has a neighbour in G, v has a neighbour in G if and only if it has a neighbour in $G \wedge xy$. If v is isolated in G, then the result follows from the first statement. We may assume v has a neighbour in G and in $G \wedge xy$. If v is non-adjacent to x and y in G, then for a neighbour z of v, we have that $(G\wedge xy)/v = (G\wedge xy\wedge zv)-v = (G\wedge zv-v)\wedge xy = (G/v)\wedge xy$. Thus $(G \wedge xy)/v$ is pivot-equivalent to G/v. If v is adjacent to x, then $(G \wedge xy)/v$ is pivot-equivalent to $(G \wedge xy \wedge yv) - v = (G \wedge xv) - v = G/v$. ◀

Here, we give the proof of Lemma 14.

▶ **Lemma 14 (restated).** *Let G be a graph containing an edge vw. If $G \wedge vw$ contains $\overline{BW_3}$ as an induced subgraph, then G contains $3P_1$, W_4 or $\overline{BW_3}$ as an induced subgraph.*

Proof. Again, we proceed by the induction on $|V(G)|$. By the induction hypothesis, we may assume that $G \wedge vw = \overline{BW_3}$ or $(G \wedge vw) - v = \overline{BW_3}$ or $(G \wedge vw) - v - w = \overline{BW_3}$. Let $U_1 = \{a_1, a_2, a_3\}$ and $U_2 = \{b_1, b_2, b_3, b_4\}$ be two disjoint cliques in $\overline{BW_3}$ such that $\{a_1b_1, a_2b_2, a_3b_3\}$ is a matching.

Suppose G has no induced subgraph isomorphic to $3P_1$, W_4, or $\overline{BW_3}$. By Lemma 11(iv), $S_1 \cup S_4$ and $S_3 \cup S_4$ are cliques of $G \wedge vw$. It is trivial that $G \wedge vw \neq \overline{BW_3}$.

We first claim that

(A) if $a_1, a_2 \in S_2$ and $b_1, b_2 \neq w$, then $a_3 \notin S_1 \cup S_3$.
Suppose not. We may assume that $a_3 \in S_1$. Then $b_1, b_2 \notin S_1 \cup S_4$ because $S_1 \cup S_4$ is a clique. Then a_1, a_2, b_2, b_1 is a cycle in $S_2 \cup S_3$, contradicting Lemma 11(v). Similarly

(B) if $b_1, b_2 \in S_2$ and $a_1, a_2 \neq w$, then $b_3, b_4 \notin S_1 \cup S_3$.
We also claim that

(C) if $a_1, a_2 \in S_4$ and $b_1, b_2 \in S_i$ for some $i = 1, 2, 3$, then $a_3 \in S_i \cup S_4 \cup \{w\}$,
(D) if $b_1, b_2 \in S_4$ and $a_1, a_2 \in S_i$ for some $i = 1, 2, 3$, then $b_3, b_4 \in S_i \cup S_4 \cup \{w\}$.
To see (C), if not, then $\{a_1, a_2, b_1, b_2\}$ induces a cycle of length 4 dominated by a_3 in G. A similar argument shows (D) as well. Here are other useful claims whose proofs are trivial.

(E) If $a_1 \in S_4$, $a_2 \in S_1$, $b_1, b_4 \in S_2$, then $a_3 \notin S_3$.
(F) If $b_1 \in S_4$, $a_1, a_2 \in S_2$, and $b_3 \in S_1$, then $b_4 \notin S_3$.
(G) If $b_1 \in S_4$, $a_1, a_2 \in S_2$, and $b_3 \in S_3$, then $b_4 \notin S_1$.
(H) If $a_1, a_2 \in S_1$ and $b_1, b_2 \in S_3$, then $a_3, b_3, b_4 \notin S_1 \cup S_3$.

Let us assume that $(G \wedge vw) - v - w = \overline{BW_3}$. We may assume that $S_1 \cup S_3 \neq \emptyset$ because otherwise $G \wedge vw = G$.

Suppose $a_1 \in S_4$. Then $b_2, b_3, b_4 \in S_2$ by Lemma 11(iv). By (B), $b_1 \notin S_1 \cup S_3$. By symmetry, we may assume that $S_1 \neq \emptyset$ and $a_2 \in S_1$. By Lemma 11(v), $a_3 \in S_3 \cup S_4$. By (E), $a_3 \notin S_3$ and so $a_3 \in S_4$, contradicting (C). Thus we may assume that $a_1, a_2, a_3 \notin S_4$ by symmetry.

Suppose $b_1 \in S_4$. Then $a_2, a_3 \in S_2$ because $S_1 \cup S_4$, $S_3 \cup S_4$ are cliques. Then by (A), $a_1 \in S_2$. If $b_2 \in S_1$, then $b_3 \notin S_1 \cup S_2$ by Lemma 11(v) and $b_3 \notin S_4$ by (D). By Lemma 11(i) for $\{b_2, b_3, b_4\}$, $b_4 \notin S_2$. By (F), $b_4 \notin S_3$. By (G), $b_4 \notin S_1$ and therefore $b_4 \in S_4$. Then $\{a_1, b_1, b_2, b_3, b_4\}$ induces W_4 in G, a contradiction. Thus we may assume that $b_2, b_3 \notin S_1$ and $b_2, b_3 \notin S_3$ by symmetry. So

$b_2, b_3 \in S_2 \cup S_4$. We may assume that $b_4 \in S_1$ as $S_1 \cup S_3 \neq \emptyset$. By Lemma 11(v) with $a_2a_3b_3b_2$, we may assume $b_2 \in S_4$. This contradicts (D). Therefore we may assume that $b_1, b_2, b_3 \notin S_4$ by symmetry.

Suppose $b_4 \in S_4$. Since $S_1 \cup S_4$ and $S_3 \cup S_4$ are cliques, it follows that $a_1, a_2, a_3 \in S_2$. We may assume that $b_1 \in S_1$ because $S_1 \cup S_3 \neq \emptyset$. By Lemma 11(v), $b_2, b_3 \in S_3$. Then $\{a_2, a_3, b_3, b_2\} \subseteq S_2 \cup S_3$ induces a cycle in $G \wedge vw$, contradicting Lemma 11(v). Therefore we may assume that $S_4 = \emptyset$.

We claim that S_2 is a clique. Suppose not. We may assume that $a_1 \in S_2$ by symmetry and a_1 has a non-neighbour in S_2. If $b_2 \in S_2$, then by Lemma 11(ii), $a_2, b_1 \in S_2$. But this contradicts Lemma 11(v). So we may assume that $b_2, b_3 \notin S_2$. Furthermore by symmetry we may assume that if a_2 or a_3 is in S_2, then $b_1 \notin S_2$. If $b_4 \in S_2$, then by Lemma 11(ii), $b_1 \in S_2$. We may assume that $b_2 \in S_1$, as $S_1 \cup S_3 \neq \emptyset$ and therefore $a_2, a_3 \notin S_2$. As $a_2a_3b_3b_2$ is a cycle in $S_1 \cup S_3$ in $G \wedge vw$, we may assume that $a_2, a_3 \in S_3$ and $b_2, b_3 \in S_1$. In this case, $\{a_1, a_2, a_3, b_1, b_2, b_3, b_4\}$ induces $\overline{BW_3}$ in G. Therefore we may assume that S_2 is a clique.

Then $V(\overline{BW_3})$ is partitioned into 3 cliques and there exist distinct $i, j \in \{1, 2, 3\}$ such that a_i, a_j are in one clique and b_i, b_j are in one clique. We may assume $i = 1$ and $j = 2$. By Lemma 11(v), we may assume that $a_1, a_2 \in S_1$ and $b_1, b_2 \in S_3$. By (H), $a_3, b_4 \in S_2$, contradicting to the assumption that S_2 is a clique. This completes the proof for the case that $(G \wedge vw) - v - w = \overline{BW_3}$.

Now let us consider the case that $(G \wedge vw) - v = \overline{BW_3}$. Note that w is adjacent to every vertex in S_1 and S_2 and non-adjacent to every vertex in S_3 and S_4 in $(G \wedge vw) - v$.

We claim that S_2 is a clique. Suppose not. If $w \in U_1$, then we may assume that $a_1 = w$ and $a_3, b_1 \in S_2$ by symmetry. By Lemma 11(ii), $b_3 \in S_4$. Then $a_2 \in S_2$ because $S_1 \cup S_4$ and $S_3 \cup S_4$ are cliques. By Lemma 11(ii), $b_2 \in S_4$. By (D), $b_4 \in S_2 \cup S_4$, contradicting $S_1 \cup S_3 \neq \emptyset$. If $w \in U_2$, then we may assume that $b_1 = w$. If $a_1 \in S_2$ and $b_3 \in S_2$ are non-adjacent, then by Lemma 11(ii), $a_3 \in S_4$. Then $b_2, b_3 \in S_2$ because $S_1 \cup S_4$ is a clique. Since $S_1 \cup S_3 \neq \emptyset$, we have $a_2 \in S_3$, contradicting Lemma 11(ii) because $a_1, b_2 \in S_2$ and $a_2 \in S_3$. This completes the claim.

If $w \in U_1$, then by symmetry we may assume that $a_1 = w$. Suppose $b_3 \in S_4$. Then, by Lemma 11(v), $a_2 \in S_2$ and so $a_3 \in S_2$, $b_1 \in S_1$ because S_2 is a clique. By (D), $b_2 \in S_3$. Since b_4 is not a neighbour of w, b_4 is in S_3 or S_4. By (F), $b_4 \in S_4$. Then a_3, b_2, b_4, b_1 and b_2 induces W_4 in G. Thus, $b_2, b_3 \notin S_4$, which implies that $b_2, b_3 \in S_3$. If $b_4 \in S_4$, then $a_2, a_3 \in S_2$ because $S_1 \cup S_4$ and $S_3 \cup S_4$ are cliques. This contradicts Lemma 11(v). So we may assume that $b_4 \in S_3$. By Lemma 11(v), $a_2, a_3 \in S_1$ and $b_1 \in S_2$, a contradiction to (H). Therefore, we may assume that $w \in U_2$.

Suppose $w = b_1$. If $S_4 \neq \emptyset$, then we may assume that $a_2 \in S_4$ by symmetry. Since $S_1 \cup S_4$ is a clique, we have $b_3, b_4 \in S_2$. Then $a_1 \in S_1$ and $b_2 \in S_2$. By (C), $a_3 \notin S_4$ and so $a_3 \in S_3$. This contradicts (E). If $S_4 = \emptyset$, then $a_2, a_3 \in S_3$. Then a_3, a_2, b_2, b_3 and b_4 induces W_4 whenever $b_2, b_3, b_4 \in S_1$ or $b_2, b_3, b_4 \in S_2$. Thus, we can conclude that $w \notin \{b_1, b_2, b_3\}$, which implies that $w = b_4$. If $S_4 \neq \emptyset$, then we may assume that $a_2 \in S_4$ by symmetry. Then $b_1, b_2, b_3 \in S_2$ because $S_1 \cup S_4$ and S_2 are cliques. Since $S_3 \neq \emptyset$, by symmetry, $a_1 \in S_3$. By (C), $a_3 \in S_3$, which contradicts Lemma 11(v). If $S_4 = \emptyset$, then $a_1, a_2, a_3 \in S_3$. By the pigeonhole principle, two of b_1, b_2, b_3 are in the same S_i for some $i = 1, 2$. This contradicts Lemma 11(v). ◀