# Graphs of small rank-width are pivot-minors of graphs of small tree-width

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(Joint work with Sang-il Oum)

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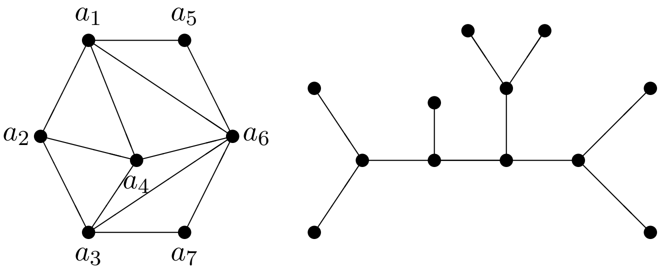
## Preliminaries

- For a graph $G$, $V(G)$ denote the vertex set of $G$ and $E(G)$ denote the edge set of $G$.
- A tree is subcubic if every non-leaf vertex has degree $3$.
- A tree is caterpillar if there is a path in the tree such that every vertex in the tree is incident with a vertex in that path.

## Rank-decomposition (05, Oum)

- A rank-decomposition $(T, L)$ of $G$ consists of
  a subcubic tree $T$,
  bijective function $L$ from $V(G)$ to leaves of $T$.



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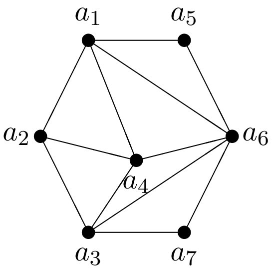


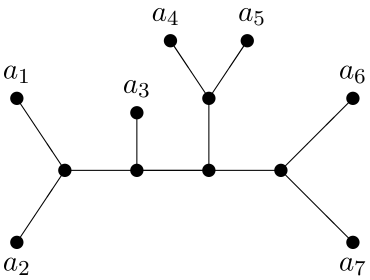

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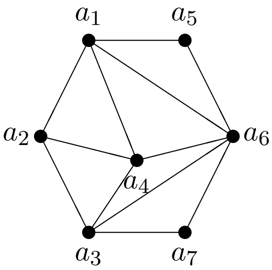
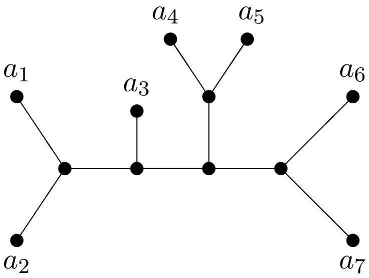
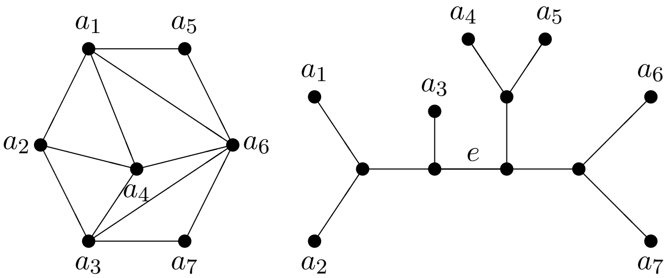
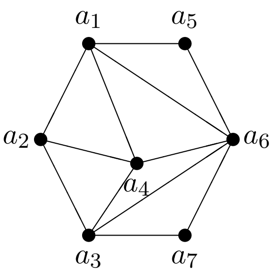
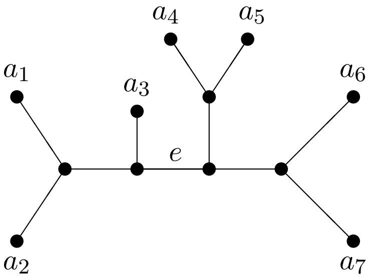




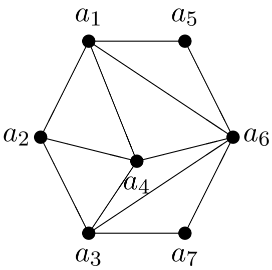

- Width of an edge of $T$: the rank of the matrix with the partition induced by the edge.

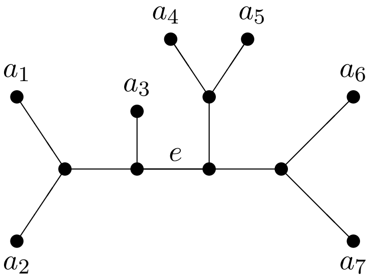



- Width of an edge of $T$: the rank of the matrix with the partition induced by the edge.

$$\text{Width of } e = \text{rank}\left(\begin{array}{cccc} & a_1 & a_2 & a_3 \\ \begin{array}{c} a_4 \\ a_5 \\ a_6 \\ a_7 \end{array} & \multicolumn{3}{c}{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}} \end{array}\right) = 3$$

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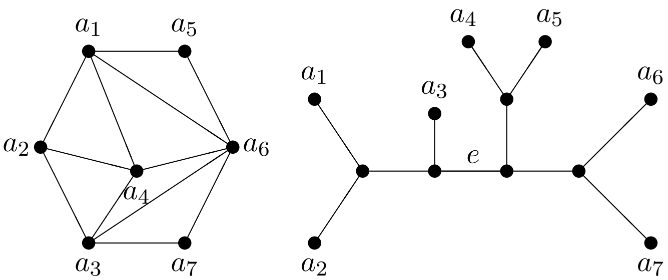
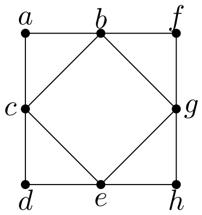


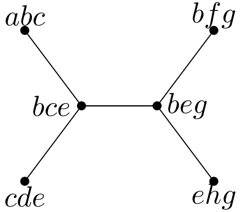

- Width of an edge of $T$: the rank of the matrix with the partition induced by the edge.
- Width of $(T, L)$: maximum width of all edges in $T$
- Rank-width of $G$: minimum width of all rank-decompositions of $G$
- If we restrict to use only caterpillar subcubic trees, then we call it the linear rank-width of $G$

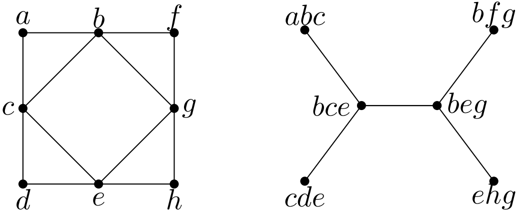
# Tree-decomposition (84, Robertson and Seymour)

- A tree-decomposition $(T, \{B_v\}_{v \in V(T)})$ of $G$ consists of
  a tree $T$,
  mapping from each vertex $v$ of $T$ to a subset $B_v$ of $V(G)$.
  and it satisfies following axioms.
  (1) Two vertices of an edge must be contained in a bag.
  (2) If $x$ is the path from $v$ to $w$ in $T$, then $B_v \cap B_w \subseteq B_x$.

$B_v$ is called a bag.



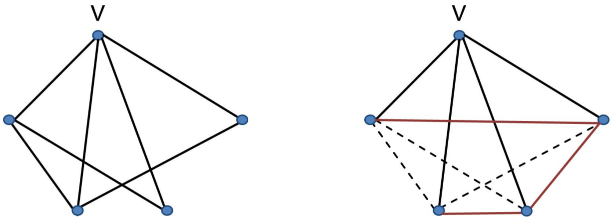



- Width of $(T, \{B_v\}_{v \in V(T)})$: $\max\{|B_v| - 1 : v \in V(T)\}$
- Tree-width of $G$: mimimum width of all tree-decompositions of $G$
- If we restrict to use only paths, we call it the path-width of $G$

# Local complementation and pivoting

We are interested in two operations.

- *Local complementation* on a vertex $v \in V$
  $G * v = (V, E\Delta\{xy : x, y \in \delta(v)\})$



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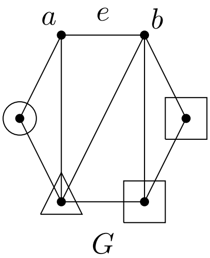

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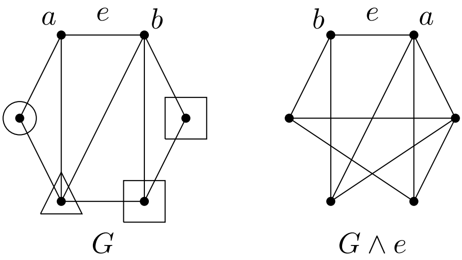

# Why these operations?

- These operations preserve the rank of the matrix induced by a partition of the graph.
- They also preserve the rank-width of the graph.

Definition(Oum, 05)

$H$ : a *vertex-minor* of a graph $G$
if $H$ is obtained from $G$ by applying a sequence of
local complementations and vertex deletions.

$H$ : a *pivot-minor* of a graph $G$
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- If $H$ is a vertex-minor or a pivot-minor of $G$,
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# Known results

## Rank-width and tree-width of some graph classes

| | rank-width | tree-width |
|---|---|---|
| tree | 1 | 1 |
| $n \times n$ grid | $n - 1$ (Jelinek 10) | $n$ (RS 91) |

**Theorem(Oum 08)**

For a graph $G$, $\mathrm{rw}(G) \leq \mathrm{tw}(G) + 1$.

In general, tree-width cannot be bounded by a function of rank-width.

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## Observation

We observe that $K_n$ is a vertex-minor of a path of length $3n$.

Question

If a graph has small rank-width,
then it can be a vertex-minor or a pivot-minor of a graph of small tree-width??

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# Main result

Theorem(K, Oum 12)

| | | |
|---|---|---|
| $G$ has rank-width $\leq k$ | $\Rightarrow$ | $G$ is a pivot-minor of a graph of tree-width $\leq 2k$ |
| $G$ has linear rank-width $\leq k$ | $\Rightarrow$ | $G$ is a pivot-minor of a graph of path-width $\leq k+1$ |

Given a graph $G$ and a rank-decomposition $(T, L)$ of width $k$, we explicitly construct a graph $H$, called a rank-expansion, such that

- tree-width of $H$ is at most $2k$
- $G$ is a pivot-minor of $H$
- $|V(H)| \leq (2k+1)|V(G)| - 6k$

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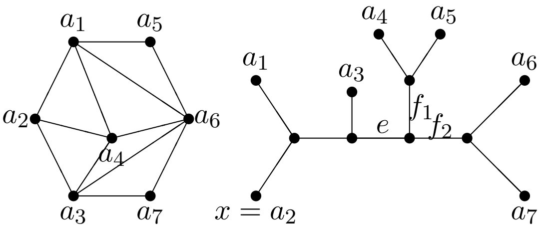

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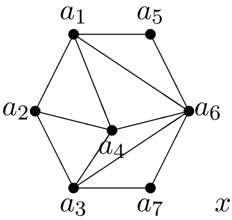


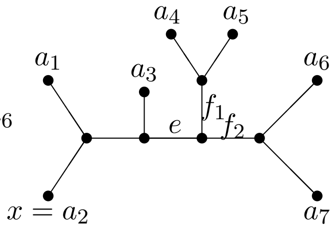


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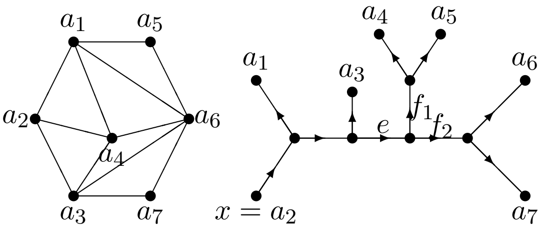


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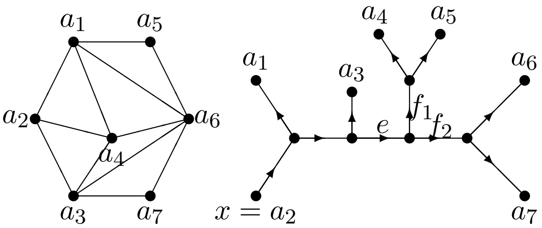


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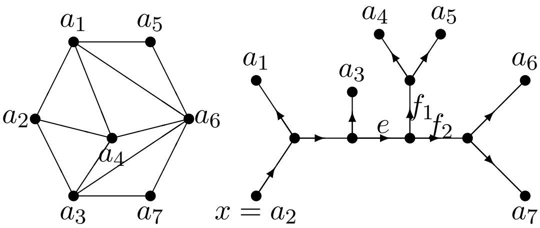


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$$A(G)[A_e, B_e] = \begin{array}{c} \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{array} \begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

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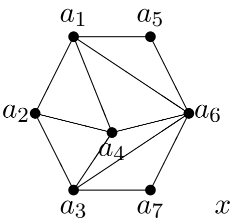


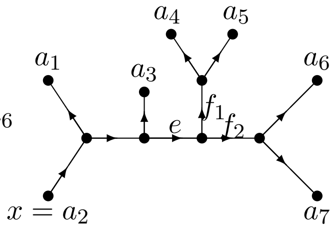


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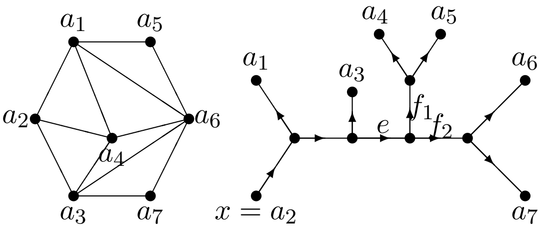


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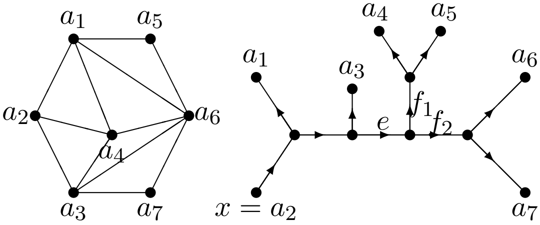
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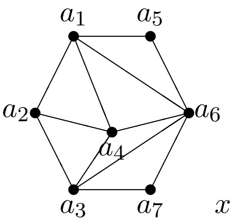
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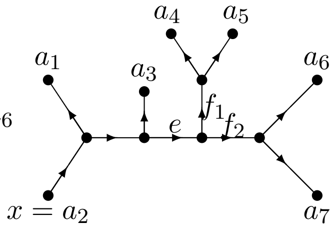
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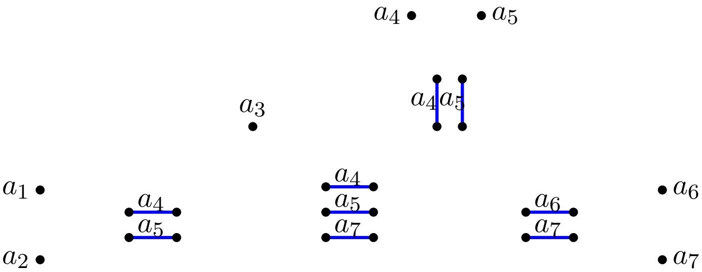
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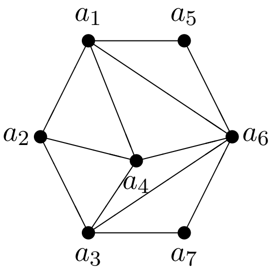
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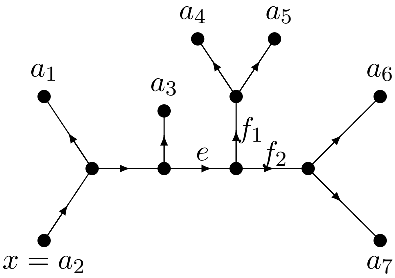
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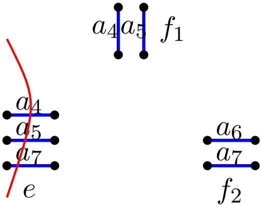
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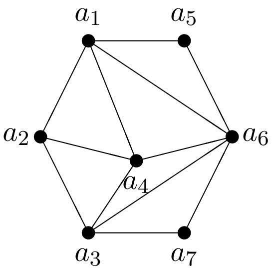
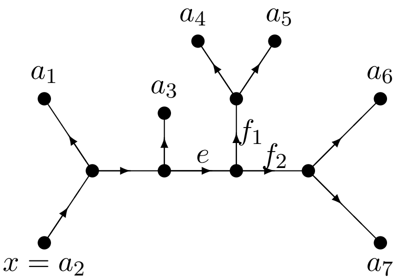
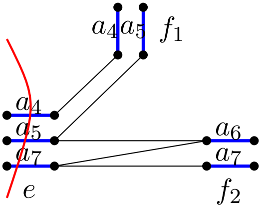
$a_7$
$f_2$

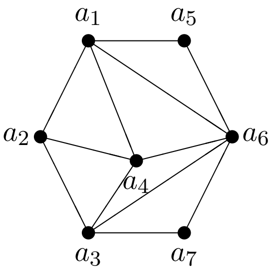


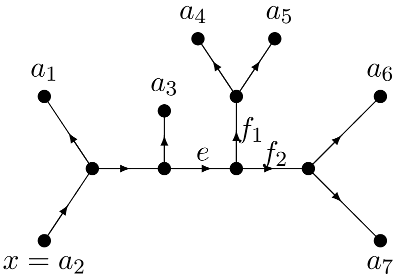



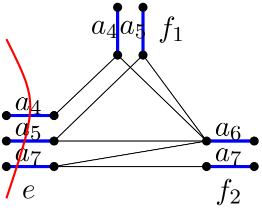




$$R_{a_6} = R_{a_5} + R_{a_7}$$

in the matrix corresponding to $e$

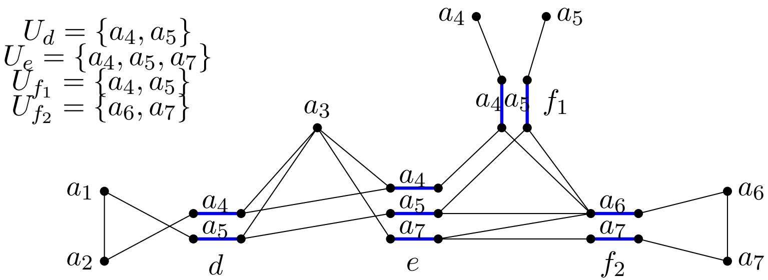



$a_4$ and $a_5$ are adjacent to $a_6$

# Sketch of proof

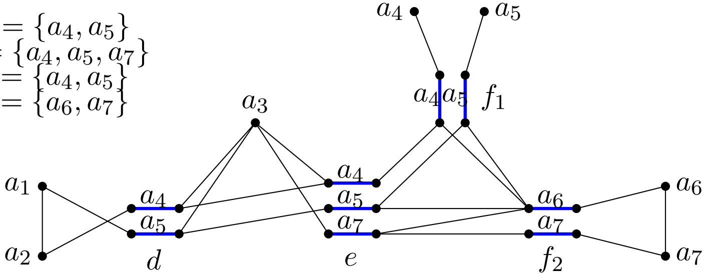



1. After pivoting every blue edges and delete them with vertices, we get the graph $G$.

2. The tree-width of this rank-expansion is at most $2k$.

# Sketch of proof

$U_d = \{a_4, a_5\}$
$U_e = \{a_4, a_5, a_7\}$
$U_{f_1} = \{a_4, a_5\}$
$U_{f_2} = \{a_6, a_7\}$

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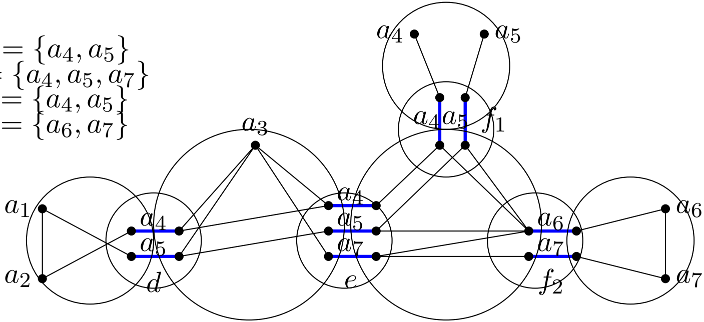


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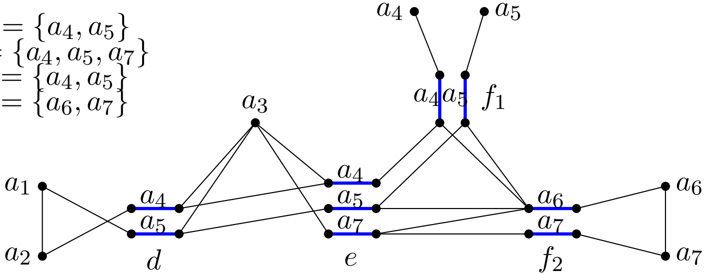


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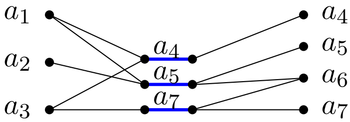
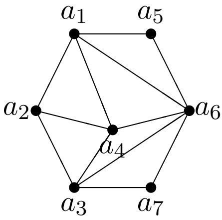
# Key lemma

## Lemma

For a given graph $G$, we make a graph as follows.
(left adjacency follows $A(G)$, and right adjacency follows basis)
Then we obtain the adjacency between partitions exactly same as the adjacency in the graph $G$ by pivoting matching edges(blue).




$$\begin{array}{c} \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{array} \begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

# Rank-width 1, linear rank-width 1

Theorem(K, Oum 12)

$G$ has rank-width $\leq 1$

# Rank-width 1, linear rank-width 1

Theorem(K, Oum 12)

$G$ has rank-width $\leq 1$ $\Leftarrow$ $G$ is a pivot-minor of a graph of tree-width $\leq 2$

# Rank-width 1, linear rank-width 1

Theorem(K, Oum 12)

$G$ has rank-width $\leq 1$

$\Leftrightarrow$ $G$ is a vertex-minor of a tree

# Rank-width 1, linear rank-width 1

Theorem(K, Oum 12)

$G$ has rank-width $\leq 1$

$\Leftrightarrow$ $G$ is a vertex-minor of a tree

$G$ has linear rank-width $\leq 1$ $\Leftrightarrow$ $G$ is a vertex-minor of a path

# Rank-width 1, linear rank-width 1

Theorem(K, Oum 12)

| | | |
|---|---|---|
| $G$ has rank-width $\leq 1$ | | |
| | $\Leftrightarrow$ | $G$ is a vertex-minor of a tree |
| $G$ has linear rank-width $\leq 1$ | $\Leftrightarrow$ | $G$ is a vertex-minor of a path |

When $G$ is bipartite,

| | | |
|---|---|---|
| $G$ has rank-width $\leq 1$ | $\Leftrightarrow$ | $G$ is a pivot-minor of a tree |
| $G$ has linear rank-width $\leq 1$ | $\Leftrightarrow$ | $G$ is a pivot-minor of a path |

To make a rank-expansion of a given graph, we need a rank-decomposition of the graph.

Question. Can we construct a graph satisfying the theorem without rank-decomposition?

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# Thank you.