A single-exponential FPT algorithm for Distance-Hereditary Vertex Deletion

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Joint work with
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Distance-hereditary graphs

A graph G is **distance-hereditary** if for every connected induced subgraph H of G and $u, v \in V(H)$, the distance between u and v in H is the same as the distance in G.
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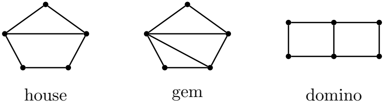
Equivalent characterizations (Bandelt, Mulder, 82):

- Every induced path is a shortest path.
- Every cycle of length at least 5 contains a pair of crossing chords.
- Graphs can be constructed from a vertex by a sequence of adding twins or adding leaf vertices.
- Having no obstructions : Induced cycles of length at least 5 +



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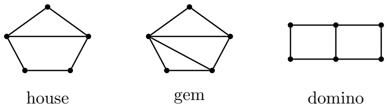

- Rank-width $\leqslant 1$ (Oum, 05).

Distance-Hereditary Vertex Deletion

Input : A graph G, an integer k
Parameter : k
Question : $\exists$? $S \subseteq V(G)$ with $|S| \leqslant k$ such that $G - S$ is distance-hereditary?

This can be seen as a counterpart of Feedback Vertex Set.

Rank-width w Vertex Deletion

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So, what is rank-width?

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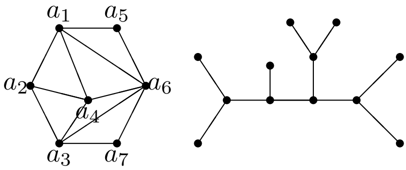
- A rank-decomposition (T, L) of G consists of a subcubic tree T, and a bijective function L from $V(G)$ to the leaves of T.



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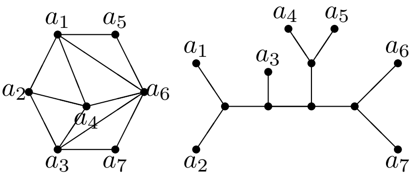

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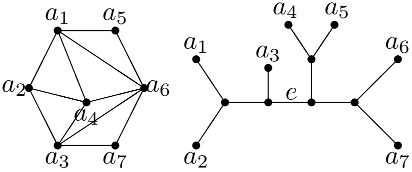

- Width of an edge of T: the rank of the matrix with the partition induced by the edge.

$$\text{Width of } e = \text{rank}\left(\begin{array}{c} \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{array}\begin{array}{c} \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \end{array}\right) = 3$$

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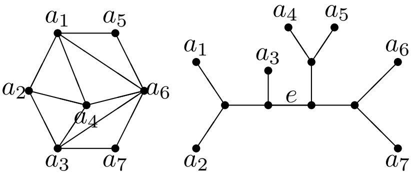

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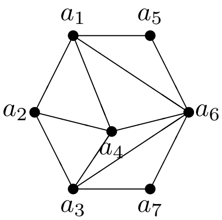
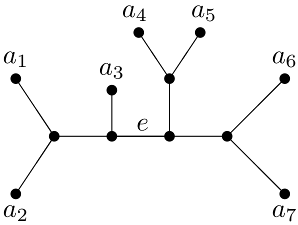
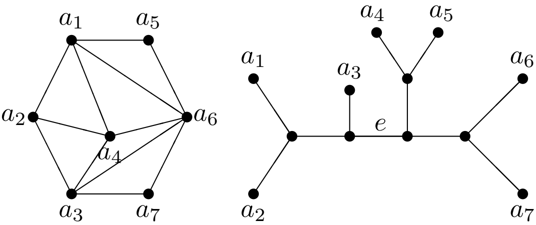




- Width of (T, L): max width over all edges in T
- **Rank-width** of G: min width over all rank-decompositions of G (Oum and Seymour 04; introduced for approximating clique-width)



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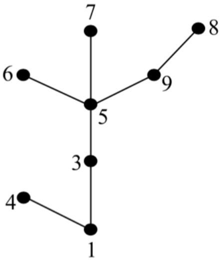
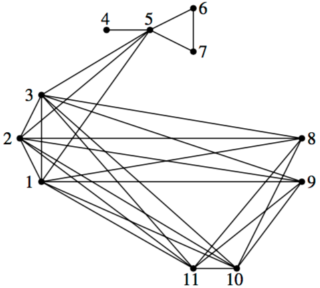
compare with clique-width : $\mathrm{rw} \leqslant \mathrm{cw} \leqslant 2^{\mathrm{rw}-1} + 1$.

- Clique-width : easier to design a dynamic programming algorithm, no direct FPT approximation algorithm, decomposition is always rooted.
- Rank-width : (3k+1)-FPT approximation algorithm, decomposition is not rooted, fit with 'vertex-minor' (every bipartite graph with large rank-width contain a grid-like vertex-minor...)



Tree (tree-width $\leqslant 1$)



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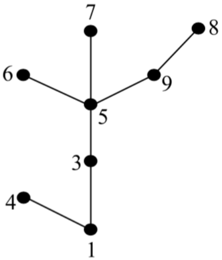

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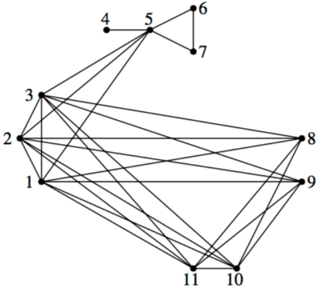

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Graphs of bounded rank-width contain dense graphs but also extend useful properties of graphs of bounded tree-width :

Courcelle, Makowsky, Rotics, 00

Every MSO_1 properties can be checked in cubic time on graphs of bounded rank-width.

Distance to graphs of bounded tree-width

graph class $\mathcal{C}$	problem
Edgeless graphs	VERTEX COVER FPT $\mathcal{O}^*(2^k)$ Kernel $2k$ vertices
Forests (graphs of tree-width $\leqslant 1$)	FEEDBACK VERTEX SET FPT $\mathcal{O}^*(3.618^k)$ [Kociumaka et al. 14] Kernel $4k^2$ vertices [Thomassé 10]
Series-parallel graphs (graphs of tree-width $\leqslant 2$)	K_4-MINOR COVER FPT $\mathcal{O}^*(c^k)$ [Kim et al. 15]
Graphs of tree-width $\leqslant w$	TREE-WIDTH w-VERTEX DELETION FPT $\mathcal{O}^*(c^k)$ (Non-uniform) Polynomial kernel [Fomin et al.12 /Kim et al. 13]

Distance to graphs of bounded rank-width?

graph class $\mathcal{C}$	problem
Disjoint union of Complete graphs	Cluster Vertex Deletion easy FPT $\mathcal{O}^*(3^k)$ (obstruction is P_3)
Block graphs	Block Graph Vertex Deletion
Thread graphs (linear rank-width $\leqslant 1$)	Linear rank-width-1 Vertex Deletion
Distance-hereditary graphs (graphs of rank-width $\leqslant 1$)	Distance-hereditary Vertex Deletion
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Distance-hereditary graphs (graphs of rank-width $\leqslant 1$)	Distance-hereditary Vertex Deletion FPT $\mathcal{O}^*(2^{\mathcal{O}(k \log k)})$ [Kim, K, manuscript]
Graphs of rank-width $\leqslant w$	Rank-width w-Vertex Deletion FPT by meta-theorem

OPEN: Rank-width w-Vertex Deletion can be solved in $\mathcal{O}^*(c^k)$-time for some constant c?

Eiben, Ganian, K, 16

DISTANCE-HEREDITARY VERTEX DELETION can be solved in time $37^k \cdot n^{\mathcal{O}(1)}$.

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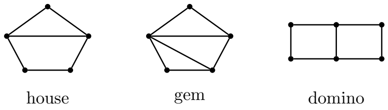
Recall :

- having no obstructions : Induced cycles of length at least 5 $+$



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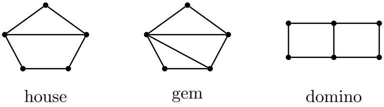

– Hitting induced cycles of length at least 5 is W[2]-hard. (Heggernes et al. 13)

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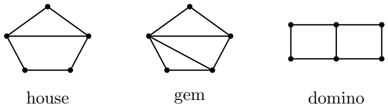

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Important property

Let G be a graph obtained from an induced path of length at least 3 by adding a vertex v adjacent to its end vertices. Then G has a DH obstruction containing v.

Eiben, Ganian, K, 16

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(1) **Iterative Compression**. Reduce to DISJOINT DISTANCE-HEREDITARY VERTEX DELETION.

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Parameter : k

Question : Is there $Q \subseteq V(G) \setminus S$ with $|Q| \leqslant k$ such that $G - Q$ is distance-hereditary?

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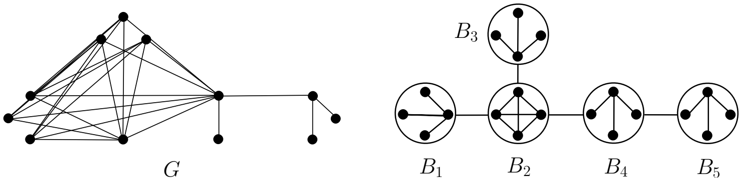
(2) **Branching Rules**. Apply two branching rules $\rightarrow$ G has no small obstructions and furthermore that certain connectivity conditions hold on $G[S]$.

(3) **Simplification of Split Decomposition**. We compute the split decomposition of $G - S$ and exploit the properties of our instance G guaranteed by branching to prune the decomposition.

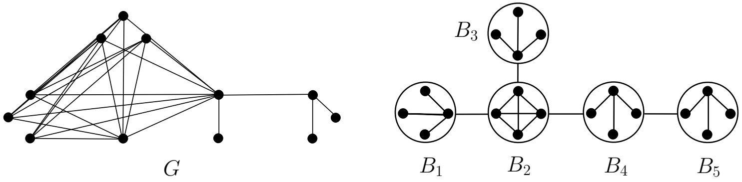
Split Decompositions



Two types of vertices : original vertices, marked vertices.
Two types of edges : marked edges, unmarked edges.

Two original vertices are adjacent if and only if they are linked by an alternating path.
(unmarked–marked–unmarked– $\cdots$ –unmarked)

Split Decompositions



Cunningham, 82

This decomposition is unique if

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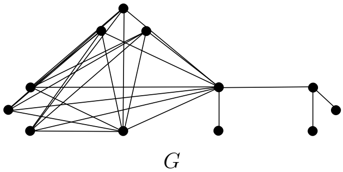
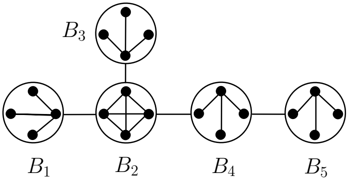
It is called the canonical split decomposition of a graph.

Split Decompositions





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Bouchet, 88

A graph is distance-hereditary if and only if every bag of its canonical split decomposition is either a star or a complete graph.

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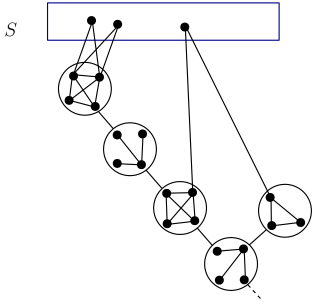
There is a given solution S and the canonical split decomposition of $G - S$.



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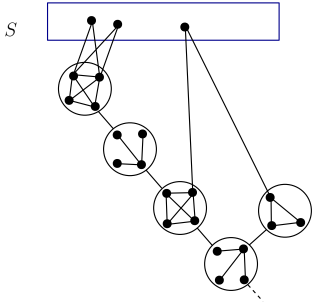

- Branching Rule 1.
  $\forall X \subseteq V(G - S)$ with $|X| \leqslant 5$, if $G[S \cup X]$ is not distance-hereditary, then we remove one of the vertices in X.
- Branching Rule 2.
  $\forall X \subseteq V(G - S)$ with $|X| \leqslant 5$ such that $G[X]$ is connected and adding X to S decreases the number of components in $G[S]$, then we either remove one of the vertices in X, or put all of them into S.

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Main ingredient : bad vertex for an induced cycle of length at least 5.

- For every induced cycle C of length at least 5 and $v \in V(C)$, a vertex $w \in S$ is a bad vertex for C and v if it is adjacent to the two neighbors of v in C.

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Main ingredient : bad vertex for an induced cycle of length at least 5.

- For every induced cycle C of length at least 5 and $v \in V(C)$, a vertex $w \in S$ is a bad vertex for C and v if it is adjacent to the two neighbors of v in C.
- It is easy to see that $v \in V(G - S)$ can be safely removed if for every induced cycle of length at least 5 containing v, there is a bad vertex for C and v.

Is it easy to find a bad vertex?

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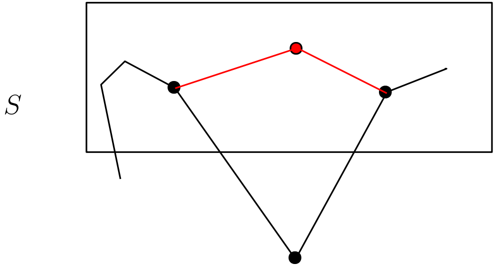

Twin Reduction Rule

If there exist two non-adjacent twins v, w in $G - S$ having at least one common neighbor in $G - S$ such that $(N_G(v) \cap S) \setminus (N_G(w) \cap S) \neq \emptyset$, then we can safely remove v.

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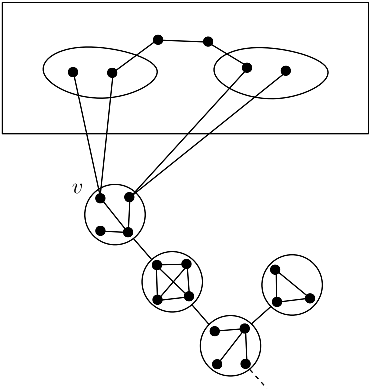

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- Claim 1. $(N_G(v) \cap S) \cap (N_G(w) \cap S) \neq \emptyset$.
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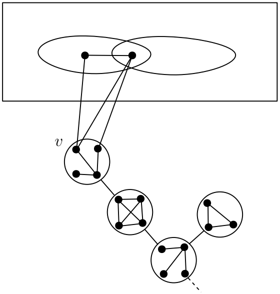
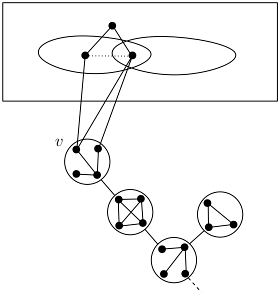
  3 types: 1) $S - v - S$ 2) $S - v - V(G - S)$ 3) $V(G - S) - v - V(G - S)$
  There is a bad vertex for C and v.





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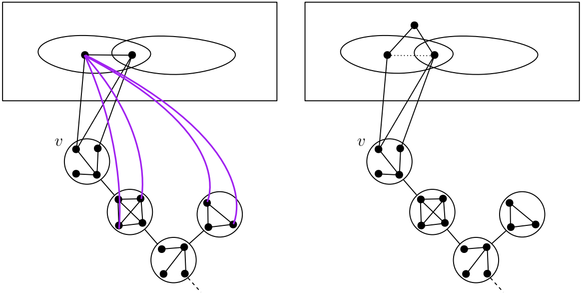

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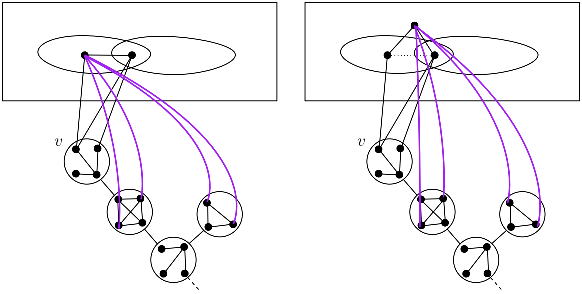

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- Rule 2. Deleting a vertex of degree 1.
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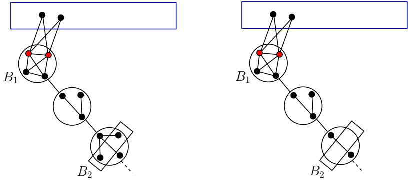

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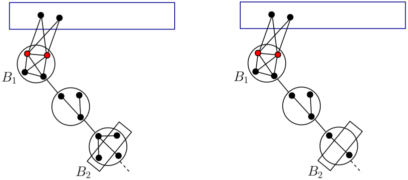
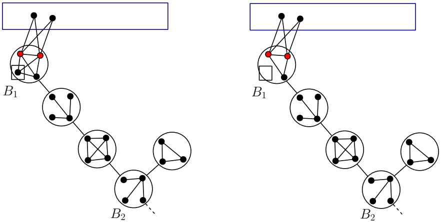


- Rule 5.



- Rule 1. D contains at most one class having neighbors on S, then we remove it.
- Rule 2. Deleting a vertex of degree 1.
- Rule 3. Twin Reduction Rule.
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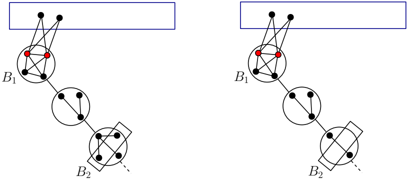

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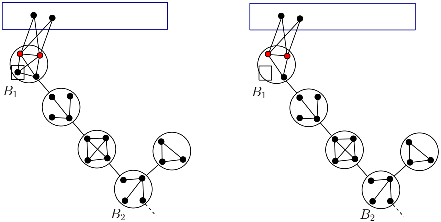
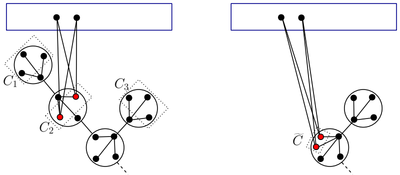


- Rule 6. Swapping the type of bags.

- Rule 7.



We obtain a property in a reduced instance:

(1) Every non-trivial branch should have a neighbor in S, and

(2) if there is a bag branching two branches, then two classes having neighbors in S in these branches have distance at most 4 in $G - S$.

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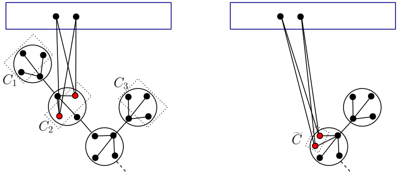

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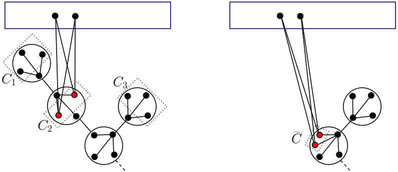

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Eventually, we will arrive at a trivial instance.

Conclusion

- Distance-Hereditary Vertex Deletion can be solved in time $\mathcal{O}^*(37^k)$. Can we improve the running time?
- Does Distance-Hereditary Vertex Deletion admit a polynomial kernel?
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- For $w \geqslant 2$, Rank-width w Vertex Deletion can be solved in time $\mathcal{O}^*(c^k)$ for some constant c?

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Thank for listening!