Binary matroids and Fundamental graphs

O-joung Kwon

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Matroid

E: set, $\mathcal{I}$: a set of subsets of E.
A pair $\mathcal{M} = (E, \mathcal{I})$ is a matroid on E
if it satisfies the following three axioms.

(T1) $\mathcal{I} \neq \emptyset$.

(T2) If $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$.

(T3) Let $X, Y \in \mathcal{I}$ and $|X| = |Y| + 1$. Then there exists $x \in X \setminus Y$ such that $Y \cup \{x\} \in \mathcal{I}$.

$X \in \mathcal{I}$: an independent set of $\mathcal{M}$.
An independent set X of $\mathcal{M}$ is a base of $\mathcal{M}$ if it is a maximal independent set.

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Binary Matroid

Let A be a $V \times W$ $(0,1)$-matrix.
The binary matroid $\mathcal{M}(A)$ of A is defined as
$(W, \{U \subseteq W : U$ represent a linearly independent set of $A\})$.

By the definition, a base of $\mathcal{M}(A)$ forms a maximally lin. independent set of column space of A.

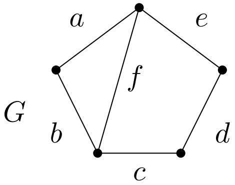
$$A = \begin{pmatrix} \overset{a}{1} & \overset{b}{0} & \overset{c}{0} & \overset{d}{1} & \overset{e}{1} \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$\{a, b, c\} \in \mathcal{I}$: a base.

Graphic Matroid

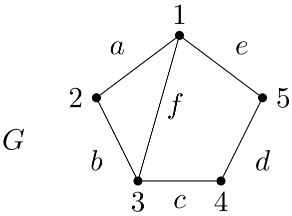
Let G be a graph.
The graphic matroid $\mathcal{M}(G)$ of G is defined as
$\mathcal{M}(G) = (E(G), \{U \subseteq E(G) : U \text{ forms a forest in } G\})$.

By the definition, a base of $\mathcal{M}(G)$ forms a spanning tree of G.



$\{a, f, e, d\} \in \mathcal{I}$: a base

Graphic Matroid



$$I_G = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Graphic and Binary Matroid

Theorem

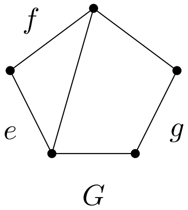
Let G be a graph and I_G be the vertex-edge$(V \times E)$ incidence matrix of G. Then $\mathcal{M}(I_G) = \mathcal{M}(G)$. Therefore every graphic matroid is binary.

0. Problem.

Graph minor

Let $G = (V, E)$ be a graph.

- G/e *Contraction* an edge $e \in E$
- $G \setminus e$ *Deletion* an edge $e \in E$
- $G \setminus v$ *Deletion* a vertex $v \in V$



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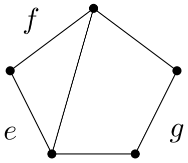

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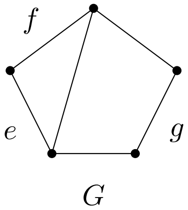

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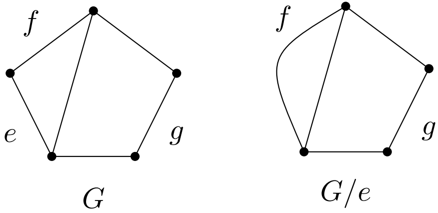

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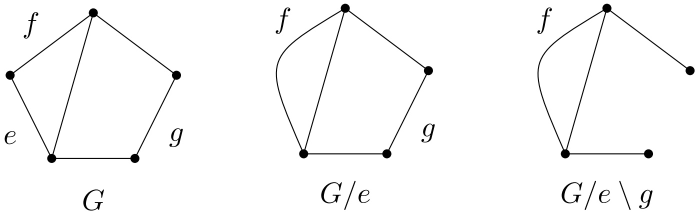

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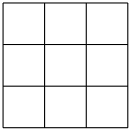
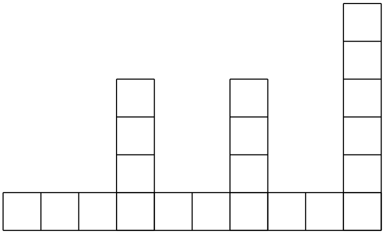
Does the right graph has the left graph as a minor???





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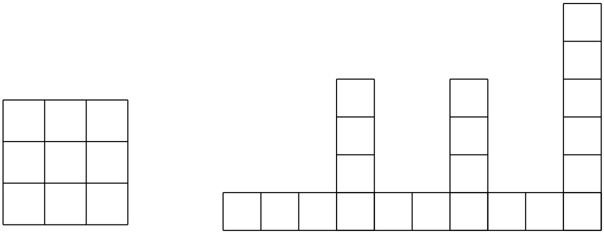

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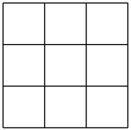
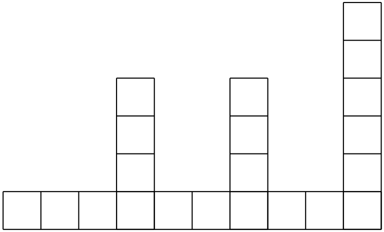


tree-width: 3



tree-width: 2

Pivot-minors of graphs

We are interested in another operation.

- $G \wedge uv$ *Pivot-operation* on an edge $uv \in E$

But in this talk, every graph has no triangle.
C does not appear.

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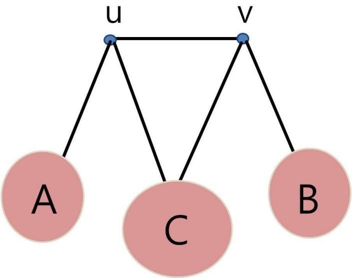

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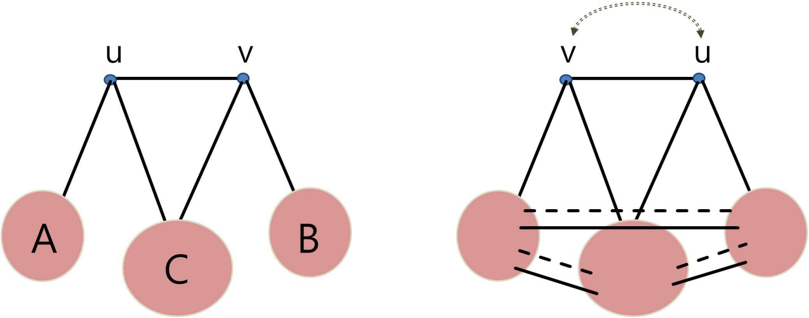

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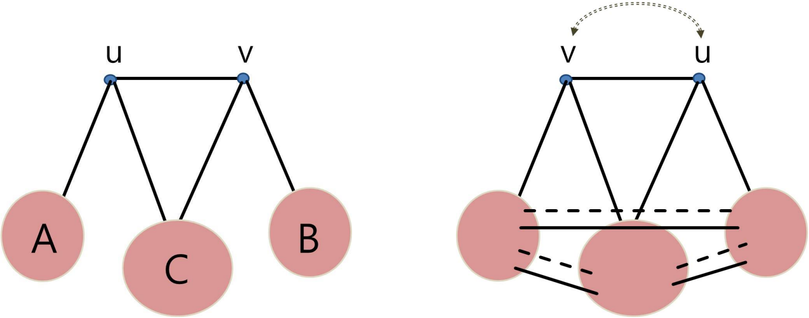

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Pivot-minors of graphs

For example,

- H : a *pivot-minor* of a graph G
  if H is obtained from G by applying a sequence of pivoting edges and vertex deletions.

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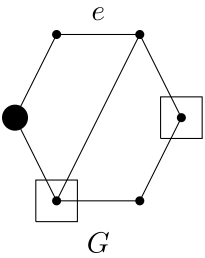

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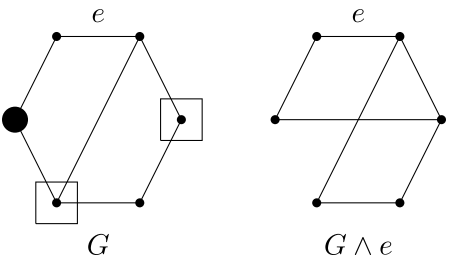

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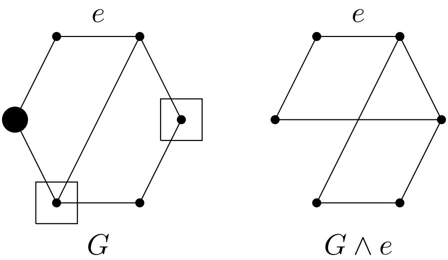

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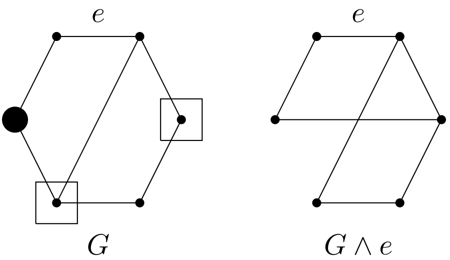

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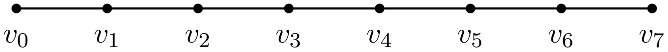

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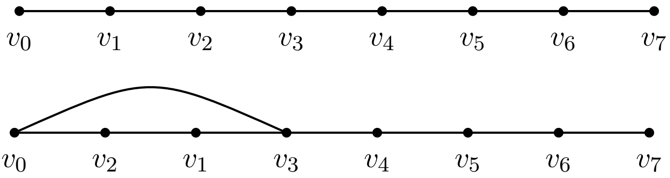

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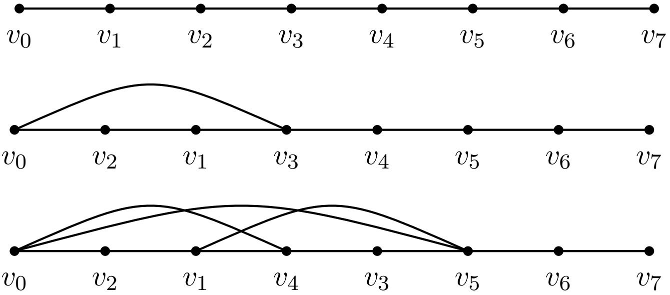

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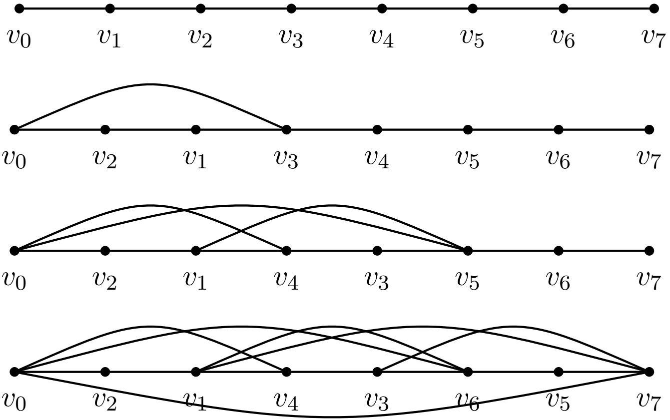

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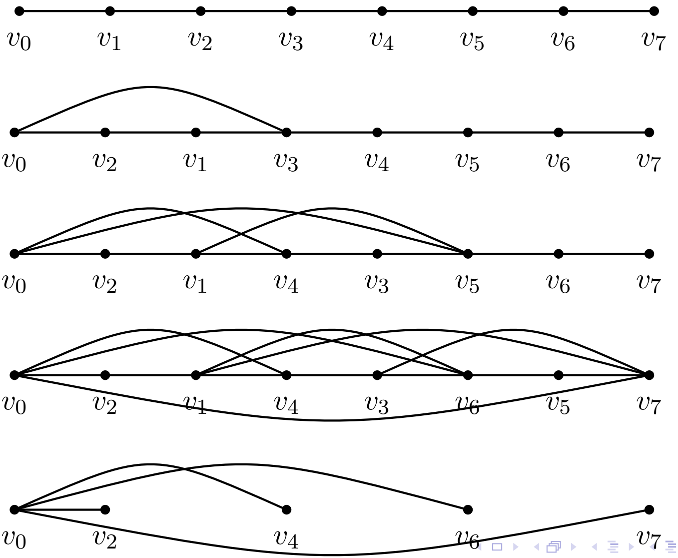
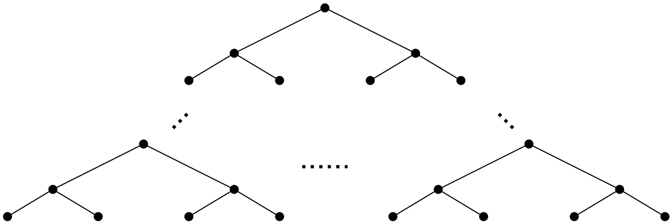
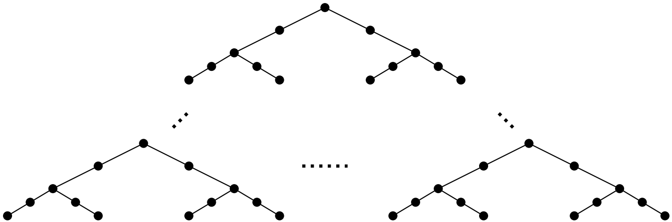


Complete binary tree

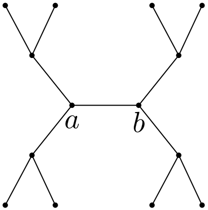


The incidence graph of a complete binary tree

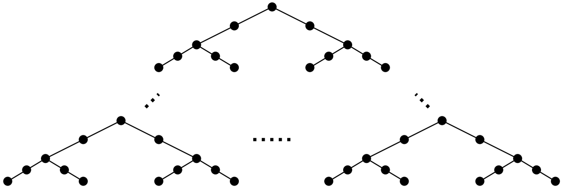


Question

Could the incidence graph of a complete binary tree have the B_6 graph as a pivot-minor???



B_6 graph



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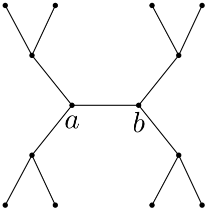
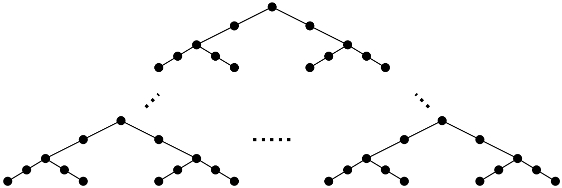


B_6 graph

rank-width: 1



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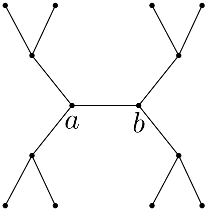
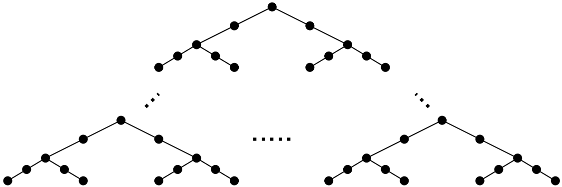


B_6 graph

rank-width: 1
linear r-w: 2



rank-width: 1
linear rank-width: $\lfloor \frac{k}{2} \rfloor + 1$

Theorem

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The graph B_6 could not be a pivot-minor of the incidence graph of a binary tree.

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We have two proofs of the theorem.

- Natural way.
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- Use matroid.

1. First proof

Sketch of proof

1. We can first every pivot, and last delete.
2. Every vertex can be appeared at most once in pivoting set.
   EX : $G \wedge ab \wedge ac \wedge bd \wedge ce \wedge ae = G \wedge ad$.
3. If $M = \begin{array}{c c} & \begin{array}{cc} X & V \setminus X \end{array} \\ \begin{array}{c} X \\ V \setminus X \end{array} & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array}$ and A is nonsingular,

   then we define

$$M * X = \begin{array}{c c} & \begin{array}{cc} X & V \setminus X \end{array} \\ \begin{array}{c} X \\ V \setminus X \end{array} & \begin{pmatrix} A^{-1} & A^{-1}B \\ -CA^{-1} & D - CA^{-1}B \end{pmatrix} \end{array}.$$

4. A graph H is obtained from G by pivoting edges, then there is $X \subseteq V(G)$ such that $A(G)[X]$ is nonsingular and $A(G) * X = A(H)$

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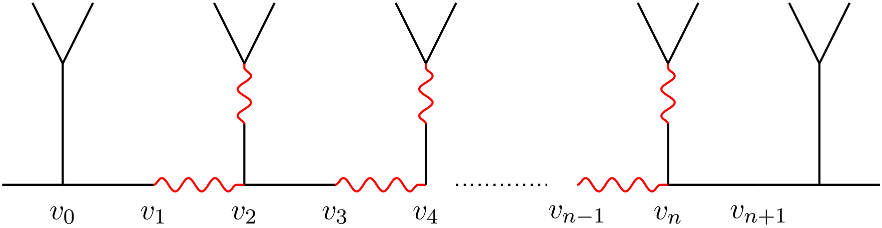
Sketch of proof

Theorem(Graphs and matrices..)

Let T be a tree.
Then $A(T)$ is nonsingular iff T has a perfect matching.

Sketch of proof

If B_6 is a pivot-minor of the incidence graph of a binary tree ...



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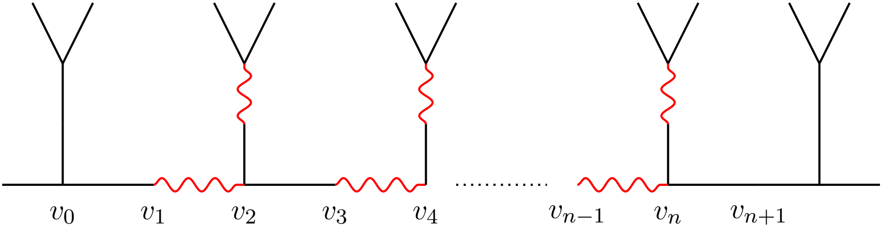


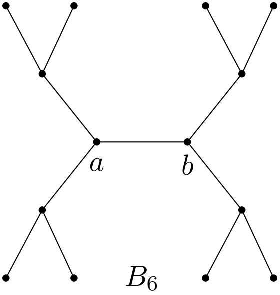
Note. Sufficient and Necessary to connect after pivoting

1. Exist an alternating path between two vertices.
2. Both are in pivoting set $\rightarrow$ (start:$\sim$, end:$\sim$)

One is in pivoting set, one is in boundary $\rightarrow$ (start:-, end:$\sim$)

Both are in boundary $\rightarrow$ (start:-, end:-)

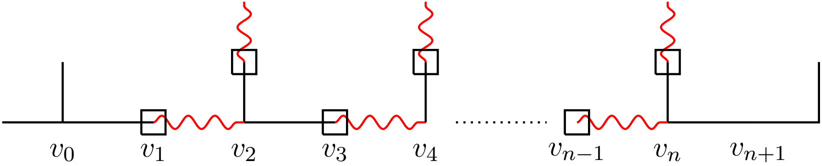
Sketch of proof



Proof of the Theorem

a or b must in a closure of X. There are 4 types of vertices.

- $X \cap C(T)$
- $X \setminus C(T)$
- $N(X) \cap C(T)$
- $N(X) \setminus C(T)$



Proof of the Theorem

a or b must in a closure of X. There are 4 types of vertices.

- $X \cap C(T)$
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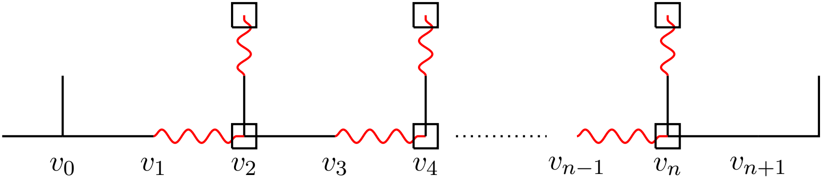

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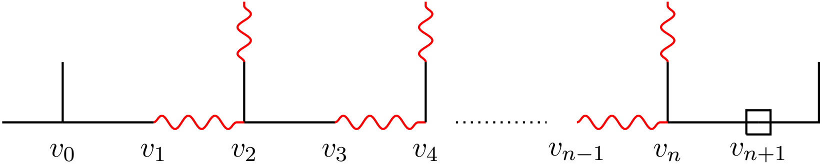

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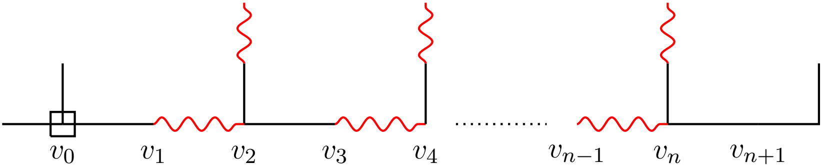
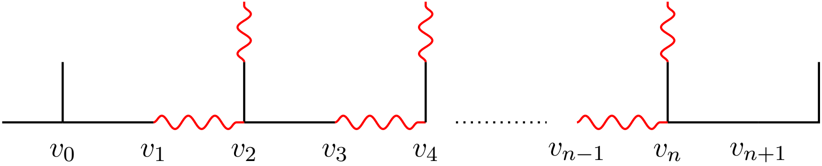


Proof of the Theorem

a or b must in a closure of X. There are 4 types of vertices.

- $X \cap C(T)$
- $X \setminus C(T)$ (X)
- $N(X) \cap C(T)$ (X)
- $N(X) \setminus C(T)$



Proof of the Theorem

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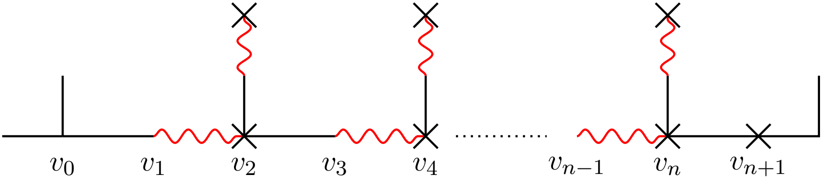


It is ok. But is there more simpler proof?

2. Second proof

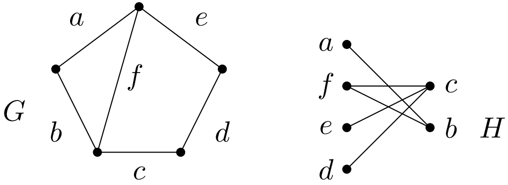
Matroid Minor

There are minor operations for a matroid. (Need to know dual..)

Note. For a graph G and H,
if $\mathcal{M}(H)$ is a minor of $\mathcal{M}(G)$ then H is a minor of G.

The fundamental graph of a matroid

B : a base in $\mathcal{M}$.
A bipartite graph H with bipartition $B \cup (E(\mathcal{M}) \setminus B)$ is a fundamental graph of $\mathcal{M}$ with respect to B
$vw \in E(H)$ if and only if $v \in B$ and $w \in E(\mathcal{M}) \setminus B$ and $B \setminus \{v\} \cup \{w\}$ is a base of $\mathcal{M}$.



$\{a, f, e, d\} \in \mathcal{I}$: a base

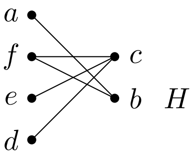
Binary Matroid from a Bipartite Graph

Let G be a bipartite graph with a bipartition $A \cup B$.

$\mathrm{Bin}(G, A, B)$ be the binary matroid on V

represented by the $A \times V$ matrix $(I_A, A(G)[A, B])$

where I_A is the $A \times A$ identity matrix.



$$(I_A, A(G)[A,B]) = \begin{array}{c} \\ a \\ f \\ e \\ d \end{array} \begin{array}{c} \begin{array}{cccccc} a & f & e & d & c & b \end{array} \\ \left(\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right) \end{array}$$

Binary Matroid from a Bipartite Graph

Lemma

Let G be a graph. Let B be a base of the matroid $\mathcal{M}(G)$ and H be the fundamental graph of the matroid $\mathcal{M}(G)$ with respect to B. Then $\mathcal{M}(G) = \mathrm{Bin}(H, B, E(G) \setminus B)$.

$$I_G = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} a & f & e & d & c & b \end{array} \\ \left(\begin{array}{cccc|cc} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right) \end{array}$$

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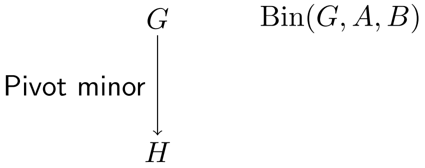
Theorem.

Theorem(Oum)

Let G be a bipartite graph with a bipartition $A \cup B = V(G)$.
If H is a pivot-minor of G,
then there is a bipartition $A' \cup B' = V(H)$ such that
$\mathrm{Bin}(H, A', B')$ is a minor of $\mathrm{Bin}(G, A, B)$.



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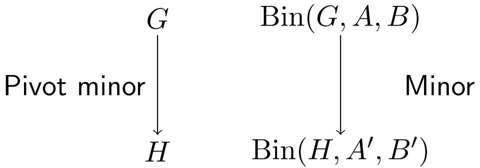

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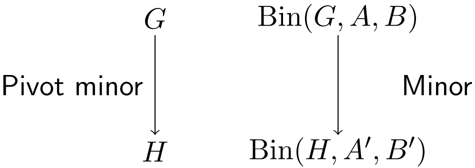
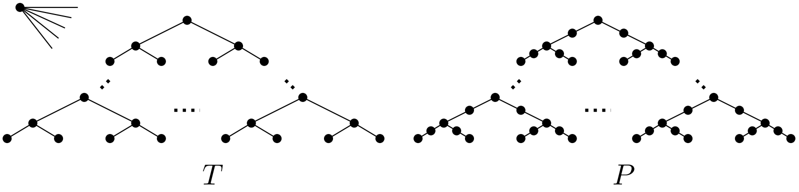


Theorem.

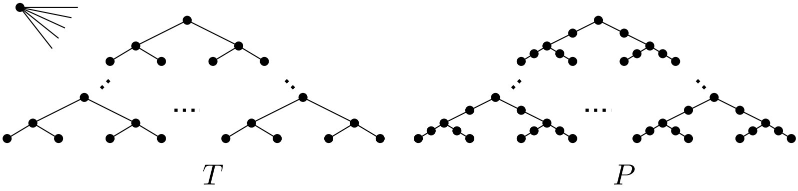


P : Incid. graph of a binary tree

↓ Pivot minor

B_6

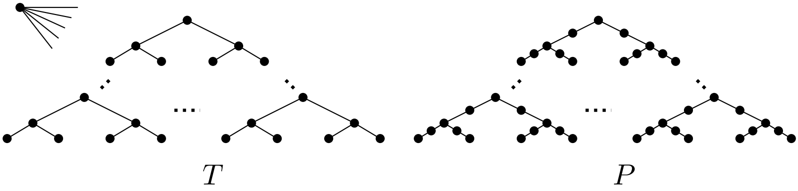
Theorem.



P : Incid. graph of a binary tree $\xrightarrow{\text{Pivot minor}}$ B_6

$\mathrm{Bin}(P, A, B)$ $\xrightarrow{\text{Minor}}$ $\mathrm{Bin}(B_6, A', B')$

Theorem.

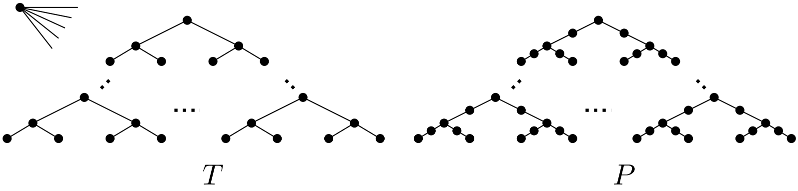


P : Incid. graph of a binary tree $\quad \mathrm{Bin}(P, A, B) = \mathcal{M}(T)$

Pivot minor: P : Incid. graph of a binary tree $\rightarrow B_6$

Minor: $\mathrm{Bin}(P, A, B) \rightarrow \mathrm{Bin}(B_6, A', B')$

Theorem.



P : Incid. graph of a binary tree

$\mathrm{Bin}(P, A, B) = \mathcal{M}(T)$

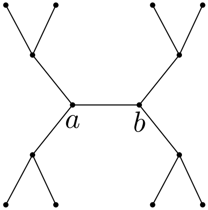
Pivot minor

Minor

B_6

$\mathrm{Bin}(B_6, A', B') = \mathcal{M}(?)$

Convert.

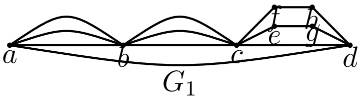


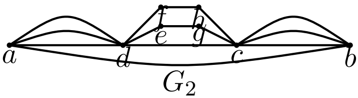
B_6 graph

Which graphs have B_6 as the fundamental graph?

Convert.

Use Whitney's 2-isomorphism theorem.

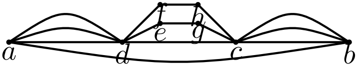




Convert.

Use Whitney's 2-isomorphism theorem.

G_1

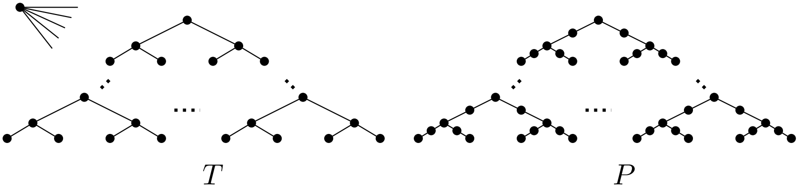


G_2

Note. G_1 and G_2 have G_3 as a minor.


G_3

Theorem.

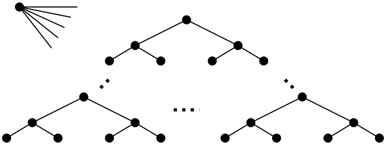


P : Incid. graph of a binary tree $\xrightarrow{\text{Pivot minor}}$ B_6

$\mathrm{Bin}(P, A, B) = \mathcal{M}(T)$ $\xrightarrow{\text{Minor}}$ $\mathrm{Bin}(B_6, A', B') = \mathcal{M}(G_1)$ or $\mathcal{M}(G_2)$

T $\xrightarrow{\text{Minor}}$ G_1 or G_2

Proof



T

G_3

Multiple edges in a minor of T must contain the apex vertex.

Therefore, T cannot have G_3 as a minor.

Done.

Discussion

- Is there a parameter that solve this problem?
- Is there another characterization of singularity(over $GF(2)$) of the adjacency matrices of graphs?
- Fundamental graph is always bipartite. Is there a way to help for working on pivot-minors of non-bipartite graphs?

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Thank you for your attention