Dynamical Systems and Ergodic Theory

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Abstract

In this KPF Physics Seminar, I want to introduce about Dynamical Systems and Ergodic Theory, which is a branch of mathematics that motivated on problems about statistical physics. Especially, this seminar focuses on 'What is the Dynamical Systems? What is the Ergodic Theory?' with some motivating physical problems. Further, I will construct some 'measures', the basic concepts for studying Dynamical Systems and Ergodic Theory. Finally, this seminar will end up with two critical theorems in this branch of mathematics.

1 Introduction : What is Dynamical Systems?

If I randomly hold on a KAIST student and ask "What is Dynamical Systems?", then the answer may be "Hmm... the branch of physics that studying dynamics? $F = ma$ blah blah blah?". However, this answer may not be true. Thus, I will emphasize that Dynamical Systems is studied by mathematicians as a branch of mathematics. Then, why does it have such physicslike name?

Accordingly, I will answer to original question by explainig why we need to consider such things. First, imagine a moving particle in a box as below.

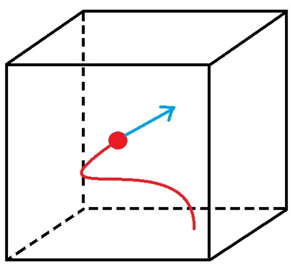

Figure 1: Moving particle in a box

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Now, how can we study about the motion of above particle? Many people may answer as “Oh, that’s easy! I can define a curve that represent motion of the particle based on $F = ma$! That is, $f(t) = (x, y, z)$!” That’s right. However, how will you solve this problem if there are N particles? How about countably many? Some people easily answer by constructing a sequence of curves $\{f_n(t) = (x, y, z)\}_{n\in\mathbb{N}}$ also based on applying $F = ma$ to each individual particles.

However, there are so many sensitive variables (Ex. too many $F = ma$) that this consideration will be really complicated, even though above situation provides very specially good conditions such as rectangularly well-coordinatedness and countable elements. Obviously, there are some situations that provide complicated structures such as not rectangularly coordinatedness and uncountable elements.

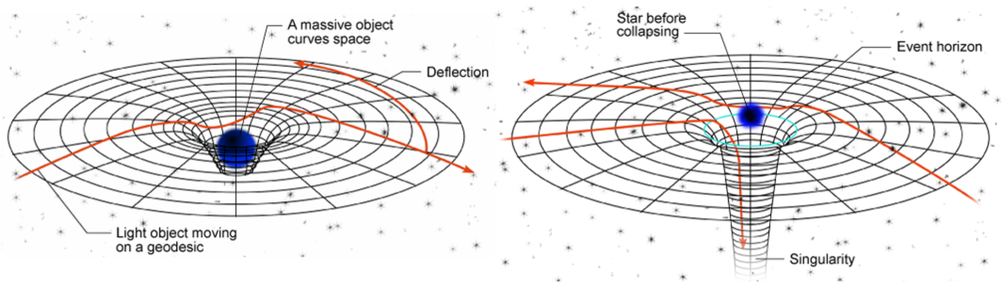

Figure 2: General relativity, the example of strangely coordinated structure

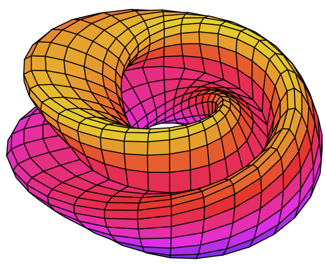

Figure 3: Heat flow, the example of uncountable elements

For such situations, we can’t apply the method as above, and most of physical models provide such complicated structure. Furthermore, this happen is not only in physics, but also in mathematics. For example, we can consider some dynamically varying mathematical phenomenon such as shifting sequence, Markov chain, etc. Hence, many mathematicians are interested in such problems and they made an effort to make some results about complicated systems. However, note that complicated systems have various structures so it seems to need different kind of approach to each different systems.

Maybe some brilliant people can deal with each of such various complicated systems. However, as you know, all mathematicians and physicists want to and need to some techniques possibly applied to general systems. This is the starting point of 'Dynamical Systems'. How can we do that?

Go to the first simple system and remind what we did. For given system that moving particle in a box, we set an appropriate rectengular coordinate, and define a function f with time parameter t based on $F = ma$. From this, we may get some idea for considering general systems.

Generally, for given system, set appropriate space X and mapping $f : X \to X$ based on dynamycal principles for that system. Then, the pair (X, f) represents given system well so that we define "Dynamical System (X, f)". According to this setting, considering dynamical variation of systems is equivalent to considering the orbit $\{f^t(X)\}_{t\in\tau}$ over the parameter space τ. Parameter space can be any appropriate set such as $\mathbb{R}$ and $\mathbb{Z}$.

Now, we know what the dynamical systems is, and how these concepts can be applied to generally given system. Here, I will finish this section with some kinds of dynamical systems according to the space X and mapping f, then where and how these concepts applied to other theory.

Some kinds of Dynamical Systems

- Differential Dynamical Systems

  Dynamical system (X, f) is *Differential Dynamical System* if the space X is differential manifold and mapping $f : X \to X$ is flow from the differential equations.

- Topological Dynamical Systems

  Dynamical system (X, f) is *Topological Dynamical System* if the space X is topological space and mapping $f : X \to X$ is continuous map.

- Ergodic Theory

  Ergodic Theory, what we will focus on, is the theory of dynamical systems (X, f) where the space X is measurable space and $f : X \to X$ is measurable fucntion.

Applications of Dynamical Systems

- In mathematics, it can be applied to everything that dynamically behaves, for instance actions over algebraic structures.
- In physics, it can be applied to physical phenomenon with dynamical system and chaotic view point. It is called "Chaos Theory".
- In information theory, (X, f) where X is the set of data and f is an algorithm is considered as dynamical system.

2 Ergodic Theory

In this section, we will focus on the motivating problems of Ergodic Theory and what it is. Further, there will be some explanations about measure, the basic concepts for studying Ergodic Theory. First, the "Ergodic Theory" is named from below Greek formula :

$$\text{Ergodic} = \text{Ergon(Work, Action)} + \text{Hodos(Way, Path)}$$

From this formula, we may infer that the Ergodic Theory is the study of path of works or actions. Indeed, the Ergodic Theory is motivated by problems of statistical physics such below.

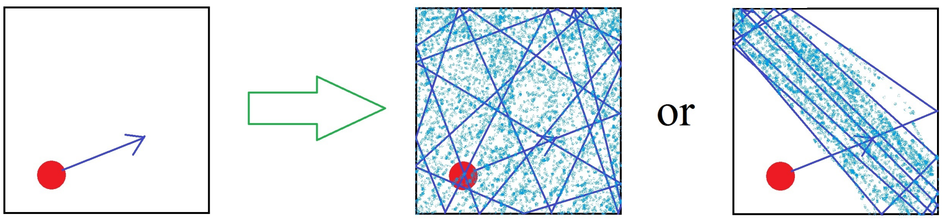

Figure 4: Motivating problem of statistical physics

That is, for given moving particle in the box, how will the orbit of particle behave after a long time? Will the orbit be distributed equally over the box(Left)? Will the orbit be distributed only in specific region(Right)? While this problem can be easily solved by middle school students, it becomes really difficult if the problem provides complicated conditions such as strangely shaped space and more particles. From such problems, mathematicians started to develop the Ergodic Theory.

In conclusion, Ergodic Theory is study of statistical properties of dynamical systems on "measure space", and its central concern is behavior of a dynamical system when it is allowed to run for a long time. And, the first result in this direction is "Poincaré Recurrence Theorem" that we will focus on.

2.1 Why do we study such way?

In dynamics, you may think that "Couldn't every problems be solved by $F = ma$?". Actually, in dynamics and also for other branches of physics, every people thaught that "We can attain PDE from appropriate modeling based on $F = ma$, then the problem will be solved." until 19C. However, there are many PDEs that cannot be solved analytically so that above approches fail and it becomes limitation of pre-existing theories. In that situation, Henri Poincaré, the French Mathematician and Phycist, changed view point as : Not 'a' particle, 'System!'. Then, fantastic developments in physics and mathematics was achieved.

2.2 Measure

As I mentioned above, 'Measure' is the basic concept for studying Ergodic Theory. Here, I will basically deal with measure over four partitions. First, the concept 'Measure' came from the question that "What size of the set?". Note that this question is pretty different to "How many elements are in the set?". This difference can be easily known from simple example : $[0, 1]$ and $[0, 2]$. Under usual sense of size, the Euclidean Geometry, size of $[0, 1]$ is 1 and size of $[0, 2]$ is 2, while they have same number of elements, that is same cardinality. Therefore, the answer of original question is not just cardinality but well-defined and geometrically reasonable measurement of set such as length, area, volume in Euclidean Geometry.

With above motivation, we have to solve several problems step-by-step to define measure as below.

1. When the set E is measurable?
2. If E is measurable, then how can we define measure?
3. What nice properties or axioms does measure or measurability obey?
4. Can we extend to abstract definitions?

2.2.1 Elementary Measure

First of all, we will consider in Euclidean space $\mathbb{R}^n$ and start to define very simple measure, the Elementary Measure. The elementary measure is the simplest measure that reflects our geometric intuition. As you know, the simplest set in $\mathbb{R}^d$ is a box, the finite cartesian product of intervals.

Definition 2.1. Interval $I \subseteq \mathbb{R}$ is $I = [a, b],\ [a, b),\ (a, b],$ or (a, b) and $|I| = b - a$. Box $B \subseteq \mathbb{R}^d$ is $B = I_1 \times \cdots \times I_d$ and $|B| = |I_1| \cdots |I_d|$. Elementary set $E \subseteq \mathbb{R}^d$ is $E = \bigcup_{i=1}^{n} B_i$.

For elementary set, we can define a measure as follows :

Definition 2.2. Clearly, for each elementary set $E \subseteq \mathbb{R}^d$, there exist some pairwisely disjoint boxes $B_1, \ldots, B_n$ such that $E = B_1 \cup \cdots \cup B_n$. Then, the elementary measure for E is $m(E) = |B_1| + \cdots + |B_n|$.

Here, the set $E \subseteq \mathbb{R}^d$ is measurable if and only if E is an elementary set. Furthermore, it is easy to prove following theorem.

Theorem 2.1.

- *For any elementary set* E, $m(E) \geq 0$.
- $m(\emptyset) = 0$
- *For pairwisely disjoint elementary sets* $\{E_i\}_{i=1}^n$, $m(\bigcup_{i=1}^n E_i) = \sum_{i=1}^n m(E_i)$.

Although the elementary measure is well-defined and intuitive measure, it is too weak so there are many non-measurable simple sets such as triangle, sphere, etc. Hence, it is necessary to define more strong measure.

2.2.2 Jordan Measure

From the elementary measure, it is natural that extending elementary measure to non-elementary measure, Jordan measure. Shortly, the Jordan measure is 'approximated elementary measure' for given set $E \subseteq \mathbb{R}^d$.

Definition 2.3. Let $E \subseteq \mathbb{R}^d$ be a bounded set.

- Jordan inner measure is
$$m_{*,(J)}(E) = \sup_{A \subseteq E,\ A\ :\ elementary} m(A).$$
- Jordan outer measure is
$$m^{*,(J)}(E) = \inf_{E \subseteq B,\ B\ :\ elementary} m(B).$$
- If $m_{*,(J)}(E) = m^{*,(J)}(E)$, then E is called Jordan measurable and the Jordan measure of E is
$$m(E) := m^{*,(J)}(E) = m_{*,(J)}(E).$$

Similar to elementary measure, it is easy to prove following theorem.

Theorem 2.2.

- *For any Jordan measurable set* E, $m(E) \geq 0$.
- $m(\emptyset) = 0$
- *For pairwisely disjoint Jordan measurable sets* $\{E_i\}_{i=1}^n$, $m(\bigcup_{i=1}^n E_i) = \sum_{i=1}^n m(E_i)$.

By the definition, the Jordan measure is well-extended measure from the elementary measure so that some non-elementary sets are Jordan measurable. However, it is weak yet since some slightly complicated sets such as $[0,1] \setminus \mathbb{Q}$ are not Jordan measurable. As a result, it is a natural requirement to define more extended measure.

2.2.3 Lebesgue Measure

From the Jordan measure, we can extend to more strong measure, Lebesgue Measure. Lebesgue Measure is very important concept in real analysis, especially integration theory. Before defining the Lebesgue measure, note $m^{*,(J)}(E) = \inf_{E \subseteq B,\ B\ :\ elementary} m(B)$. From the finite additivity and subadditivity of elementary measure, it is equivalent to write

$$m^{*,(J)}(E) = \inf_{E \subseteq \bigcup_{i=1}^{n} B_i,\ B_i's\ are\ boxes} |B_1| + \cdots + |B_n|.$$

Now, we are ready to define Lebesgue measure as follows.

Definition 2.4. The Lebesgue outer measure is

$$m^*(E) = \inf_{E \subseteq \bigcup_{i=1}^{\infty} B_i,\ B_i's\ are\ boxes} \sum_{i=1}^{\infty} |B_i|.$$

For curious reader, it may be a natural question that "Why don't we define Lebesgue inner measure?". This is just becuase Lebesgue inner measure doesn't gain any increase in power in the Jordan inner measure. Anyway, from the definition, clearly $m^*(E) \leq m^{*,(J)}(E)$. Now, we can define Lebesgue measurable set as follows.

Definition 2.5. A set $E \subseteq \mathbb{R}^d$ is Lebesgue measurable if for any $\varepsilon > 0$, there exists an open set $U \subseteq \mathbb{R}^d$ with $E \subseteq U$ such that $m^*(U \setminus E) \leq \varepsilon$.

For Lebesgue measurable sets, Lebesgue measure is well-defined as follows.

Definition 2.6. If $E \subseteq \mathbb{R}^d$ is Lebesgue measurable, define $m(E) = m^*(E)$ as Lebesgue measure. It may be ∞.

Similar to previous measures, it is easy to prove following theorem.

Theorem 2.3.

- *For any Lebesgue measurable set E, $m(E) \geq 0$.*
- $m(\emptyset) = 0$
- *For pairwisely disjoint Lebesgue measurable sets $\{E_i\}_{i \in \mathbb{N}}$, $m(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m(E_i)$.*

Now, relations between previous measures are summarized as following diagram.

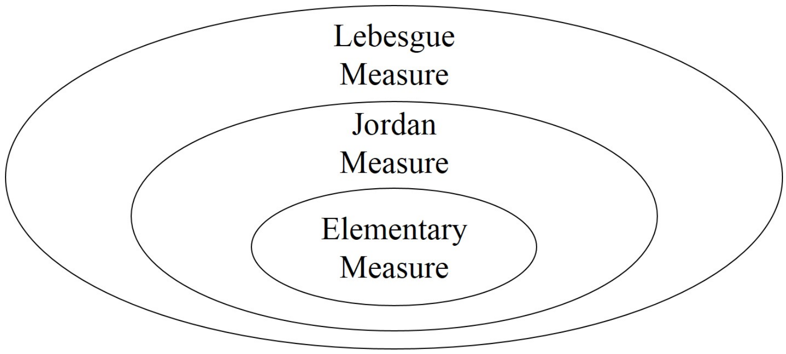

Figure 5: Relations between previous measures

2.2.4 Abtract Measure

Until now, we defined some measures in the Euclidean space. However, as you may know, there are not only situations that be able to considered in Euclidean space, but also many others that require new sense of measure. Further, this problem will be arised even if we define new specific measure in specific space. Therefore, it is necessary to define general, or abstract, measure that can be well-defined in arbitrarily given space. Then, if we successfully define abtract measure, setting appropriate measure for each given situation leads to attain powerful results by applying properties of abstract measure. To sum up, the motivating question for abstract measure is :

"Elementary, Jordan, Lebesgue measures are measure. OK. So, what is measure?"

To define abstract measure, an additional structure should be defined for given space X. That is,

Definition 2.7. Let X be a set. Then, $\Sigma \subseteq \mathcal{P}(X)$ is $\sigma - algebra$ if it satisfies following axioms :

- $X \in \Sigma$
- $E \in \Sigma \Rightarrow X \setminus E \in \Sigma$
- $\{E_i\}_{i \in \mathbb{N}} \subseteq \Sigma \Rightarrow \bigcup_{i=1}^{\infty} E_i \in \Sigma$

From definition, it is easy to prove following theorem.

Theorem 2.4. *For any set X and its $\sigma - algebra$ Σ,*

- $\emptyset \in \Sigma$
- $\{E_i\}_{i\in\mathbb{N}} \subseteq \Sigma \Rightarrow \bigcap_{i=1}^{\infty} E_i \in \Sigma$

Now, it's time to define measure what we critically wanted. The motivations for definition of abstract measure are common properties of specific measures that we previously considered, well-reflecting our geometric intuition.

Definition 2.8. For a set X and its $\sigma - algebra$ Σ, a function $\mu : \Sigma \to \bar{\mathbb{R}}$ is a measure if

- $\forall E \in \Sigma,\ \mu(E) \geq 0$
- $\mu(\emptyset) = 0$
- $\{E_i\}_{i\in\mathbb{N}} \subseteq \Sigma$ and they are pairwisely disjoint $\Rightarrow \mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$

In definition, some readers may think the second axiom can be derived from other two axioms as follows :

$$\mu(E) = \mu(E \cup \emptyset) = \mu(E) + \mu(\emptyset) \Rightarrow \mu(\emptyset) = 0$$

However, this is reasonable logic only when $\mu(E) < \infty$. For explicitly, if we set $\Sigma = \{\emptyset,\ X\}$ and $\mu(X) = \infty$, then cancellatoin law fails so that there is no guarantee for $\mu(\emptyset) = 0$. As we did for dynamical system, we define some pairs.

Definition 2.9. For a set X, its $\sigma - algebra$ Σ, and a measure $\mu : \Sigma \to \bar{\mathbb{R}}$,

- Each element in Σ is a measurable set.
- (X, Σ) is a measurable space.
- (X, Σ, μ) is a measure space.

Also, following theorem about monotonicity and countable additivity for abstract measure is easy to prove.

Theorem 2.5. *For a measure space* (X, Σ, μ),

- $E \subseteq F$ *and* $E, F \in \Sigma \Rightarrow \mu(E) \leq \mu(F)$
- $\{E_i\}_{i\in\mathbb{N}} \subseteq \Sigma \Rightarrow \mu(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu(E_i)$

From now, we will define some basic concepts in measure space, especially a mapping between measure spaces and some characteristics of them.

Definition 2.10. For two measurable spaces (X, Σ) and (Y, T), a funciton $f : X \to Y$ is a measurable function if

$$\forall E \in \mathrm{T},\ f^{-1}(E) \in \Sigma.$$

Definition 2.11. For a measure space (X, Σ, μ), a measurable function $T : X \to X$ is a measure preserving transformation if

$$\forall E \in \Sigma,\ \mu(T^{-1}(E)) = \mu(E).$$

Definition 2.12. For a measurable space (X, Σ) and a measurable function $f : X \to X$, a measure μ on (X, Σ) is invariant under f if

$$\forall E \in \Sigma,\ \mu(f^{-1}(E)) = \mu(E).$$

In the case of dynamical system (X, T, φ) where (X, Σ) is a measurable space, T is a monoid(i.e. parameter space), and $\varphi : T \times X \to X$ be a flow map, a measure μ on (X, Σ) is an invariant measure if

$$\forall t \in T \text{ and } E \in \Sigma,\ \mu(\varphi_t^{-1}(E)) = \mu(E).$$

Definition 2.13. For a measure preserving transformation $T : X \to X$ on a measure space (X, Σ, μ), T is ergodic if

$$\forall E \in \Sigma,\ T^{-1}(E) = E \Rightarrow \mu(E) \in \{0,\ \mu(X)\}.$$

For example, consider a unit circle $S^1 = [0, 1]/\sim$ where $\sim$ indicates that 0 and 1 are identified, and the rotation R_α of S^1 by angle $2\pi\alpha$ for $\alpha \in \mathbb{R}$. Then, R_α is ergodic if and only if $\alpha \notin \mathbb{Q}$.

3 Poincaré Recurrence Theorem

As I mentioned above, the first result about long time behavior of dynamical system is Poincaré Recurrence Theorem. This theorem gives a beautiful conclusions for such following problem, called recurrence problem :

Will the particle come back to initial poisition after it depart in randomly-shaped box? If it will comback, how often? If it won't come back, how close will it come back?

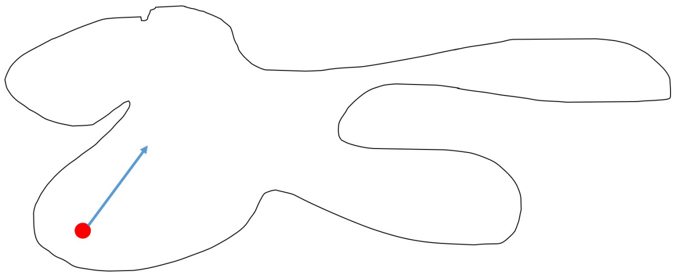

Figure 6: Recurrence problem

Actually, it is not only applied to mechanical systems but also any system that provides measure theoritic structure. Shortly, certain system will, after a sufficiently long but finite time, return very close to the initial state.

Theorem 3.1. *Poincaré Recurrence Theorem*

Let (X, Σ, μ) be a measure space with $\mu(X) < \infty$ and $T : X \to X$ be a measure preserving transformation. Then, for any $E \in \Sigma$ with $\mu(E) > 0$ and $\mu - a.e.$ $x \in E$, the orbit $\{T^n(x)\}_{n\in\mathcal{T}}$ returns to E infinitely often. Namely,

$$\mu(\{x \in E \mid \exists N > 0 \ s.t. \ T^n(x) \notin E \ for \ all \ n > N\}) = 0.$$

Proof. From now, consider for such a measure space (X, Σ, μ), a measure preserving transformation $T : X \to X$, and a measurable set E.

Let $A = \{x \in E \mid T^n(x) \in E$ for infinitely many $n > 0\}$. Then, it suffices to show that $\mu(E \setminus A) = 0$. Also, let $F = \{x \in E \mid T^n(x) \notin E$ for all $n > 0\}$. Now,

$$E \setminus A = \bigcup_{t=0}^{\infty} T^{-t}(F) \cap E.$$

Thus, the following inequality holds by the monotonicity and subadditivity for measure.

$$\mu(E \setminus A) = \mu(\bigcup_{t=0}^{\infty} T^{-t}(F) \cap E) \leq \mu(\bigcup_{t=0}^{\infty} T^{-t}(F)) \leq \sum_{t=0}^{\infty} \mu(T^{-t}(F))$$

Now, claim that $T^{-n}(F) \cap T^{-m}(F) = \emptyset$ if $n \neq m$. To show this, suppose, to contrary, that $y \in T^{-n}(F) \cap T^{-m}(F)$ for WLOG $n > m$.

Then, since $y \in T^{-n}(F)$, $T^{n-m}(T^m(y)) = T^n(y) \in F \subseteq E$. From $n > m$, $n - m > 0$ so it contradicts to $y \in T^{-m}(F)$ which implies $T^m(y) \in F$. As a result, the claim follows, hence $T^{-t}(F)$'s are pairwisely disjoint. Thus, from the definition of measure,

$$\sum_{t=0}^{\infty} \mu(T^{-t}(F)) = \mu(\bigcup_{t=0}^{\infty} T^{-t}(F)) \leq \mu(X) < \infty.$$

Since T is measure preserving, $\mu(T^{-t}(F)) = \mu(F)$ for all $t \in \mathbb{N} \cup \{0\}$. Applying to previous results, we attain

$$0 \leq \mu(E \setminus A) \leq \sum_{t=0}^{\infty} \mu(F) = \sum_{t=0}^{\infty} \mu(T^{-t}(F)) < \infty.$$

Since countably infinite sum of a constant $\mu(F)$ is finite, $\mu(F) = 0$.

$$\therefore \mu(E \setminus A) = 0$$

□

Note that this theorem deals with arbitrary measure space and measure preserving transformation. From this generality, for each given dynamical system, modeling a measure space and a measure preserving transformation appropriately leads to conclude some results with Poincaré Recurrence Theorem. This is why the Poincaré Recurrence Theorem is a powerful result in Dynamical Systems and Ergodic Theory. For example, this theorem is applied to isolated mechanical system subject to some constraints, such as the system that all particles must be bound to a finite volume, similar to the above problem.

4 Ergodic Theorem

More precise informations are provided by Ergodic Theorem. Here, I want to introduce an Ergodic Theorem without proof, and an explicit example for applying Ergodic Theorem. Shortly, Ergodic Theorem says that under certain conditions, time average is equal to space average.

Now, let $T : X \to X$ be a measure preserving transformation on a measure space (X, Σ, μ) and $f \in L^1(X, \mu)$.

Definition 4.1.

- Time average is an average over iterations of T starting from an initial point $x \in X$.

$$\hat{f}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x)$$

  Note that it may not exist.

- Space average is defined when $0 < \mu(X) < \infty$ as follows.

$$\bar{f} = \frac{1}{\mu(X)} \int_X f \, d\mu$$

With above definition, the Ergodic Theorem is formally as below.

Theorem 4.1. *Ergodic Theorem*

If T is ergodic and μ is invariant, $\hat{f}(x)$ exists and $\hat{f} = \bar{f}$ for $\mu - a.e.\ x$. Namely,

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} f(T^i x) = \frac{1}{\mu(X)} \int_X f \, d\mu \text{ for } \mu - a.e.\ x.$$

This theorem will provide various powerful meanig according to the measure space, the measure preserving transformation, and the L^1 function. I recommend you to consider this theorem in probability space. Now, I will finish this paper with an explicit example that I told above. That is, for Lebesgue measure space (X, Σ, μ) where $X = [0, 1)$, integrating a special function $f : X \to \mathbb{R}$ defined as below.

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & \text{otherwise} \end{cases}$$

This problem is easy exercise for students studying Lebesgue Integration Theory. However, in here, I want to show that this problem is can also be solved by Ergodic Theorem.

Theorem 4.2. *For such* (X, Σ, μ) *and* f*,*

$$\int_X f \ d\mu = 1.$$

Proof. Fix any $\alpha \notin \mathbb{Q}$, and let $\Lambda = \{q - m\alpha \mid q \in \mathbb{Q},\ m \in \mathbb{Z}\}$. Then, R_α is ergodic and $\forall x \in X \setminus \Lambda$,

$$1 - \frac{1}{n} \leq \frac{1}{n}\sum_{i=0}^{n-1} f(R_\alpha^i x) \leq 1$$

for all $n \in \mathbb{N}$ so that it converges to 1.

By Ergodic Theorem,

$$\int_X f \ d\mu = \frac{1}{\mu(X)} \int_X f \ d\mu = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} f(R_\alpha^i x)$$

for $\mu - a.e.\ x$. Since Λ is countable, it is measure zero so that the Ergodic Theorem forces the integration to be 1.

$$\therefore \int_X f \ d\mu = 1$$

□

References

[1] *An Introduction to Measure Theory*, Terence Tao

[2] *Introduction to Dynamical Systems*, M. Brin & G. Stuck