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1.

Let $p : \mathbb{S}^2 \to \mathbb{P}^2$ be the covering map induced by $p(-x) = p(x)$ with $\mathbb{S}^2 \subset \mathbb{R}^3$. And let f be a map of $\mathbb{P}^2$ to itself with $f_\#(\pi_1(\mathbb{S}^2, e)) \neq 1$, where $e = (1, 0, 0)$. Choose $e' \in p^{-1}(f(p(e)))$. Then, $p : (\mathbb{S}^2, e') \to (\mathbb{P}^2, f(p(e)))$ is also a covering map. Put $g = f \circ p$. Since $\mathbb{S}^2$ is simply connected, by the lifting theorem, there exists a lift $\bar{g}$ of g such that $\bar{g}(e) = e'$.

From $p\circ\bar{g} = g = f\circ p$, it follows that $\bar{g}(-x) = \pm\bar{g}(x)$ for each $x \in \mathbb{S}^2$. Then, since $\mathbb{S}^2$ is connected and a mapping $x \in \mathbb{S}^2 \mapsto < \bar{g}(-x), \bar{g}(x) > \in \{-1, 1\}$ is continuous, $x \mapsto < \bar{g}(-x), \bar{g}(x) >$ is a constant map -1 or 1, where $< \cdot, \cdot >$ is the usual inner product.

Assume that $\bar{g}(-x) = +\bar{g}(x)$ for every $x \in \mathbb{S}^2$. Then, $\bar{g}$ factors through $\mathbb{P}^2$. Since p is a quotient map, $\bar{g} = h \circ p$ defines a continuous map $h : (\mathbb{P}^2, p(e)) \to (\mathbb{S}^2, e')$. Then, h is a lift of f since p is surjective, and so $f_\#(\pi_1(\mathbb{P}^2, p(e))) \subseteq p_\#(\pi_1(\mathbb{S}^2, e')) = p_\#(1) = 1$. This contradicts that $f_\#(\pi_1(\mathbb{S}^2, e)) \neq 1$.

Hence, $\bar{g}(-x) = -\bar{g}(x)$ for every $x \in \mathbb{S}^2$. This yields the map T.

2.

Let $\pi : \mathbb{R} \to \mathbb{S} \subset \mathbb{C}$ be the exponential map and let $f : (\mathbb{S}, 1) \to (\mathbb{S}, 1)$ be a map. Suppose that f is of degree 1. This implies that every lift of $f \circ \pi|_{[0,1]}$ having value 0 at 0 has value 1 at 1.

Now, since $\mathbb{R}$ is simply connected, by the lifting theorem, there exists a lift $g : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ of $f \circ \pi$. Then, $g|_{[0,1]}$ forms a lift of $f \circ \pi|_{[0,1]}$ with $g|_{[0,1]}(0) = 0$, and so $g(1) = 1$. By the mapping of π, we may find that $g(x + 1) - g(x) \in \mathbb{Z}$ for each $x \in \mathbb{R}$. Since $\mathbb{R}$ is connected and a mapping $x \in \mathbb{R} \mapsto g(x + 1) - g(x) \in \mathbb{Z}$ is continuous, together with $g(0) = 0$ and $g(1) = 1$, we have $g(x + 1) - g(x) = 1$ for every $x \in \mathbb{R}$.

Put $h(x) = g(x) - x$. Then, h is periodic with period 1 and factors through $\mathbb{S}$. Since π is an identification map, $h = h' \circ \pi$ defines a continuous map $h' : \mathbb{S} \to \mathbb{R}$. Note that $(f \circ \pi)(x) = (\pi \circ g)(x) = (\pi \circ h)(x)\pi(x) = (\pi \circ h' \circ \pi)(x)\pi(x)$ and so $f(z) = z(\pi \circ h')(z)$ since π is surjective.

From the fact that $\mathbb{R}$ is contractible, it follows that $h' \simeq c_0$ and so $\pi \circ h' \simeq c_1$. By multiplying a homotopy between $\pi \circ h'$ and c_1 by z, we may obtain a homotopy between f and $id_{\mathbb{S}}$, that is, $f \simeq id_{\mathbb{S}}$.

1.

Suppose that these actions coincide. To show that $\pi_1(Y, y_0)$ is abelian, choose α and β there. Then, for every x in a fiber $F = p^{-1}(y_0)$, $x \cdot (\alpha\beta) = \Theta(\alpha\beta) \cdot x = (\Theta(\alpha) \circ \Theta(\beta)) \cdot c = \Theta(\alpha) \cdot (\Theta(\beta) \cdot x) = \Theta(\alpha) \cdot (x \cdot \beta) = (x \cdot \beta) \cdot \alpha = x \cdot (\beta\alpha)$, or

$x \cdot (\alpha\beta\alpha^{-1}\beta^{-1}) = x$, which means that $\alpha\beta\alpha^{-1}\beta^{-1} \in p_{\#}(\pi_1(X, x)) = p_{\#}(1) = 1$. Thus, $\alpha\beta = \beta\alpha$ and so $\pi_1(Y, y_0)$ is abelian.

Conversely, suppose that $\pi_1(Y, y_0)$ is abelian. Choose $x_0 \in F = p^{-1}(y_0)$. For every $x \in F$, there exists $\beta \in \pi_1(Y, y_0)$ such that $x_0 \cdot \beta = x$ since the monodromy action of $\pi_1(Y, y_0)$ on F is transitive. Then, we have $\Theta(\alpha) \cdot x = \Theta(\alpha) \cdot (x_0 \cdot \beta) = (\Theta(\alpha) \cdot x_0) \cdot \beta = (x_0 \cdot \alpha) \cdot \beta = x_0 \cdot (\alpha\beta) = x_0 \cdot (\beta\alpha) = (x_0 \cdot \beta) \cdot \alpha = x \cdot \alpha$. Here, we used Proposition 6.2 and our supposion. The actions coincide.

1.

Write $G = \{g_1 = e, g_2, \cdots, g_n\}$, where g_i's are all distinct. Let $x \in X$. Then, $G(x) = \{g_1 \cdot x, g_2 \cdot x, \cdots, g_n \cdot x\}$. If $g_i \cdot x = g_j \cdot x$, then $(g_j^{-1} g_i) \cdot x = x$ and this implies $g_j^{-1} g_i = e$, i.e., $g_i = g_j$. So, all of $g_i \cdot x$ are distinct. Since X is Hausdorff, there exist mutually disjoint open sets $U_1, U_2, \cdots, U_n$ in X such that $g_i \cdot x \in U_i$ for every $i = 1, 2, \cdots, n$. Put $U = \cap_{i=1}^n g_i^{-1}(U_i)$. Obviously, U is an open set in X containing x. Suppose $g_k(U) \cap U \neq \emptyset$. Then, we have $\emptyset \neq g_k(U) \cap U \subseteq g_k(g_k^{-1}(U_k)) \cap U_1 = U_k \cap U_1$. By the choice of U_i's, we obtain $k = 1$. Thus, $g_k = g_1 = e$. The action of G is properly discontinuous.

4.