20143698 Junho, Choe

2.

In order to show $\pi_1(\mathbb{P}^2) \cong \mathbb{Z}_2$, the following suffices: $\pi_1(\mathbb{P}^2)\setminus 1 \neq \emptyset$ and $[f]\cdot[g] = 1$ whenever $[f], [g] \in \pi_1(\mathbb{P}^2) \setminus 1$.

Let $p : \mathbb{S}^2 \to \mathbb{P}^2$ be the quotient map induced by $p(-x) = p(x)$ with $\mathbb{S}^2 \subset \mathbb{R}^3$. We know that p is a covering map. Let $e = (1, 0, 0) \in \mathbb{S}^2$. Considering a path on $\mathbb{S}^2$ which starts form e and ends to $-e$, one can obtain a loop on $\mathbb{P}^2$ not contained in $p_\#(\pi_1(\mathbb{S}^2, e)) = p_\#(1) = 1$. Let f and g be maps from $(I, \partial I)$ to $(\mathbb{P}^2, p(e))$ with $[f], [g] \in \pi_1(\mathbb{P}^2, p(e)) \setminus 1$. By the path lifting property, together with $p^{-1}(p(e)) = \{e, -e\}$, there exist liftings $\bar{f}$ and $\bar{g}$ of f and g, respectively, such that $\bar{f}(0) = e$ and $\bar{g}(0) = -e$.

Then, we may find that for a loop $h : (I, 0) \to (\mathbb{P}^2, p(e))$, $[h] = 1$ if a lift $\bar{h}$ of h is a loop. It is because $[h] \in p_\#(\pi_1(\mathbb{S}^2, \bar{h}(0) = \bar{h}(1))) = p_\#(1) = 1$.

Hence, $f(1) = -e = g(0)$ and $g(1) = e$ since $\bar{f}$ and $\bar{g}$ satisfy $\bar{f}(1) = \pm e$ and $\bar{g}(1) = \pm e$ as liftings of loops. So, $\bar{f} * \bar{g}$ exists and forms a lift of $f * g$ which is a loop. We have $[f] \cdot [g] = [f * g] \in p_\#(\pi_1(\mathbb{S}^2, e)) = p_\#(1) = 1$.

4.

Let α and β in $\pi_1(E, 0')$ satisfy that α contains a loop f running one lap counter-clockwise around the right-sided cycle of E and β does a loop g doing so around the left-sided one, where E means the figure eight space and $0'$ is the 'junction point' of E. Consider a loop $h : (I, \partial I) \to (E, 0')$ such that $h = f * g * f^{-1} * g^{-1}$. We notice that if $[h] \neq 1$, then $\alpha\beta\alpha^{-1}\beta^{-1} \neq 1$ and so $\pi_1(E, 0')$ is not abelian.

Let us show $[h] \neq 1$. Consider the covering map and the covering space given in Figure III-5. Write the preimage of $0'$ as $\{1', 2', 3'\}$, where $1'$ is the 'lowest' point and $3'$ is the 'highest' point in Figure III-5. By the path lifting property, there exists a lift $\bar{h}$ of h such that $\bar{h}(0) = 3'$. Then, since $\bar{h}$ is a lift of $f * g * f^{-1} * g^{-1}$, obviously $\bar{h}(1) = 1'$ and so $\bar{h}$ is not a loop. We obtain $[h] \notin p_\#(\pi_1(\bar{E}, 3')) \ni 1$, that is, $[h] \neq 1$, where $\bar{E}$ is the covering space,

1.

We know that the quotient map $p : \mathbb{S}^m \to \mathbb{P}^m$ induced by $p(-x) = p(x)$, is a covering map. Let $f : (\mathbb{S}^n, e_n) \to (\mathbb{P}^m, p(e_m))$ be a map, where $e_l = (1, 0, \cdots, 0) \in \mathbb{R}^{l+1}$, $l = n, m$. If $[f] = 1$, then the desired result follows.

Note that $f_\#(\pi_1(\mathbb{S}^n, e_n)) = f_\#(1) = 1 \subseteq p_\#(\pi_1(\mathbb{S}^m, e_m))$ due to $n > 1$. The lifting theorem gives that there exists a lift $\bar{f}$ of f with $\bar{f}(e_n) = e_m$. Then, $[\bar{f}] \in \pi_n(\mathbb{S}^m, e_m) = 1$ since $m > n$. Finally, we have $[f] = [p \circ \bar{f}] = p_\#[\bar{f}] = p_\#(1) = 1$.

2.

Let $f : \mathbb{P}^2 \to \mathbb{S}^1$ be a map. Let $p : (\mathbb{S}^2, e) \to (\mathbb{P}^2, p(e))$ be the covering map induced by $p(-x) = p(x)$, where $\mathbb{S}^2 \subset \mathbb{R}^3$ and $e = (1, 0, 0)$. Write $z_0 = f(p(e))$. Let $q : (\mathbb{R}, 0) \to (\mathbb{S}^1, z_0)$ be defined by $q(x) = z_0 \exp(2\pi i x)$, where $\mathbb{S}^1 \subset \mathbb{C}$. Then, similarly to the exponential map $x \mapsto \exp(2\pi i x)$, q is also a covering map. Put $g = f \circ p$. Note that $\mathbb{S}^2$ is simply connected. Thus, by the lifting theorem, there exists a lift $\bar{g}$ of g such that $q \circ \bar{g} = g = f \circ p$ and $\bar{g}(e) = 0$.

We claim that $\bar{g}$ factors through $\mathbb{P}^2$. Note that $(q \circ \bar{g})(-x) = (f \circ p)(-x) = (f \circ p)(x) = (q \circ \bar{g})(x)$ for every $x \in \mathbb{S}^2$, and that $\bar{g}(-x) - \bar{g}(x) \in \mathbb{Z}$ for every $x \in \mathbb{S}^2$ by the mapping of q. Since $\mathbb{S}^2$ is connected and a mapping $x \in \mathbb{S}^2 \mapsto \bar{g}(-x) - \bar{g}(x) \in \mathbb{Z}$ is continuous, there exists an integer n such that $\bar{g}(-x) - \bar{g}(x) = n$ for every $x \in \mathbb{S}^2$. Then, $n = \bar{g}(-x) - \bar{g}(x) = -\{\bar{g}(-(-x)) - \bar{g}(-x)\} = -n$, i.e., $n = 0$. So, $\bar{g}(-x) = \bar{g}(x)$ for $x \in \mathbb{S}^2$.

Since p is an identification map, $\bar{g} = h \circ p$ defines a continuous map $h : (\mathbb{P}^2, p(e)) \to (\mathbb{R}, 0)$. We have $f \circ p = q \circ \bar{g} = q \circ h \circ p$ and $f = q \circ h$ since p is surjective. Then, $h \simeq c_0$ follows from the fact that $\mathbb{R}$ is contractible. $f = q \circ h$ implies $f \simeq c_{z_0}$.