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1.

One may say the following: $f \bullet g$ means that f and g go together, but $f * g$ does that f goes twice faster and then rests but g rests and then goes twice faster. This yields the following homotopy.

Let $\mathbb{S}^1 \subset \mathbb{C}$ be a pointed space $(\mathbb{S}^1, 1)$. Define $H : \mathbb{S}^1 \times I \to G$ by

$$H(\exp(\theta i), t) = f(\exp(s_1(\frac{t+1}{2}\theta)i))g(\exp(s_2(\frac{t+1}{2}\theta + 2\pi(\frac{1-t}{2}))i))$$

for each $0 \le \theta < 2\pi$, where

$$s_1(x) = \begin{cases} 2x & 0 \le x < \pi \\ 2\pi & \pi \le x \le 2\pi \end{cases}$$

and

$$s_2(x) = \begin{cases} 0 & 0 \le x < \pi \\ 2(x-\pi) & \pi \le x \le 2\pi. \end{cases}$$

Since the right-hand side in the definition of H is obviously continuous at (θ, t) for every $(\theta, t) \in [0, 2\pi] \times I$, together with $s_1(0) = 0$, $s_1(\frac{t+1}{2}2\pi) = 2\pi$, $s_2(2\pi(\frac{1-t}{2})) = 0$ and $s_2(\frac{t+1}{2}2\pi + 2\pi(\frac{1-t}{2})) = 2\pi$, H is continuos on $\mathbb{S}^1 \times I$ with $H(1, t) = e, t \in I$. Then,

$$\begin{aligned} H(\exp(\theta i), 0) &= f(\exp(s_1(\frac{1}{2}\theta)i))g(\exp(s_2(\frac{1}{2}\theta + \pi)i)) \\ &= f(\exp(\theta i))g(\exp(\theta i)) \\ &= (f \bullet g)(\exp(\theta i)) \end{aligned}$$

and

$$\begin{aligned} H(\exp(\theta i), 1) &= f(\exp(s_1(\theta)i))g(\exp(s_2(\theta)i)) \\ &= \begin{cases} f(\exp(2\theta i))e & 0 \le \theta < \pi \\ eg(\exp(2(\theta - \pi)i)) & \pi \le \theta 2\pi \end{cases} \\ &= (f * g)(\exp(\theta i)). \end{aligned}$$

By the homotopy, we find that $fg \simeq f * g \; rel 1$.

2.

Modifying the above homotopy, we may obtain a homotopy between $g \bullet f$ and $f * g$. With the above setting, put

$$G(\exp(\theta i), t) = g(\exp(s_2(\frac{t+1}{2}\theta + 2\pi(\frac{1-t}{2}))i))f(\exp(s_1(\frac{t+1}{2}\theta)i)).$$

In other words, G is given by interchanging the $f(\cdots)$ term and the $g(\cdots)$ term. Similarly to H, G is a homotopy with $G(1,t) = e, t \in I$. Thus, we have $g \bullet f \simeq f * g$ $rel1$. Then, by the result of problem 1, we also have $g \bullet f \simeq g * f$ $rel1$. Finally, it follows that $f * g \simeq g \bullet f \simeq g * f$ $rel1$, which means that the fundamental group $\pi_1(G, e)$ is abelian.

3.

Write $\mathbb{K}^2 = p(\mathbb{R}^2)$, where p is a quotient map induced by $(x+2\pi, y) \sim (x, y), (x, y+2\pi) \sim (-x, y)$. For each $z = \exp(\theta i) \in \mathbb{S}^1 \subset \mathbb{C}$, let $a(z) = p(0, \theta)$ and let $b(z) = p(\theta, 0)$. Since the maps $t \mapsto p(0,t)$ and $t \mapsto p(t,0)$ are continuous, and since $p(0, t+2\pi) = p(-0, t)$ and $p(t+2\pi, 0) = p(t, 0)$ for every $t \in \mathbb{R}$, a and b are well-defined and continuous with $a(1) = p(0,0) = b(1)$.

Then, we claim that $\pi_1(\mathbb{K}^2)$ is generated by $[a]$ and $[b]$. Let $[f] \in \pi_1(\mathbb{K}^2, p(0,0))$. It is well known that $\mathbb{K}^2$ is a 2-dimensional manifold. By Theorem 11.8 in Chapter 2, there exists a smooth map $\bar{f}$ with $\bar{f} \simeq f$ $rel1$, where $1 \in (\mathbb{S}^1, 1)$ with $\mathbb{S}^1 \subset \mathbb{C}$. Also, Since each smooth loop must miss a point, the smooth loop $\bar{f}$ must do a point $q \in \mathbb{K}^2$. Moreover, obviously we can choose such a point q so that q is not in images of a and b. Deleting the point q yields a 'hole'. We 'expand' the 'hole' leaving a and b fixed, and obtain a space homeomorphic to $\mathbb{K}^2 \setminus \{q\}$ with a loop $\bar{f}^*$ corresponding to $\bar{f}$. This space is given by the following diagram:

Focus on the straps A and B in the preceding diagram. If the first part of the loop $\bar{f}^*$ turns a lap around strap A as the loop a (a^{-1}), then it is indeed homotopic to a (a^{-1}) relatively to 1. On the other hand, if the first part does so around strap B as the loop b (b^{-1}), the it is so to b (b^{-1}) relatively to 1. Consider the second part starting from the end of first part. We may do a similar thing. Continue this process. Since $\bar{f}$ is smooth and $\mathbb{S}^1$ is compact, the process ends with finite steps, which means that $[f]$ is a finite product of copies of $[a]$,$[a]^{-1}$,$[b]$ and $[b]^{-1}$. $[a]$ and $[b]$ generate $\pi_1(\mathbb{K}^2, p(0,0))$.

Now, by taking a homotopy, we will show $[a][b][a]^{-1} = [b]^{-1}$. Let $\mathbb{S}^1 \subset \mathbb{C}$. Define $H : \mathbb{S}^1 \times I \to \mathbb{K}^2$ by

$$H(\exp(\theta i), t) = \begin{cases} p(0, (t(3\theta))) & 0 \leq \theta < \frac{2\pi}{3} \\ p(-3(\theta - \frac{2\pi}{3})), 2\pi t) & \frac{2\pi}{3} \leq \theta < 2 \cdot \frac{2\pi}{3} \\ p(2\pi, 2\pi t - t(3(\theta - 2 \cdot \frac{2\pi}{3}))) & 2 \cdot \frac{2\pi}{3} \leq \theta < 2\pi. \end{cases}$$

Here, the mapping $(\theta, t) \mapsto (2\pi, 2\pi t - t(3(\theta - 2 \cdot \frac{2\pi}{3})))$ on $\mathbb{R}^2$ is continuous everywhere with value $(2\pi, 0)$ at $(2\pi, t), t \in I$, and $p(2\pi, 0) = p(0, 0)$. Thus, H is continuous. And we have

$$H(\exp(\theta i), 0) = \begin{cases} p(0, 0) & 0 \leq \theta < \frac{2\pi}{3} \\ p(-3(\theta - \frac{2\pi}{3})), 0) & \frac{2\pi}{3} \leq \theta < 2 \cdot \frac{2\pi}{3} \\ p(0, 0) & 2 \cdot \frac{2\pi}{3} \leq \theta < 2\pi, \end{cases}$$

$$H(\exp(\theta i), 1) = \begin{cases} p(0, 3\theta) & 0 \leq \theta < \frac{2\pi}{3} \\ p(3(\theta - \frac{2\pi}{3}), 0) & \frac{2\pi}{3} \leq \theta < 2 \cdot \frac{2\pi}{3} \\ p(0, -3(\theta - 2 \cdot \frac{2\pi}{3})) & 2 \cdot \frac{2\pi}{3} \leq \theta < 2\pi \end{cases}$$

and $H(1, t) = p(0, 0)$. Hence, we find that $[C_{p(0,0)} * b^{-1} * C_{p(0,0)}] = [a * b * a^{-1}]$, that is, $[a][b][a]^{-1} = [b]^{-1}$.