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4.

Let N be a neighborhood of e in G. There exists an open set U in G with $e \in U \subset N$. Define $f : G \times G \to G$ by $f(x, y) = xyx^{-1}$. As a composition of multiplications and an inversion, f is continuous. Take $V = f(G \times (U^c))^c$.

We may prove that V is open with $e \in V \subset N$ as follows. Suppose that $e \in V^c$. Then, for some $g \in G$ and some $u' \in U^c$, $e = gu'g^{-1}$ and so $e = u' \in U^c$. But $e \in U$. So, e is in V. Let $v \in V$. Assume that $v \notin U$. Then, $(e, v) \in G \times (U^c)$, which means that $v \in f(G \times (U^c)) = V^c$. This is a contradiction. We find that $V \subset U$ and thus $V \subset N$. And we notice that U^c is compact as a closed subset of the compact space G, and that $G \times (U^c)$ is also compact as a finite Cartesian product of compact spaces, which implies that $f(G \times (U^c))$ is compact since f is continuous. Note that the topological group G is Hausdorff. Hence, $f(G \times (U^c))$ is closed in G and its complement V is open in G.

Now, we claim that V is invariant under conjugation. Let $w \in V$ and let $g_1 \in G$. Suppose that $g_1 w g_1^{-1} \notin V$. We have $g_1 w g_1^{-1} = g_2 u'' g_2^{-1}$ for some $g_2 \in G$ and some $u'' \in U^c$. Then, $w = (g_1^{-1} g_2) u'' (g_1^{-1} g_2)^{-1} = f(g_1^{-1} g_2, u'') \in f(G \times (U^c))$. However, w was chosen in V and we obtain a contradiction. So, $g_1 V g_1^{-1} \subset V$ for every $g_1 \in G$. Also, for each $g_1 \in G$, $V = g_1(g_1^{-1} V (g_1^{-1})^{-1}) g_1^{-1} \subset g_1 V g_1^{-1}$. Therefore, $g_1 V g_1^{-1} = V$ for every $g_1 \in G$.

5.

Let X be an open and closed subset of G. Replacing X by X^c if necessary, we can suppose that X intersects with the nonempty set H. Then, $X \cap H$ is nonempty, open and closed in the connected space H. So, $X \cap H = H$ and thus $X \supset H$. Also, since $X \neq \emptyset$, $p^{-1}(p(X)) = XH = X$ and $p^{-1}(p(X)^c) = X^c$, we find that $p(X)$ is nonempty, open and closed in the connected space G/H, which means that $p(X) = G/H$ and thus $X = p^{-1}(p(X)) = p^{-1}(G/H) = G$. Actually, by the replacement, we have $X = \emptyset$ or G. Therefore, G is connected.

8.

To prove that $\mathbb{SO}(n)$ is connected, one can use the mathematical induction. First, $\mathbb{SO}(1)$ is connected as an one-point space $\{[1]\}$. Next, suppose that $\mathbb{SO}(n-1)$ is connected for some positive integer $n \geq 2$. For this n, we know that $\mathbb{SO}(n)/\mathbb{SO}(n-1) \cong \mathbb{S}^{n-1}$, and that $\mathbb{S}^{n-1}$ is connected. By a theorem, $\mathbb{SO}(n)$ is connected. The induction gives the connectedness of $\mathbb{SO}(n)$ for each positive integer n.

$I_n \in \mathbb{SO}(n)$ is trivial. And, since $\mathbb{SO}(n)$ is connected, $\mathbb{SO}(n) \subset C$ for some component C of $\mathbb{O}(n)$. Note that det is a continuous map with $+1 \in \det(C) \subset \{-1, +1\}$, and that $\{+1\}$ is open and closed in $\{-1, +1\}$. Thus, $(\det|_C)^{-1}(+1)$ is C as a nonempty, open and closed subset of the component C. We have $C = \mathbb{SO}(n)$, and so $\mathbb{SO}(n)$ is a component of $\mathbb{O}(n)$.

11.

Obviously, $\mathbb{O}(n+1)$ acts on $\mathbb{RP}^n$ by $A \cdot p(x) = p(Ax), A \in \mathbb{O}(n+1), x \in \mathbb{R}^{n+1} \setminus \{0\}$, where $p : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ is the quotient map. Then, the action is transitive. For showing, choose $p(x)$ and $p(y)$ in $\mathbb{RP}^n$ with $p(x) \neq p(y)$. Here, x and y are linearly independent by the mapping of p. Then, by Gram-Schmidt orthonormalization, there exists an orthonormal basis $v_1, v_2, \cdots, v_{n+1}$ for $\mathbb{R}^{n+1}$ with $v_1 = x/\|x\|$ and $v_2 = (y - (v_1 \cdot y)v_1)/\|(y - (v_1 \cdot y)v_1)\|$. Put $L(v) = (v_1 \cdot v)u_1 + (v_2 \cdot v)u_2 + \sum_{i=3}^{n+1}(v_i \cdot v)v_i$ for each $v \in \mathbb{R}^{n+1}$, where $u_1 = \cos\theta v_1 + \sin\theta v_2$, $u_2 = -\sin\theta v_1 + \cos\theta v_2$ and $\theta = \cos^{-1}((x \cdot y)/\|x\|\|y\|)$. We observe that $u_1, u_2, v_3, \cdots, v_{n+1}$ form an orthonormal basis for $\mathbb{R}^{n+1}$, and that L is a linear transformation mapping between orthonormal bases. Hence, $L \in \mathbb{O}(n+1)$. And a simple calculation gives $L(x) = (\|x\|/\|y\|)y$, which implies that $L \cdot p(x) = p(L(x)) = p((\|x\|/\|y\|)y) = p(y)$.

By a theorem, we have $\mathbb{O}(n+1)/\mathbb{O}(n+1)_{p(e)} \cong \mathbb{O}(n+1)(p(e)) = \mathbb{RP}^n$, where $e = (0, 0, \cdots, 0, 1) \in \mathbb{R}^{n+1}$. Let us find $\mathbb{O}(n+1)_{p(e)}$. Take S in $\mathbb{O}(n+1)_{p(e)}$. Then, $Se = \lambda e$ for some nonzero real number λ. If $c_1, c_2, \cdots, c_{n+1}$ are the column vectors of S, then thay are orthonormal with $c_{n+1} = \lambda e$. Thus, $\lambda = \pm 1$, and $c_1, c_2, \cdots, c_n$ are orthonormal in $\mathbb{R}^n \times \{0\}$. So, $S = diag(S', \pm 1)$ for some $S' \in \mathbb{O}(n)$. Writing $\mathbb{O}(n) \times \mathbb{O}(1) = \{diag(A_1, A_2) | A_1 \in \mathbb{O}(n), A_2 \in \mathbb{O}(1)\}$, we obtain $\mathbb{O}(n+1)_{p(e)} \subset \mathbb{O}(n) \times \mathbb{O}(1)$. Evidently, the reverse inclusion also holds. $\mathbb{O}(n+1)_{p(e)} = \mathbb{O}(n) \times \mathbb{O}(1)$. In conclusion, $\mathbb{RP}^n \cong \mathbb{O}(n+1)/\mathbb{O}(n) \times \mathbb{O}(1)$.

Similarly, we can prove that $\mathbb{CP}^n \cong \mathbb{U}(n+1)/\mathbb{U}(n) \times \mathbb{U}(1)$. But we use the Hermitian inner product $< \cdot, \cdot >$ instead of the Euclidean inner product. And, modify L by $L(v) = < v_1, v > (< v_1, y/\|y\| > v_1 + < v_2, y/\|y\| > v_2) + < v_2, v > (-\overline{< v_2, y/\|y\| >} v_1 + \overline{< v_1, y/\|y\| >} v_2) + \sum_{j=3}^{n+1} < v_j, v > v_j$. Moreover, it follows that $|\mu| = 1$ from the corresponding equation: $Te = \mu e$, $T \in \mathbb{U}(n+1)$, $\mu \in \mathbb{C} \setminus \{0\}$.