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4.

Put $f(z) = z^2$. Write C_f as $p_1((S^1_1 \times I) \bigsqcup S^1_2)$. Here, p_1 is the quotient map, $S^1_1 = S^1_2 = S^1 \subset \mathbb{C}$, and $\bigsqcup$ means the disjoint union. Define a map $g : (S^1_1 \times I) \bigsqcup S^1_2 \to \mathbb{P}^2 = p_2(D^2)$ by

$$\begin{cases} g(x,t) = p_2((1-t)x) \text{ if } (x,t) \in S^1_1 \times I, \\ g(y) = p_2(\sqrt{y}) \text{ if } y \in S^1_2, \end{cases}$$

where p_2 is the quotient map of the 2-dimensional disk $D^2 \subset \mathbb{C}$ onto $\mathbb{P}^2$, and $\sqrt{y} = \exp(\frac{i}{2}\arg(y))$, $-\pi < \arg < \pi$. Since arg is continuous on $S^1_2 \setminus \{-1\}$, so is g. And we have $g(y) = p_2(-i\exp(\frac{i}{2}\arg'(y)))$ for every $y \in S^1_2 \setminus \{1\}$, where $0 < \arg' < 2\pi$. So, g is continuous on $S^1_2 \setminus \{1\}$. By the gluing lemma for open subsets, we find that it is continuous on S^1_2. Evidently, g is continuous on $S^1_1 \times I$ and thus continuous everywhere.

Now, $g(x,1) = p_2(0)$ for every $x \in S^1_1$, and if $y = f(x)$ for some $y \in S^1_2$ and some $x \in S^1_1$, then $x = \pm\sqrt{y}$ and so $g(x,0) = g(y)$. We can take a canonical map h of C_f into $\mathbb{P}^2$ with $h \circ p_1 = g$.

Then, h is a homeomorphism between C_f and $\mathbb{P}^2$. Since g is continuous and surjective, so is h. To check that h is injective, suppose $h(z) = h(w)$ for some $z, w \in C_f$. Since $p_1(S^1_1 \times I) = C_f$, we write $z = p_1(x_1, t_1)$ and $w = p_1(x_2, t_2)$ for some $x_j \in S^1_1, t_j \in I, j = 1, 2$. We have $p_2((1-t_1)x_1) = p_2((1-t_2)x_2)$. By the definition of p_2 and by the usual norm on D^2, we obtain the following result:

$$t_2 = 0 \text{ and } x_2 = \pm x_1 \text{ if } t_1 = 0,$$

and

$$t_2 = t_1 \text{ and } x_2 = x_1 \text{ if } t_1 \neq 0.$$

Thus, we have $z = p_1(x_1, t_1) = p_1(x_2, t_2) = w$, that is, h is one-to-one. Note that C_f is compact due to the compact space $(S^1_1 \times I) \bigsqcup S^1_2$, and that $\mathbb{P}^2$ is a T_2 space. By a theorem, h is a homeomorphism between C_f and $\mathbb{P}^2$.

6.

For $0 \leq \theta < 2\pi$, put

$$H(\exp(\theta i), t) = \begin{cases} \exp(4\theta i) & \text{if } 0 \leq \theta < \pi/2, \\ \exp(4t(\theta - \pi/2)i) & \text{if } \pi/2 \leq \theta < \pi, \\ \exp(-2t(\theta - \pi)i + 2\pi t i) & \text{if } \pi \leq \theta < 2\pi, \end{cases}$$

and put $f_j(\exp(\theta i)) = H(\exp(\theta i), j), j = 0, 1$. Since H is a homotopy of maps from S^1 to S^1, we have $f_1 \simeq f_0$. Then, f_0 is obviously homotopic to 1_{S^1}. Explicitly, the following map $G : S^1 \times I \to S^1$ is a homotopy between f_0 and 1_{S^1}:

$$G(\exp(\theta i), t) = \begin{cases} \exp(\frac{2\pi}{M(t)}\theta i) & \text{if } 0 \leq \theta < M(t), \\ 1 & \text{if } M(t) \leq \theta < 2\pi, \end{cases}$$

where $0 \leq \theta < 2\pi$, and $M(t) = \pi/2 + (3\pi/2)t$. Then, by a theorem, we have $C_{f_1} \simeq C_{1_{S^1}}$.

Let us consider C_{f_1}. We observe that $f^{-1}(\{\exp(\theta i)\}) = \{\exp((\theta/4)i), \exp((\pi/2)i + (\theta/4)i), \exp((2\pi - \theta/2)i)\}$ for each $\theta \in [0, 2\pi)$. Hence, we obtain the following diagram:

This means that C_{f_1} is homeomorphic to the dunce cap space. On the other hand, $C_{1_{S^1}} \cong D^2$ can be explained as follows:

In conclusion, the dunce cap space is contractible since it is homotopy equivalent to the contractible space D^2.

8.

Let X be a contractible space, and let A be a retract of X with a retraction $f : X \to A$. By the choice of X and by a theorem, there exists a homotopy $H : X \times I \to X$ with a point $x_0 \in X$, such that $H(x,0) = 1_X(x)$ and $H(x,1) = x_0$ for every $x \in X$. Take $G = f \circ (H|_{A\times I})$. Trivially, G is a continuous mapping of $A \times I$ into A. Also, $G(a,0) = 1_A(a)$ and $G(a,1) = f(x_0)$ for every $a \in A$. Here, $f(x_0)$ is a point fixed in A. Therefore, A is contractible.

9.

Put

$$\bar{g}(x) = \frac{g(x) - (f(x)\cdot g(x))f(x)}{\|g(x) - (f(x)\cdot g(x))f(x)\|},$$
$$w(x) = \cos^{-1}(f(x)\cdot g(x))$$

and

$$H(x,t) = \begin{cases} \cos(w(x)t)f(x) + \sin(w(x)t)\bar{g}(x) & \text{if } f(x) \neq g(x), \\ f(x) & \text{if } f(x) = g(x), \end{cases}$$

for every $x \in X$ and every $t \in I$. Now, since $f(x) \neq -g(x)$ in S^n for any $x \in X$, $g(x) - (f(x)\cdot g(x))f(x) \neq 0$ on Y, and we find that $\bar{g}|_Y$ is a well-defined continuous map, where $Y = \{x \in X | f(x) \neq g(x)\}$. Note that $\|H(y,t) - f(y)\| \leq \frac{1}{2}(w(y)t)^2 + w(y)t$ for every $y \in Y$ and every $t \in I$, and thus that $w(y) \to 0$ and so $\|H(y,t) - f(y)\| \to 0$ as $y \in Y$ goes outside Y. Hence, H is continuous. By a simple calculation, together with $\sin(w(x)) = \|g(x) - (f(x)\cdot g(x))f(x)\|$, we have $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for every $x \in X$. It follows that $f \simeq g$.