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1.

Let i be the inclusion map of S^n into $\mathbb{R}^{n+1}\backslash\{0\}$. Then, there exists a map j of $S^n/{\sim_1} = \pi_1(S^n)$ into $\mathbb{R}^{n+1}\backslash\{0\}/{\sim_2} = \pi_2(\mathbb{R}^{n+1}\backslash\{0\})$, defined by $j \circ \pi_1 = \pi_2 \circ i$, where π_1 and π_2 are the quotient maps.

To show that j is continuous, choose an open set U in $\mathbb{R}^{n+1}\backslash\{0\}/{\sim_2}$. Since π_2 and i is continuous, and $\pi_1^{-1}(j^{-1}(U)) = i^{-1}(\pi_2^{-1}(U))$ holds, we find that $\pi_1^{-1}(j^{-1}(U))$ is open in S^n. And $j^{-1}(U)$ is open in $S^n/{\sim_1}$, for π_1 is an identification map. Thus j is continuous.

If $y \in \mathbb{R}^{n+1}\backslash\{0\}$, then we have $j(\pi_1(y/\|y\|)) = \pi_2(y)$. And, if both z and tz are in S^n for some $t \in \mathbb{R}\backslash\{0\}$, then $t = \pm 1$, and so $\pi_1(z) = \pi_1(tz)$. Consequently, j is bijective.

We claim that $\mathbb{R}^{n+1}\backslash\{0\}/{\sim_2}$ is a T_2 space. Let $\pi_2(a)$ and $\pi_2(b)$ be contained in $\mathbb{R}^{n+1}\backslash\{0\}/{\sim_2}$ with $\pi_2(a) \neq \pi_2(b)$. We can suppose that $\|a\| = 1 = \|b\|$. Put $f(x) = \|x - (x \cdot a)a\| - \|x - (x \cdot b)b\|$ on $\mathbb{R}^{n+1}\backslash\{0\}$. Then f is continuous. Letting $V_1 = f^{-1}((-\infty, 0))$ and $V_2 = f^{-1}((0, \infty))$, we have $\pi_2^{-1}(\pi_2(V_i)) = V_i$, $i = 1, 2$, and we obtain two open sets $\pi_2(V_1)$ and $\pi_2(V_2)$ in $\mathbb{R}^{n+1}\backslash\{0\}/{\sim_2}$ with $\pi_2(a) \in \pi_2(V_1), \pi_2(b) \in \pi_2(V_2)$ and $\pi_2(V_1) \cap \pi_2(V_2) = \emptyset$. So, $\mathbb{R}^{n+1}\backslash\{0\}/{\sim_2}$ is a T_2 space. And obviously, $S^n/{\sim_1}$ is compact. Since j is a bijective continuous map, j is thus open.

In conclusion, $S^n/{\sim_1} \cong \mathbb{R}^{n+1}\backslash\{0\}/{\sim_2}$.

2.

Let O be a open set in $A \times B$. Then $O = \bigcup_{\alpha\in\Phi} U_\alpha \times V_\alpha$ for some U_α open in A, V_α open in B. Now we have

$$(f \times g)^{-1}(O) = \bigcup (f \times g)^{-1}(U_\alpha \times V_\alpha) = \bigcup (f^{-1}(U_\alpha) \times g^{-1}(V_\alpha)).$$

Since f and g are continuous, $f^{-1}(U_\alpha) \times g^{-1}(V_\alpha)$'s are open in $X \times Y$. Hence, $(f \times g)^{-1}(O)$ is open, and so $f \times g$ is continuous.

Let $P = \bigcup_{\beta\in\Psi} W_\beta \times \Omega_\beta$ be open in $X \times Y$. Note that each of $f(W_\beta)$ (respectively, $g(\Omega_\beta)$), $\beta \in \Psi$, is open in A (respectively, B) since f (respectively, g) is an open map. Therefore, $(f \times g)(P) = \bigcup f(W_\beta) \times g(\Omega_\beta)$ is open, which means that $f \times g$ is an open map.

We conclude that $f \times g$ is an open identification map.

5.

Let us see the following diagram.

6.

Assume that $\mathbb{R}/\sim$ is countable. Write it as $\{\pi(x_i) : i \in A\}$, where π is the quotient map, and $A = \emptyset, \{1, 2, \cdots, n\}$ or $\mathbb{Z}^+$. We know that $\mathbb{R} = \bigcup_{i\in A} \pi(x_i)$. Since $\mathbb{Q}$ is countable, we find that each of $\pi(x_i)$'s is countable. Thus, $\mathbb{R}$ is a countable union of countable sets, and so it is countable. But $\mathbb{R}$ is uncountable. Hence, $\mathbb{R}/\sim$ is uncountable.

Let us show that the quotient topology on $\mathbb{R}/\sim$ is indiscrete. Let U be a nonempty open set in $\mathbb{R}/\sim$, and let $\pi(a) \in U$. Note that $\pi^{-1}(U)$ is open in $\mathbb{R}$ containing the point a in $\mathbb{R}$. By the usual topology on $\mathbb{R}$, there exists a positive real number ϵ such that $(a-\epsilon, a+\epsilon) \subset \pi^{-1}(U)$. So, $\pi((a-\epsilon, a+\epsilon)) \subset U$. Let $\pi(b) \in \mathbb{R}/\sim$. By the density of $\mathbb{Q}$, there exists $q \in \mathbb{Q}$ such that $|(a-b)-g| < \epsilon$. Therefore, $\pi(b) = \pi(b+q) \in \pi((a-\epsilon, a+\epsilon)) \subset U$, for $|(b+q)-a| < \epsilon$ holds. Finally, we have $U = \mathbb{R}/\sim$, and thus $\emptyset$ and $\mathbb{R}/\sim$ are all of open sets in $\mathbb{R}/\sim$.

7.

Assume that $B \cong A = p_1(\mathbb{R})$ with a homeomorphism $f : B \to A$, where p_1 is the quotient map. We observe that $B\backslash\{(0,0)\}$ consists of countably many connected components, and that each of components has exactly one of points $(1/n, 1/n), n = 1, 2, \cdots$. Put $X = \{(1/n, 1/n) : n = 1, 2, \cdots\}$. We know that $B\backslash\{(0,0)\} \cong A\backslash\{f(0,0)\}$ with a homeomorphism $f|_{B\backslash\{(0,0)\}}$. By the observation, $f(0,0)$ must be $p_1(0)$, and each element of $f(X)$ is contained in exactly one of $p_1((m, m+1)), m \in \mathbb{Z}$. Then, $(0,0) = \lim_{n\to\infty}(1/n, 1/n)$ is not in X. So, X is not closed in B. But, considering the subspace topology on $\mathbb{R}\backslash\mathbb{Z} \subset \mathbb{R}$, we find that $f(X)$ is closed in A, which is a contradiction. We conclude that $B \ncong A$. Similary, we find that $C \ncong A \ncong D$.

C is noncompact since C is unbounded in $\mathbb{R}^2$. But the bounded and closed set B in $\mathbb{R}^2$ is compact. Also, the fact that S^1 in $\mathbb{C}$ is compact, ensures that so is D. Thus, $B \ncong C \ncong D$.

Let g be a mapping of S^1 into B, defined by

$$g(\exp(\theta i)) = \begin{cases} (0,0) & , \theta = 0, \\ (\frac{1}{n+1}\cos(a_n(\theta)), \frac{1}{n+1}\{1+\sin(a_n(\theta))\}) & , \frac{\pi}{n+1} \le \theta < \frac{\pi}{n}, n \in \mathbb{Z}^+, \\ (\cos(2\theta - \pi/2), 1 + \sin(2\theta - \pi/2)) & , \pi \le \theta < 2\pi; \end{cases}$$

for $\theta \in [0, 2\pi)$, where

$$a_n(\theta) = \frac{3\pi}{2} + 2\pi \frac{\theta - \frac{\pi}{n+1}}{\frac{\pi}{n} - \frac{\pi}{n+1}}.$$

$g|_L$ is obviously continuous, where $L = \{\exp(\theta i) : \pi \le \theta \le 2\pi\}$. Also, since $|g(z)| \le 2/(n+1)$ for every $z = \exp(\theta i), 0 \le \theta \le \pi/n$, $g|_U$ is continuous at the point 1, where $U = \{\exp(\theta i) : 0 \le \theta \le \pi\}$. From the gluing lemma for closed subsets, together with the continuity at the point 1 and the Archimedean property, it follows that $g|_U$ is continuous at each point in U. Again from the gluing lemma, it comes that g is continuous. Since $g(\exp((\pi/n)i)) = (0, 0)$ for every $n \in \mathbb{Z}^+$, there exists a canonical map h of $D = p_2(S^1)$ into B with $h \circ p_2 = g$, where p_2 is the quotient map. Then, h is a continuous surjection, for g is so and p_2 is an identification map. And it is easily shown that h is injective. Note that $p_2(S^1)$ is compact due to the compact space S^1, and that B is a T_2 space as a subspace of $\mathbb{R}^2$. Thus, h is a homeomorphism, and so $D \cong B$.