Density of specialization sets and Hasse principle in families of twisted Galois covers

(joint with Francois Legrand)

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Regular Inverse Galois problem

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Does there exist a k-regular Galois extension $E/k(t)$ with group G, for each finite group G?
(Equivalently, does there exist a Galois cover $X \to \mathbb{P}^1$ of smooth projective curves, with group G, defined over k?)

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Motivation: Positive answer to **RIGP** implies positive answer to **IGP**, by Hilbert's irreducibility theorem.
Concretely: Given $E/k(t)$ Galois with group G, there are infinitely many $t_0 \in k$ such that the **specialization** E_{t_0}/k (=residue field extension at prime $t \mapsto t_0$) has Galois group G.

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- Is there even a cover of $\mathbb{P}^1_k$, resp., of $\mathbb{P}^d_k$ which lifts all G-extensions at once? (**"Parametric extension"**)

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- More generally, given a G-cover over k: what is the structure of the set of specializations?

Examples

- $k(\sqrt{t})/k(t)$ is a parametric extension for the group C_2. It is even **generic** (i.e., parametric over all extensions of k).
- $E/\mathbb{Q}(t)$ the function field of an elliptic curve: This specializes to a quadratic field $\mathbb{Q}(\sqrt{d})$ if and only if the d-th quadratic twist of the curve has a non-trivial $\mathbb{Q}$-point.
  So this extension is not parametric. Moreover, when counting by discriminant the fields $\mathbb{Q}(\sqrt{d})$ which occur as specializations, Goldfeld's ("average rank 1/2") conjecture predicts that 50% of them occur.

Previous results

Theorem (K.-Legrand 2018, K.-Legrand-Neftin 2019)

For "many" finite groups G: There is no $\mathbb{Q}$-regular G-extension $E/k(t)$ which specializes to all G-extensions of k.

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Computational evidence suggests more: Compared with the set of all G-extensions, the set of specializations of a given regular extension seems "very small".

The Malle conjecture

Let G be a finite group, k be a number field and $B \in \mathbb{N}$. Let $N(G,k,B)$ be the number of G-extensions L/k such that the discriminant $\Delta(L/k)$ is of norm at most B.

Conjecture (Malle; 2002, 2004)

There are constants C_1 (depending on G and k) and C_2 (depending on G, k and $\epsilon > 0$) such that

$$C_1 B^{1/\alpha(G)} \leq N(G,k,B) \leq C_2 B^{1/\alpha(G)+\epsilon}.$$

Here $\alpha(G) := \frac{p-1}{p}|G|$, where p is the smallest prime divisor of $|G|$.

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The conjecture is known to hold for all nilpotent groups G (but of course open in general, since it implies the inverse Galois problem over k).

Specialization and "lower bound Malle"

Theorem (Dèbes, 2017)

Let $E/k(t)$ be a k-regular G-extension. Then the number of G-extensions of k, with discriminant of norm $\leq B$, **and which arise as specializations of** *$E/k(t)$, is $\gg B^{\alpha(E)}$, where $\alpha(E)$ is an (explicitly given) positive constant depending on $E/k(t)$.*

Remarks:

- In fact, $\alpha = \beta(G)/R$, where $\beta(G)$ depends only on G and R is the number of branch points of $E/k(t)$.
- Obvious question: Can we also give a (reasonably non-trivial) **upper bound** exponent?

The *abc*-conjecture

Definition

The **radical** $rad(N)$ of a positive integer N is the product of all prime divisors of N, without multiplicities.

abc-conjecture

For every $\epsilon > 0$, there are only finitely many triples (a, b, c) of coprime positive integers with $a + b = c$, such that

$$rad(abc) \leq c^{1-\epsilon}.$$

A result about quadratic twists of hyperelliptic curves

Let C be a hyperelliptic curve over $\mathbb{Q}$, given by $Y^2 = f(T)$ ($f \in \mathbb{Z}[T]$ separable). Recall that the d-th quadratic twist C_d of C is given by $dY^2 = f(T)$.

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Theorem (Granville, 2007)

Assume that the abc-conjecture holds.
Let C be a hyperelliptic curve over $\mathbb{Q}$ of genus $g \geq 2$. Then the number of squarefree integers $d \in [-N, ..., N]$ such that the d-th quadratic twist C_d of C has a non-trivial rational point is asymptotically smaller than $N^{1/(g-1)+\epsilon}$.
In particular, if $g \geq 3$, the density of squarefree integers such that C_d has a non-trivial rational point is 0.

A result about quadratic twists of hyperelliptic curves

- Translation into specialization of Galois covers: Given $C : Y^2 = f(T)$, with $f \in \mathbb{Z}[T]$ separable, C_d has a non-trivial rational point if and only if the hyperelliptic (degree-2) cover $C \to \mathbb{P}^1$ has the field $\mathbb{Q}(\sqrt{d})$ as a specialization.
- In total, there are of course $\Omega(N)$ quadratic fields of discriminant (of absolute value) $\leq N$.
- So Granville's result says that (conditionally on abc), a hyperelliptic curve of genus $g \geq 3$ has "very few" quadratic fields as specializations. In particular, the proportion of such fields is 0.

Twists of Galois covers

Let $f : X \to \mathbb{P}^1$ be a Galois cover with group G, defined over k. Let $\varphi : Gal_k \to G$ be a continuous epimorphism (yielding a G-extension F/k). Then there exists a cover $f^\varphi : \tilde{X} \to \mathbb{P}^1$, defined over k (but not necessarily Galois), with the following properties:

- A fiber $(f^\varphi)^{-1}(t_0)$ contains a rational point if and only if the specialization of f at t_0 equals φ.
- After extension of constants from k to F, the covers f^φ and f become isomorphic.

f^φ is called the **twisted cover** of f by φ.
Special case: If $G = C_2$ and $f : C \to \mathbb{P}^1_{\mathbb{Q}}$ is a hyperelliptic cover, then the twist of f by (the epimorphism corresponding to) $\mathbb{Q}(\sqrt{d})/\mathbb{Q}$ yields the d-th quadratic twist of C.

Rational points on twisted covers

Theorem (K., Legrand (submitted, 2019))

Let $f : X \to \mathbb{P}^1$ be a Galois cover defined over $\mathbb{Q}$, with group G, with R branch points. If the abc-conjecture holds, then the number of distinct specializations $L/\mathbb{Q}$ of f of discriminant $\leq B$ is asymptotically bounded from above by

$$B^{\frac{2}{\tilde{\alpha}(f)(R-4)}+\epsilon}.$$

Here $\tilde{\alpha}(f) := \frac{p-1}{p}|G|$, where p is the smallest prime divisor of any ramification index of f.

(In particular, as soon as $R \geq 5$, we have an upper bound $B^{\gamma(G)/R}$, where $\gamma(G)$ depends only on G.)

Outline of proof

Important tool: Ramification in specializations of covers:

Theorem (Beckmann, 1991)

Let $f : X \to \mathbb{P}^1$ be a Galois cover defined over a number field k, with Galois group G, function field extension $E/k(t)$, and with branch points $t_1, \dots, t_r \in \overline{k}$. Then there is a finite set $\mathcal{S}$ of primes of k ("bad primes") such that for all primes $p \notin \mathcal{S}$, the following holds:
p can only ramify in E_{t_0}/k of t_0 and t_i meet modulo p (for some $i \in \{1, \dots, r\}$), and in this case, the inertia group at p in E_{t_0}/k is generated by $x^{\nu_p(t_0 - t_i)}$, where x generates an inertia group at $t \mapsto t_i$ in $E/k(t)$.

Outline of proof

- Let $t_1, ..., t_r \in \mathbb{P}^1(\overline{\mathbb{Q}})$ be the branch points of f and let $F(X,Y) = \prod_{i=1}^{r}(X - t_i Y) \in \mathbb{Z}[X,Y]$ the homogenized product of their minimal polynomials ($\deg(F) = R$).
  Let $t_0 = \frac{r}{s} \in \mathbb{Q}$, $n := \max\{|r|, |s|\}$.
- By Beckmann's theorem, the specialization $f_{t_0}/\mathbb{Q}$ is ramified at most at the prime divisors of $F(r,s)$, and possibly some fixed finite set ("bad primes").
  Conversely, if a "good" prime q divides $F(r,s)$, but q^p does not divide, then q is ramified in $f_{t_0}/\mathbb{Q}$, and with ramification index at least p.
  Well-known discriminant formula then gives $|\Delta(f_{t_0}/\mathbb{Q})| > C \cdot (\prod q)^{\tilde{\alpha}(f)}$ (with some constant $C > 0$), where the product is over all primes dividing $F(r,s)$ such that q^p does not divide.

Outline of proof

- Now use a consequence of the abc-conjecture:

Theorem (Langevin, Granville)

Let $F(X, Y)$ be a homogeneous polynomial of degree d over $\mathbb{Z}$, and let $\epsilon > 0$. Then

$$rad(F(r,s)) \geq \max\{|r|,|s|\}^{d-2-\epsilon}$$

for all but finitely many coprime $r, s \in \mathbb{Z}$.

- This gives bounds from below on the product P of all primes which divide $F(r,s)$ to a power less than p, namely

$$P \geq n^{R-2-2/(p-1)-\epsilon} (\geq n^{R-4-\epsilon}).$$

- Substituting in the above discriminant formula, with some elementary manipulations, gives the assertion.

Some consequences

Corollary

Assume the abc-conjecture and the Malle conjecture for the group G. If $f : X \to \mathbb{P}^1$ is a Galois cover with group G over $\mathbb{Q}$, with at least 7 branch points, then f cannot be parametric. More precisely, the proportion of G-extensions of $\mathbb{Q}$ which arise as specializations from f converges to 0 (when counted by discriminant).

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Worded differently, the set of all twists of f (by G-extensions of $\mathbb{Q}$) which have an unramified rational point is of density 0 (when counted by discriminant).

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In particular, the case $G = C_2$ regains Granville's result on twists of hyperelliptic curves.

Some consequences

Combination with the lower bound results (Dèbes) yield that two G-covers f_1, f_2 with "sufficiently different" branch point number must have different sets of specializations (conditional on abc).

Corollary

Let G be a finite group. Assume that the abc-conjecture holds. Then there exists a constant $N \in \mathbb{N}$ (depending on G) such that for every Galois cover $f : X \to \mathbb{P}^1$, defined over $\mathbb{Q}$ and with at least N branch points, the following holds: The proportion of G-extensions of $\mathbb{Q}$ which arise as specializations from f converges to 0 (when counted by discriminant).

What we expect vs what we know

Conjecture I

Let $f : X \to \mathbb{P}^1$ be an arbitrary Galois cover with group G over $\mathbb{Q}$ (or indeed, over any number field), of genus ≥ 2. Then G specializes to 0% of all G-extensions of $\mathbb{Q}$.

Our evidence:

- Conditional on abc, we show Conjecture I, with the bound $g \geq 2$ replaced by $g \geq 2|G| - 1$.
- In joint work with Dèbes, Legrand and Neftin, we showed (unconditionally!) a **geometric analog** of this conjecture, where the field $\mathbb{Q}$ is replaced by $\mathbb{C}(t)$, the notion of specialization is replaced by "rational pullback", and the "density" notion is replaced by a notion in the Zariski topology on moduli spaces of Galois covers.

What we expect vs what we know

Definition

Let $f : X \to \mathbb{P}^1$ be a connected cover, given by an equation $F(t, X) = 0$. Let $T(U) \in \mathbb{C}(U)$ be a non-constant rational function. Then the cover f^U given by $F(T(U), X) = 0$ is called the rational pullback of f by $T(U)$.

Pullback can be viewed as specialization from $\mathbb{C}(U)(T)$ into $\mathbb{C}(U)$, where the initial cover was **isotrivial**.

Definition (Hurwitz space)

Let $\underline{C}$ be a class vector of length r in the group G. The set of all Galois covers of $\mathbb{P}^1$ with ramification type $\underline{C}$ is called the Hurwitz space $\mathcal{H}(G, \underline{C})$. It is a finite (possibly empty) union of r-dimensional varieties over $\mathbb{C}$.

(Recall:) Conjecture I

Let $f : X \to \mathbb{P}^1$ be an arbitrary Galois cover with group G over $\mathbb{Q}$ (or indeed, over any number field), of genus ≥ 2. Then G specializes to 0% of all G-extensions of $\mathbb{Q}$.

Theorem (Dèbes-K.-Legrand-Neftin (2018))

Let $g \geq 2$, and let $\mathcal{S}_g$ be **any** *set of genus-g covers $X \to \mathbb{P}^1$ with group G. Then $\mathcal{S}_g$ pulls back to 0% of all G-covers in the following sense: Given any sufficiently long class vector $\underline{C}$ of G with non-empty Hurwitz space, the set of all covers in $\mathcal{H}(G, \underline{C})$ which are rational pullbacks from $\mathcal{S}_g$ (by* **any** *rational function) is contained in the complement of a Zariski-dense open subset of $\mathcal{H}(G, \underline{C})$.*

What we expect vs what we know

Conjecture II

Let f, g be two Galois covers of $\mathbb{P}^1$ with group G, defined over $\mathbb{Q}$, of genus ≥ 2, and not "equivalent". Then f and g have different sets of specializations (in other words, the set of specializations identifies the cover!).

- Once again, we showed a weaker statement: If f and g have "sufficiently different" branch point numbers, then their specialization sets are different.
- The "pullback" analog of this conjecture is trivial: Since f is a pullback of itself, f and g could only have the same set of pullbacks if they are mutual pullbacks of each other. Riemann-Hurwitz genus formula shows that such a thing is impossible, unless the pullback maps are trivial.

Hasse principle

Hasse principle (for curves)

Let C be a curve over $\mathbb{Q}$ with a non-singular point over every $\mathbb{Q}_p$ (including the infinite prime). Then C has a rational point.

- Hasse principle is known to hold for some important special cases (e.g., quadratic forms), but fails in general.
- E.g., Bhargava et al. have shown that a positive proportion of hyperelliptic curves of a fixed genus fail the Hasse principle.

Hasse principle

Hasse principle (for covers)

Let f be a Galois cover over $\mathbb{Q}$ with an unramified $\mathbb{Q}_p$-point for all p. Then f has an unramified rational point.

Question

How many twists of a given Galois cover fail the above Hasse principle, i.e., have an (unramified) point everywhere locally, but no (unramified) $\mathbb{Q}$-point?

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- In the hyperelliptic case: Conditional on abc, infinitely many quadratic twists of a given genus ≥ 3 hyperelliptic curve violate the Hasse principle (Clark, Watson 2018).

An unconditional result:

Theorem (K., in preparation)

Let G be a finite abelian, but non-cyclic group, and k be a number field. Let f be a G-cover of $\mathbb{P}^1$, defined over k. Then the proportion of twists of f by G-extensions of k which do not have a point everywhere locally equals 100%, when extensions are counted by **conductor***.*
In other words, for 100% of G-extensions L/k, there is a prime p of k such that f does not specialize to L/k, even after base field extension to k_p.

So Hasse principle holds (but trivially) for 100% of twists. However:

Theorem (K.-Legrand (2019))

Let G be an abelian group, f be a G-cover of $\mathbb{P}^1$, defined over $\mathbb{Q}$, with ≥ 7 branch points. Conditional on abc, 0% of those twists of f which have a point everywhere locally, also have a $\mathbb{Q}$-point.

Remarks:

- The result remains (essentially) true for arbitrary groups G with non-trivial center, under some technical extra assumptions of the cover f.
- In particular, we can generate a huge amount of curves failing the Hasse principle, via twists, starting from a "relatively" general curve.

Idea of proof

- Need to count how many G-extensions $L/\mathbb{Q}$ have the following property: For every prime p, there is a specialization of f which locally at p behaves the same as $L/\mathbb{Q}$.
- For (most) *unramified* primes of $L/\mathbb{Q}$, the following observation suffices: If p is sufficiently large (depending on f), then every unramified behaviour occurs at p in a suitable specialization of f. This is due to:

Theorem (Dèbes, Ghazi (2012))

Let $f : X \to \mathbb{P}^1$ be a G-cover defined over $\mathbb{Q}$. Then for every prime p outside some finite set S_0 (depending only on E) and for every **unramified** *extension $K_p/\mathbb{Q}_p$ with Galois group embedding into G, there are specializations of f whose completion at p equals $K_p/\mathbb{Q}_p$.*

Idea of proof

- So for sufficiently **large** primes p, we always have a $\mathbb{Q}_p$-point on the twist of f by $L/\mathbb{Q}$, as soon as p is unramified in L.
- If p is (large and) ramified in L, then we use (a special case) of a recent result on local behaviour in specializations:

Theorem ((Special case of) K.-Legrand-Neftin, 2019)

Let $f : X \to \mathbb{P}^1$ be a G-cover defined over $\mathbb{Q}$, and $t_i \in \mathbb{P}^1(\overline{\mathbb{Q}})$ be a branch point of f, with inertia group $I \leq G$. Let p be a prime which splits completely in the residue extension of f at t_i. Then every I-extension of $\mathbb{Q}_p$ is a specialization of f $(\otimes\mathbb{Q}_p)$.

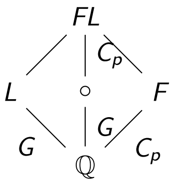
- If p is **small**, then we know less. But at least there is **some** G-extension $L/\mathbb{Q}$ whose twist as a $\mathbb{Q}_p$-point.

Idea of proof

- Therefore, it suffices to estimate the number of G-extensions with the following two conditions:
  i) For some fixed finite set S_0 of primes p, the local behaviour at p is prescribed (according to some specialization of f).
  ii) All further **ramified primes** are in some prescribed positive density set (namely, in the set of primes that ramify in some specialization of f, and completely split in some prescribed number field), and with a prescribed inertia group.

Idea of proof

- Our idea is now to grab **one** such G-extension (namely, a suitable specialization of f), and then change it "slightly" by twisting with a suitable C_p-extension ($C_p \leq Z(G)$), say $F/\mathbb{Q}$, without destroying the local conditions (i.e., without altering the behaviour at the set S_0, and without introducing "forbidden" ramified primes).
- These C_p-extensions can be constructed very explicitly inside certain cyclotomic field $\mathbb{Q}(\zeta_q)$ (introducing only one ramified prime q at a time!), and their discriminant is "under control".



Idea of proof

- Then it can be shown that the proportion of “good” G-extensions of discriminant $\leq N$ (i.e., such that the corresponding twist of f has a point everywhere locally) is at least $1/($ polylogarithmic expression in $N)$.
- On the other hand, due to our assumptions on f, and conditionally on abc, the proportion of twists of f with a **rational** point is $< N^{\alpha}$, for some $\alpha > 0$.