Robin Hartshorne's Algebraic Geometry Solutions

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Chapter V Section 5 Birational Transformations

5.8. *A surface singularity.* **Let k be an algebraically closed field, and let X be the surface in $\mathbb{A}^3_k$ defined by the equation $x^2+y^3+z^5=0$. It has an isolated singularity at the origin $P=(0,0,0)$.**

(a). **Show that the affine ring $A=k[x,y,z]/(x^2+y^3+z^5)$ of X is a unique factorization domain, as follows. Let $t=z^{-1}$; $u=t^3x$, and $v=t^2y$. Show that z is irreducible in A; $t\in k[u,v]$, and $A[z^{-1}]=k[u,v,t^{-1}]$. Conclude that A is a UFD.**

Claim. *z is irreducible in A.*

Proof. Notice that

$$\begin{gathered} z \text{ is irreducible in } A. \\ \Leftrightarrow (z) \text{ is a prime ideal in } A. \\ \Leftrightarrow A/(z), \text{ which is } k[x,y]/(x^2+y^3), \text{ is an integral domain.} \\ \Leftrightarrow x^2+y^3 \text{ is irreducible in } k[x,y]. \end{gathered}$$

We prove the last statement. Suppose that for some f,g in $k[x,y]$, we have

$$fg=x^2+y^3.$$

(1) (case 1) Assume that $\deg_x(f)$, the degree of f in x, is zero. Then f is a polynomial in y, and we can write $g=cx^2+ax+b$ for some c in $k^\times$ and a,b in $k[y]$. Thus,
$$x^2+y^3=fg=cfx^2+fax+fb.$$
This implies that $f=1/c$, which is a unit in $k[x,y]$.

(2) (case 2) Assume that $\deg_x f=1$. Then, by multiplying a suitable constant in $k^\times$, we may assume that $f=x+a$ and $g=x+b$ for some a,b in $k[y]$. Then,
$$x^2+y^3=fg=x^2+(a+b)x+ab$$
so that $a+b=0$ and $ab=y^3$. Then, $b^2=-y^3$ and since y is irreducible in $k[y]$, $b=yb'$ for some b' in $k[y]$. Hence $y^2(b')^2=-y^3$, thus $(b')^2=-y$, which is impossible because we then have $2\deg_y(b')=1$.

(3) (case 3) Assume that $\deg_x(f)=2$. Then, by symmetry, (case 1) shows that g must be a unit.

Hence x^2+y^3 is irreducible in $k[x,y]$, and thus z is irreducible in A. □

Claim. *$t\in k[u,v]$.*

Proof. The equality $x^2+y^3+z^5=0$ implies that in the fraction field we have $-x^2/z^6-t^3/z^6=1/z$. This is equivalent to $t=-u^2-v^3$. □

Claim. *$A[z^{-1}]=k[u,v,t^{-1}]$.*

Proof. The equalities

$$\begin{cases} x=(t^{-1})^3t^3x=(t^{-1})^3u, \\ y=(t^{-1})^2t^2y=(t^{-1})^2v, \\ z=t^{-1} \end{cases}$$

show that $A\subset k[u,v,t^{-1}]$. By the previous claim $z^{-1}=t\in k[u,v]$, thus $A[z^{-1}]\subset k[u,v,t^{-1}]$.

Conversely, we have

$$\begin{cases} u = (z^{-1})^3 x \in A[z^{-1}], \\ v = (z^{-1})^2 y \in A[z^{-1}], \\ t^{-1} = z \in A \end{cases}$$

so that $A[z^{-1}] \supset k[u, v, t^{-1}]$. This finishes the proof. □

Claim. *A is a UFD.*

Notice that, being a polynomial ring, $A[z^{-1}] = k[u, v, t^{-1}]$ is a UFD.

Lemma (1)**.** *Let f be an irreducible element in A, that is not in (z). Then, f is irreducible as an element in $A[z^{-1}]$.*

Proof. Suppose that $(g/z^m)(h/z^n) = f/1$ for some integers $m, n \geq 0$ and g, h in A, so that $gh = z^{m+n} f$. If $m > 0$ or $n > 0$, then since the quantity $gh = z^{m+n} f$ is in the ideal (z), either $g \in (z)$ or $h \in (z)$. By canceling a suitable number of z's if necessary, we may assume that $m = n = 0$. Thus, $gh = f$ in A. But, since f is irreducible in A, g or h must be a unit in A. Hence $g/1$ or $h/1$ is a unit in $A[z^{-1}]$, thus, $f/1$ is irreducible in $A[z^{-1}]$. □

Lemma (2)**.** *If a nonzero element $f/1 \in A[z^{-1}]$ is irreducible for some f in A, then $f = z^m g$ for some integer $m \geq 0$ and an irreducible element g in A, where $g \notin (z)$.*

Proof. Since f is nonzero, we can write $f = z^m g$ for some integer $m \geq 0$ and an element $g \in A$ that is not in the ideal (z). We need to check that this g is irreducible in A.

If not, then for some nonunits p, q in A, the equality $g = pq$ holds. Thus, $f/1 = (z^m/1)(g/1) = (z^m/1)(p/1)(q/1)$. Since $f/1$ is irreducible in $A[z^{-1}]$ and z^m is a unit, one of $p/1$ and $q/1$ must be a unit element in $A[z^{-1}]$, say $p/1$, without loss of generality. Thus, for some r in A and an integer $n \geq 0$, we have $(p/1)(r/z^n) = 1$, that is, $pr = z^n$. Thus, $pr \in (z)$, and z being irreducible either $p \in (z)$ or $r \in (z)$. But since $g \notin (z)$ and $g = pq$, the element p must not be in (z). Hence $r \in (z)$. Thus, by repeating this argument, we may assume that $n = 0$. Then, we have the equality $pr = 1$ in A, contradicting the assumption that p is not a unit in A. □

We now prove that A is a UFD. For any nonzero $f \in A$, since the ring $A[z^{-1}]$ is a UFD, we have a factorization of $f/1$

$$\frac{f}{1} = \frac{u}{z^m} \frac{f_1}{z^{m_1}} \cdots \frac{f_n}{z^{m_n}}$$

for some nonnegative integers $m, m_1, \cdots, m_n$, a unit u in A, and $f_1, \cdots, f_n$ in A, where f_i/z^{m_i} are irreducible in $A[z^{-1}]$. Since each z^{m_i} is a unit, by replacing $m + m_1 + \cdots + m_n$ by m, we may simplify the above equation as

$$\frac{f}{1} = \frac{u}{z^m} \frac{f_1}{1} \cdots \frac{f_n}{1}$$

where f_i are irreducible in $A[z^{-1}]$. Thus, $z^m f = u f_1 \cdots f_n$ in A. By the Lemma (2), each $f_i = z^{r_i} g_i$ for some integer $r_i \geq 0$ and an irreducible element $g_i \in A$, where $g_i \notin (z)$, so that

$$z^m f = u z^{r_1 + \cdots + r_n} g_1 \cdots g_n.$$

Note that we must have $m \geq r_1 \cdots r_n$ since all g_i is not in (z). Thus, $f = u z^s g_1 \cdots g_n$, where $s = r_1 + \cdots + r_n - m \geq 0$, gives a factorization of f into a product of irreducible elements of A.

To show that this factorization is unique, suppose that we have two such factorizations

$$g = u g_1 \cdots g_n = v h_1 \cdots h_m,$$

where u, v in A are units, and g_i, h_j in A, $1 \leq i \leq n$, $1 \leq j \leq m$, are irreducible. Since $f = u g_1 \cdots g_n$ is in the ideal (h_1), for some i, the element g_i must be in (h_1). We may assume that $g_1 \in (h_1)$ so that $g_1 = h_1 k$ for some k in A. Since g_1 is irreducible and h_1 is not a unit (being irreducible), k must be a unit in A. Hence, continuing in this way, by suitably renumbering them if necessary, we must have $m = n$ and each irreducible element g_i is a unit multiple of h_i. This finishes the proof.