Algebraic Geometry by Robin Hartshorne

Exercises solutions by Jinhyun Park

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Chapter 4.Curves, Section 1.Riemann-Roch Theorem.

1. Choose $Q \in C$. Choose n big enough so that deg $n(2P - Q) > 2g - 2,\ g,\ 1.$ $\Rightarrow h^0(n(2P-Q)) = 1 - g + n(2P - Q) > 1 \Rightarrow \exists$ effective divisor $D \in |n(2P - Q)| \Rightarrow$ $\exists f \in K(C)$ such that $D + nQ - 2nP = (f)$. Since deg $D = n$, so D cannot cancel $-2nP$ i.e. f has a pole only at P.//

2. Let $F = \{P_1, \cdots, P_r\}$. Multiplying functions of Ex.IV.1.1 might give cancellation of poles and zeros, so we need slightly different approach.
Let $Q \in C - F$. Consider $D' = n(P_1 + \cdots P_r - (r-1)Q)$. Choose $n > 2g - 2,\ g$. Then, $\exists D \in |D'|$ i.e. $\exists f \in K(c)$ such that $D + (r-1)Q - nP_1 - \cdots - nP_r = (f)$. Note that deg $D = n$. Each P_i occurs with order $-n$ so, if $P_i \in \mathrm{Supp}D$, then either (i) $\mathrm{ord}_{P_i} D < n$ or (ii) $D = nP_i$ for some i, in this case, WMA $i = 1$ WLOG.

For (i) there is no problem.
For (ii) f has ples only at $P_2, \cdots, P_r$ not at P_1. By Ex.IV.1.1, $\exists g \in K(C)$ which has a pole only at P_1. Let $\mathrm{ord}_{P_i} g = n_i$ $2 \le i \le r$. Then, if we choose $m > 1, n_2, \cdots, n_r$, then, $f^m g$ has poles at and only at F.//

3. Proof 1) By I-(6.10), there is a projective nonsingular curve $\bar{X}$ over k such that X is an open subset of $\bar{X}$, i.e. $\bar{X} - X$ is a finite set, say, $\{P_1, \cdots, P_r\} \neq \phi$ because X is not proper.
Then, by Ex.IV.1.2, $\exists f \in k(\bar{X}) = k(X)$ such that f has poles only at $P_1, \cdots, P_r$.
We can consider $f \in k(\bar{X})$ as a morphism $f : \bar{X} \to \mathbb{P}^1$.Then, $f^{-1}(\mathbb{A}^n) = X$, so, $g = f|_X : X \to \mathbb{A}^1$ is a morphism.
$f(\bar(X))$ is proper over k(because $\bar{X}$ is proper) and irreducible and, $f(\bar{X}) \neq$a point. Hence, $f(\bar{X}) = \mathbb{P}^1$. And by (II-6.8), f is a finite morphism, in particular, affine morphism. Hence, $f^{-1}(\mathbb{A}^1) = X$ is affine.

Proof 2) As above, let $F = \{P_1, \cdots, P_r\} = \bar{X} - X$. Choose m such that $mr > 2g$ Then, $D = m(P_1 + \cdots P_r$ has a degree $> 2g$, so by (3), it is very ample. Then, it gives an embedding of $\bar{X}$ into a projective space $\mathbb{P}^N$ for some N, and $D = \bar{X}.H$ for a hyper plane H of $\mathbb{P}^N$. Then, $\bar{X} - F =$ a closed subscheme of $\mathbb{A}^N = \mathbb{P}^N - H$ which is affine, so, $X = \bar{X} - F$ is also affine.//