Robin Hartshorne's Algebraic Geometry Solutions

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CHAPTER III SECTION 9 FLAT MORPHISMS

9.1.

9.2.

9.3.

9.4.

9.5.

9.6.

9.7.

***9.8. Let A be a finitely generated k-algebra. Write A as a quotient of a polynomial ring P over k, and let J be the kernel:**

$$0 \to J \to P \to A \to 0.$$

Consider the exact sequence of (II, 8.4A)

$$J/J^2 \to \Omega_{P/k} \otimes_P A \to \Omega_{A/k} \to 0.$$

Apply the functor $\mathrm{Hom}_A(\cdot, A)$, and let $T^1(A)$ be the cokernel:

$$\mathrm{Hom}_A(\Omega_{P/k} \otimes A, A) \to \mathrm{Hom}_A(J/J^2, A) \to T^1(A) \to 0.$$

Now use the construction of (II, Ex. 8.6) to show that $T^1(A)$ classifies infinitesimal deformations of A, i.e., algebras A' flat over $D = k[t]/t^2$, with $A' \otimes_D k \simeq A$. It follows that $T^1(A)$ is independent of the given representation of A as a quotient of a polynomial ring P. (For more details, see Lichtenbaum and Schlessinger [1].)

Proof. Suppose that $P = k[x_1, \cdots, x_n]$ is a polynomial k-algebra of which A is a quotient with the kernel J. Let $P_2 := k[x_1, \cdots, x_n, y_1, \cdots, y_n]$.

For each infinitesimal deformation A' of A, we can define a k-algebra homomorphism $f : P_2 \to A'$ so that we obtain the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & J \arrow[r, "t"] \arrow[d] & K \arrow[r] \arrow[d] & J \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P \arrow[r, "t"] \arrow[d] & P_2 \arrow[r] \arrow[d, "f"] & P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "t"] \arrow[d] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

where K is an ideal of $P - 2$.

Notice that to give a k-algebra A' with the required properties is equivalent to give an ideal K, and the ambiguity is given by the choice of the k-algebra homomorphism f. Thus, the set of equivalence classes of infinitesimal deformations A' of A is equal to

$$\frac{\{\text{ choices of an ideal } K\}}{\{\text{ choices of } f\}}.$$

We will identify the numerator and the denominator.

Claim. $\{$ *choices of an ideal* $K\} \simeq \operatorname{Hom}_P(J, A)$ *as sets.*

Notice that the middle row splits via the natural inclusion $P \to P_2$ of the right hand side P. So that as modules, $P_2 = P \oplus tP$.

Suppose an ideal K was chosen. For each $x \in J$, lift it to $\tilde{x} \in K$. Since $P_2 = P \oplus tP \supset K$, $\tilde{x} = x + t(y)$ for some $y \in P$. Two liftings of x differ by an image of tz for some $z \in I$, thus, $y \in P$ is not uniquely determined by x, but $\bar{y} \in A$ is uniquely determined. Thus, it defines a map in $\operatorname{Hom}_P(J, A)$ that sends $x \mapsto \bar{y}$.

Conversely, suppose that $\phi \in \operatorname{Hom}_P(J, A)$. Define an ideal K of P_2 by

$$K = \{x + ty | x \in J, y \in P \text{ such that } \bar{y} = \phi(x) \text{ in } A\}.$$

It is easy to see that K is an ideal of P_2, and the image of K in P is J so that

$$0 \to J \to K \to J \to 0$$

is exact. It defines $A' := P_2/K$, and here f is the canonical quotient map. Thus, it shows the claim.

Claim. $\{$ *choices of* $f\} \simeq \operatorname{Der}_k(P, A)$ *as sets.*

A choice of $f : P_2 \to A'$ gives after composing with $t : P \to P_2$, a lifting of $P \to A$ to $P \to A'$. Thus, Ex. II-8.6-(a) shows the assertion. This proves the claim.

Hence, the obvious identities

$$\operatorname{Hom}_P(J, A) \simeq \operatorname{Hom}_A(J/J^2, A), \text{ and}$$

$$\operatorname{Der}_k(P, A) \simeq \operatorname{Hom}_P(\Omega_{P/k}, A)$$

show that the set of isomorphism classes of infinitesimal deformations are in one-to-one correspondence with the coker $\big(\operatorname{Hom}_P(\Omega_{P/k}, A) \to \operatorname{Hom}_A(J/J^2, A)\big)$, which is by definition $T^1(A)$. This finishes the proof. $\square$

Remark. In fact, via a natural map

$$T_1(A) \supset \operatorname{Ext}^1_A(\Omega_{A/k}, A),$$

where the natural map will be apparent from the following discussion.

For the exact sequence

$$J/J^2 \to \Omega_{P/k} \otimes_P A \to \Omega_{A/k} \to 0,$$

let L be the kernel of the second map so that we have a natural projection $J/J^2 \to L$ and a commutative diagram

$$\begin{tikzcd}
 & J/J^2 \arrow[d] \arrow[dr] & & & \\
0 \arrow[r] & L \arrow[r] & \Omega_{P/k} \otimes_P A \arrow[r] & \Omega_{A/k} \arrow[r] & 0.
\end{tikzcd}$$

Then, it induces a commutative diagram with exact rows

$$\begin{tikzcd}
\mathrm{Hom}_P(\Omega_{P/k}, A) \arrow[r] & \mathrm{Hom}_P(J/J^2, A) \arrow[r] & T^1(A) \arrow[r] & 0 \\
\mathrm{Hom}_P(\Omega_{P/k}, A) \arrow[r] \arrow[u, "="] & \mathrm{Hom}_P(L, A) \arrow[r] \arrow[u] & \mathrm{Ext}^1_A(\Omega_{A/k}, A) \arrow[r] & \mathrm{Ext}^1_P(\Omega_{P/k}, A)
\end{tikzcd}$$

First of all, since P is smooth over k, $\mathrm{Ext}^1_P(\Omega_{P/k}, A) \simeq 0$, $\Omega_{P/k}$ being projective. Hence, by diagram chasing we can define a map

$$\mathrm{Ext}^1_A(\Omega_{A/k}, A) \to T^1(A)$$

and furthermore, the diagram implies that it must be injective.

It is known that this map becomes an isomorphism when

(1) k is a perfect field, and

(2) A is a reduced k-algebra of finite type,

according to Lichtenbaum and Schlessinger.

9.9.

9.10.

9.11.