Robin Hartshorne's Algebraic Geometry Solutions
by Jinhyun Park

Chapter III Cohomology Section 5 The Cohomology of Projective Space

5.1.

5.2.

5.3.

5.4.

5.5.

5.6.

(a) Identify $Q = \mathbb{P}^1 \times \mathbb{P}^1$ and let $|Y| = \mathbb{P}^1 \times *$, $|Z| = * \times \mathbb{P}^1$. First, observe that Q is birational to $\mathbb{P}^2$ and $h^1(X, \mathcal{O}_X)$ is a birational invariant, so, $h^1(Q, \mathcal{O}_Q) = h^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) = 0$.

Claim (1)**.** *Let $p > 0$. Then, $H^1(Q, \mathcal{O}_Q(p, 0)) = 0$.*

Proof. Let $Y = \mathbb{P}^1 \times \{p\text{-points}\}$. Then, we have a short exact sequence $0 \to \mathcal{O}_Q(-p, 0) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0$. Tensor it by $\mathcal{O}_Q(p, 0)$ then, we obtain $0 \to \mathcal{O}_Q \to \mathcal{O}_Q(p, 0) \to \mathcal{O}_Y(p, 0) \to 0$. Then, from the cohomology long exact sequence, we obtain

$$0 = H^1(Q, \mathcal{O}_Q) \to H^1(Q, \mathcal{O}_Q(p, 0)) \to \bigoplus_p H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(p|Y|^2)) \to 0,$$

but, $|Y|^2 = 0$, so, by Serre duality, $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2))^* = 0$. Hence, $H^1(Q, \mathcal{O}_Q(p, 0)) = 0$ for $p > 0$. This finishes the proof of Claim 1. $\square$

By symmetry, we also have $H^1(Q, \mathcal{O}_Q(0, q)) = 0$ for $q > 0$.

Claim (2)**.** *For all $p \geq 0, q \geq 0$, $H^1(Q, \mathcal{O}_Q(p, q)) = 0$.*

Proof. If $(p, q) = (0, 0)$ or $p = 0$ or $q = 0$, then, we already know this result, so, assume that $p, q > 0$. Tensor the sequence $0 \to \mathcal{O}_Q(-p, 0) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0$ with $\mathcal{O}_Q(p, q)$ to obtain a short exact sequence $0 \to \mathcal{O}_Q(0, q) \to \mathcal{O}_Q(p, q) \to \mathcal{O}_Y(p, q) \to 0$. Then, from the cohomology long exact sequence we have

$$H^1(Q, \mathcal{O}_Q(0, q)) \to H^1(Q, \mathcal{O}_Q(p, q)) \to \bigoplus_p H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(p|Y|^2 + q|Y|.|Z|))$$

but $p|Y|^2 + q|Y|.|Z| = q$ and by Serre duality, $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(q)) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-q - 2))^* = 0$ as $-q - 2 < 0$. By Claim 1, we know that $H^1(Q, \mathcal{O}_Q(0, q)) = 0$ so, $H^1(Q, \mathcal{O}_Q(p, q)) = 0$ consequently. This proves the result. $\square$

Claim (3)**.** *For any $p \in \mathbb{Z}$, $H^1(Q, \mathcal{O}_Q(p, -1)) \simeq H^1(Q, \mathcal{O}_Q(0, -1))$.*

Proof. If $p = 0$, it is obvious. First consider the case when $p > 0$. From the sequence $0 \to \mathcal{O}_Q(-p, 0) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0$, by tensoring with $\mathcal{O}_Q(p, -1)$, we obtain $0 \to \mathcal{O}_Q(0, -1) \to \mathcal{O}_Q(p, -1) \to \mathcal{O}_Y(p, -1) \to 0$. Hence, the long exact sequence gives

$$\bigoplus_p H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(p|Y|^2 + (-1)|Y|.|Z| = -1) \to H^1(Q, \mathcal{O}_Q(0, -1)) \to H^1(Q, \mathcal{O}_Q(p, -1)) \to \bigoplus_p H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)).$$

Then, $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$ and $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1))^* = 0$, so, $H^1(Q, \mathcal{O}_Q(p,-1)) \simeq H^1(Q, \mathcal{O}_Q(0,-1))$ indeed.

Now consider the case when $p < 0$. let $p' = -p > 0$ and let $Y' = \mathbb{P}^1 \times \{p'\text{-points}\}$. Then we have $0 \to \mathcal{O}_Q(-p', 0) \to \mathcal{O}_Q \to \mathcal{O}_{Y'} \to 0$ and by tensoring with $\mathcal{O}_Q(0,-1)$, we obtain $0 \to \mathcal{O}_Q(-p',-1) \to \mathcal{O}_Q(0,-1) \to \mathcal{O}_{Y'}(0,-1) \to 0$. Hence, the long exact sequence gives us

$$\bigoplus_{p'} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) \to H^1(Q, \mathcal{O}_Q(-p',-1)) \to H^1(Q, \mathcal{O}_Q(0,-1)) \to \bigoplus_{p'} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1))$$

and $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$. This shows that $H^1(Q, \mathcal{O}_Q(p,-1)) \simeq H^1(Q, \mathcal{P}_Q(0,-1))$ for $p < 0$. $\square$

Claim (4)**.** (i) *$H^1(Q, \mathcal{O}_Q(0,q)) \neq 0$ if $q \leq -2$.*
(ii) *$H^1(Q, \mathcal{O}_Q(0,-1)) = 0$.*

Proof. Let $p > 0$. From $0 \to \mathcal{O}_Q(-p,0) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0$, by tensoring with $\mathcal{O}_Q(0,q)$, we obtain $0 \to \mathcal{O}_Q(-p,q) \to \mathcal{O}_Q(0,q) \to \mathcal{O}_Y(0,q) \to 0$ so that the long exact sequence gives

$$\bigoplus_{p} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(q)) \to H^1(Q, \mathcal{O}_Q(-p,q)) \to H^1(Q, \mathcal{O}_Q(0,q)) \to \bigoplus_{p} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(q)) \to 0.$$

When $q \leq -2$, $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$ and $h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(q)) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-q-2)) > 0$ so that $H^1(0,q) \neq 0$. This proves (i) and by symmetry we also have $H^1(p,0) \neq 0$ if $p \leq -2$. This proves (3).

When $p = 1, q = 0$, we have

$$k = H^0(Q, \mathcal{O}_Q) \xrightarrow{\simeq} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = k \to H^1(Q, \mathcal{O}_Q(-1,0)) \to H^1(Q, \mathcal{O}_Q) = 0$$

so that $H^1(Q, \mathcal{O}_Q(-1,0)) = 0$. This proves (ii) and similarly we have $H^1(Q, \mathcal{O}_Q(0,-1)) = 0$. $\square$

Now, we prove (2). From V. 1.4.4, the canonical line bundle $K \simeq \mathcal{O}_Q(-2,-2)$, so, when $a, b < 0$, by Serre duality,

$$H^1(Q, \mathcal{O}_Q(a,b)) \simeq H^1(Q, \mathcal{O}_Q(-a-2,-b-2))^*.$$

If $a, b \leq -2$, then, by Claim 2, this group vanishes.

In case $(a,b) = (0,-1), (-1,0), (-2,-1), (-1,-2), (-1,-1)$, the previous claims already show it. Hence, it is 0 for any $a, b < 0$. This proves (2). (1) is trivial once we have (2) and the previous claims.

(b) (1) For Y, $0 \to \mathcal{O}_Q(-Y) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0$ is exact and $\mathcal{O}_Q(-Y) \simeq \mathcal{O}_Q(-a,-b)$. Thus,

$$0 \to H^0(Q, \mathcal{P}_Q(-a,-b)) \to H^0(Q, \mathcal{O}_Q) \to H^0(Y, \mathcal{O}_Y) \to H^1(Q, \mathcal{O}_Q(-a,-b)) \to 0$$

so, $H^0(Y, \mathcal{O}_Y) \simeq H^0(Q, \mathcal{O}_Q) \simeq k$. Hence Y has only 1 connected component, i.e. connected.

(2) Let $\mathcal{L}$ be a line bundle on Q of type (a,b) with $a > 0, b > 0$. Then, by II. 7.6.2, $\mathcal{L}$ is very ample so that it gives an embedding of Q into a projective space $\mathbb{P}^N$. Then, by Bertini's theorem (II, 8.18), there is a hyperplane $H \subset \mathbb{P}^N$ whose intersection with Q is a nonsingular projective curve Y and this $\mathcal{O}_Q(Y)$ is isomorphic to $\mathcal{L}$, i.e. Y is of type (a,b).

(3) By Ex. II 5.14-(d), $X \subset \mathbb{P}^r_A$ is projectively normal if and only if it is normal and for all $n \geq 0$, the natural map $\Gamma(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) \to \Gamma(X, \mathcal{O}_X(n))$ is surjective. We will use this.

Since we have a sequence of closed embeddings $Y \hookrightarrow Q \hookrightarrow \mathbb{P}^3$, it gives a commutative diagram

$$\begin{tikzcd}
\Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n)) \arrow[rr] \arrow[dr] & & \Gamma(Y, \mathcal{O}_Y(n)) \\
& \Gamma(Q, \mathcal{O}_Q(n)) \arrow[ur] &
\end{tikzcd}$$

so, if, $\Gamma(Q, \mathcal{O}_Q(n)) \to \Gamma(Y, \mathcal{O}_Y(n))$ is not surjective, then, $\Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n)) \to \Gamma(Y, \mathcal{O}_Y(n))$ cannot be surjective.

On the other hand, since $Q = V(xy - zw) \subset \mathbb{P}^3$, the ideal sheaf of Q $\mathcal{I}_Q \simeq \mathcal{O}_{\mathbb{P}^3}(-2)$ so that the sequence $0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_Q \to 0$ is exact. Hence, by tensoring with $\mathcal{O}_{\mathbb{P}^3}(n)$, we have $0 \to \mathcal{O}_{\mathbb{P}^3}(n-2) \to \mathcal{O}_{\mathbb{P}^3}(n) \to \mathcal{P}_Q(n) \to 0$ whose cohomology long exact sequence gives

$$H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n)) \to H^0(Q, \mathcal{O}_Q(n)) \to H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n-2)) = 0.$$

Consequently, the map $\Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n)) \to \Gamma(Q, \mathcal{O}_Q(n))$ is always surjective and it implies that $\Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n)) \to \Gamma(Y, \mathcal{O}_Y(n))$ is surjective if and only if $\Gamma(Q, \mathcal{O}_Q(n)) \to \Gamma(Y, \mathcal{O}_Y(n)))$ is surjective if and only if $Y \subset \mathbb{P}^3$ is projectively normal, because being nonsingular, Y is already normal.

Hence, it remains to show that $\Gamma(Q, \mathcal{O}_Q(n)) \to \Gamma(Y, \mathcal{O}_Y(n))$ is surjective if and only if $|a - b| \leq 1$.

($\Leftarrow$) Suppose that $|a - b| \leq 1$. Then, from $0 \to \mathcal{O}_Q(-a, -b) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0$, we obtain $0 \to \mathcal{O}_Q(n-a, n-b) \to \mathcal{O}_Q(n, n) \to \mathcal{O}_Y(n) \to 0$ which gives us

$$\Gamma(Q, \mathcal{O}_Q(n)) \to \Gamma(Y, \mathcal{O}_Y(n)) \to H^1(Q, \mathcal{O}_Q(n-a, n-b)).$$

But, $|a-b| \leq 1$ means $|(n-a)-(n-b)| \leq 1$ so, by part (a) - (1), $H^1(Q, \mathcal{O}_Q(n-a, n-b))$ vanishes and the natural map is surjective.

($\Rightarrow$) Conversely, suppose that the natural map is surjective for all $n \geq 0$. Then, the same sequence gives

$$\Gamma(Q, \mathcal{O}_Q(n)) \to \Gamma(Y, \mathcal{O}_Y(n)) \to H^1(Q, \mathcal{O}_Q(n-a, n-b)) \to H^1(Q, \mathcal{O}_Q(n, n))$$

where the last one is 0 by Claim 2 of (a) and the first map is surjective. Hence, we must have $H^1(Q, \mathcal{O}_Q(n-a, n-b)) = 0$ for all $n \geq 0$.

Toward contradiction, so, suppose that $|a - b| \geq 2$, i.e. $a \geq b+2$ or $b \geq a+2$. For the first case, when $n = b$, $n - a \leq -2$ so that by (a)- (3), we have $H^1(Q, \mathcal{O}_Q(n-a, n-b)) \neq 0$, which is a contradiction. For the second case, we will have the same contradiction. Hence $|a - b| \leq 1$.

Hence, a nonsingular $Y \subset Q$ of type (a, b) with $a, b > 0$ is projectively normal in $\mathbb{P}^3$ if and only if $|a - b| \leq 1$.

(c) First, we reduce this problem to a nonsingular Y. By part (b)-(2), Y is linearly (hence rationally) equivalent to a nonsingular projective curve lying on Q and this new curve has the same bidegree. Also, since this is a rational equivalence, they belong to the same flat family, so, the arithmetic genera are unchanged (which are defined to be $h^1(Y, \mathcal{O}_Y)$). Hence, we may replace Y by its linearly equivalent nonsingular Y. Then, for this Y, the arithmetic genus $p_a(Y) = p_g(Y)$, the geometric genus, and we can compute it in terms of a, b as follows: $\mathcal{O}_Q(Y) = \mathcal{O}_Q(a, b)$ and the first Chern class $c_1(N_{Q/Y}) = \deg_Y(N_{Q/Y}) = Y.Y = (ah+bk)^2 = ab(h.k)+ba(k.h) = 2ab$ where h, k are generators of Pic$Q \simeq \mathbb{Z} \oplus \mathbb{Z}$ with intersection product $h^2 = k^2 = 0$, $h.k = k.h = 1$.

On the other hand, $T_{\mathbb{P}^1\times\mathbb{P}^1} = \left(\Omega^1_{\mathbb{P}^1\times\mathbb{P}^1}\right)^*$ implies that $c_1\left(T_{\mathbb{P}^1\times\mathbb{P}^1}\right) = c_1\left(\wedge^2 T_{\mathbb{P}^1\times\mathbb{P}^1}\right) = c_1\left(K^*_{\mathbb{P}^1\times\mathbb{P}^1}\right) = -(c_1(K_{\mathbb{P}^1}), c_1(K_{\mathbb{P}^1})) = -(2\cdot 0-2, 2\cdot 0-2) = (2,2) = 2h+2k$ and so, $c_1(T_Q|_Y) = \deg_Y\left(T_Q \otimes_{\mathcal{O}_Q} \mathcal{O}_C\right) = \deg_Y\left(\wedge^2 T_Q \otimes_{\mathcal{O}_Q} \mathcal{O}_C\right) = [K^*_Q]\cdot(ah+bk) = (2h+2k)\cdot(ah+bk) = 2(a+b)$.

Of course, $c_1(T_Y) = -\deg_Y(K_Y) = -(2g-2) = 2-2g$. Hence the short exact sequence

$$0 \to T_Y \to T_Q|_Y \to N_{Q/Y} \to 0$$

gives $c_1(T_Y) + c_1(N_{Q/Y}) = c_1(T_Q|Y)$ and it is equivalent to $2(a+b) = 2-2g+2ab$, i.e. $g = ab-a-b+1 = (a-1)(b-1)$. This proves the result.

5.7.

5.8.

5.9.

5.10.