Robin Hartshorne's Algebraic Geometry Solutions

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Chapter III Cohomology Section 4 Cohomology of Affine spaces

4.1.

4.2.

4.3.

4.4.

4.5.

4.6.

4.7.

4.8.

4.9.

4.10. Let X be a nonsingular variety over an algebraically closed field k, and let $\mathcal{F}$ be a coherent sheaf on X. Show that there is a one-to-one correspondence between the set of infinitesimal extensions of X by $\mathcal{F}$ (II, Ex. 8.7) up to isomorphism, and the group $H^1(X, \mathcal{F} \otimes \mathcal{T})$, where $\mathcal{T}$ is the tangent sheaf of X (II, §8). [*Hint*: Use (II, Ex. 8.6) and (4.5)]

Proof. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an affine open cover of X. Note that since X is nonsingular, so is each U_i. Let $(X', \mathcal{O}_{X'})$ be an infinitesimal extension of X by $\mathcal{F}$, that is, there is a scheme X' with an ideal sheaf $\mathcal{I} \simeq \mathcal{F}$ as $\mathcal{O}_X$-modules, such that $\mathcal{I}^2 = 0$, $(X, \mathcal{O}_{X'}/\mathcal{I}) \simeq (X, \mathcal{O}_X)$ as ringed spaces, and a short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{I} \to \mathcal{O}_{X'} \to \mathcal{O}_X \to 0.$$

Since each U_i is nonsingular and affine, by Ex.II-8.7, the above short exact sequence restricts to a split exact sequence on U_i, where the splitting is given by a lifting $\alpha_i : \mathcal{O}_X|_{U_i} \to \mathcal{O}_{X'}|_{U_i}$.

On each $U_{ij} = U_i \cap U_j$, that is affine since X is separated, we have two liftings $\alpha_i|_{U_{ij}}$, $\alpha_j|_{U_{ij}} : \mathcal{O}_X|_{U_{ij}} \to \mathcal{O}_{X'}|_{U_{ij}}$, and they differs by a section β_{ij} in $\mathrm{Der}_k(\mathcal{O}_X(U_{ij}), \mathcal{I}(U_{ij}))$ so that on U_{ij} we have

$$\alpha_i - \alpha_j = \beta_{ij}.$$

Notice that β_{ij} can be seen as a section in $(\mathcal{F} \otimes \mathcal{T})(U_{ij})$ via isomorphisms

$$\mathrm{Der}_k(\mathcal{O}_X(U_{ij}), \mathcal{I}(U_{ij})) \simeq \mathrm{Hom}_{\mathcal{O}_{X/k}(U_{ij})}(\Omega_{X/k}(U_{ij}), \mathcal{I}(U_{ij})) \simeq (\mathcal{F} \otimes \Omega^*_{X/k})(U_{ij}) \simeq (\mathcal{F} \otimes \mathcal{T})(U_{ij}).$$

Restricting all the above sections onto $U_{ijk} = U_i \cap U_j \cap U_k$, we thus obtain

$$\beta_{ij} + \beta_{jk} + \beta_{ki} = (\alpha_i - \alpha_j) + (\alpha_j - \alpha_k) + (\alpha_k - \alpha_i) = 0,$$

and $\{\beta_{ij}\}$ gives a cocycle of the Čech complex $\check{C}^\bullet(\mathcal{U}, \mathcal{F} \otimes \mathcal{T})$ in degree 1.

For a different choice of liftings $\mu_i : \mathcal{O}_X|_{U_i} \to \mathcal{O}_{X'}|_{U_i}$ for each U_i, as above we have the corresponding sections β'_{ij} of $\mathrm{Der}_k(\mathcal{O}_X(U_{ij}), \mathcal{I}(U_{ij}))$ with $\mu_i - \mu_j = \beta'_{ij}$ on U_{ij}, and with $\beta'_{ij} + \beta'_{jk} + \beta'_{ki} = 0$ on U_{ijk}.

Applying the Ex. II-8.6- (a) again to the pair of liftings α_i and μ_i on U_i, we have sections ξ_i of $\mathrm{Der}_k(\mathcal{O}_X(U_i), \mathcal{I}(U_i))$ for each U_i with $\alpha_i - \mu = \xi_i$, that can also be seen as a section of $\mathcal{F} \otimes \mathcal{T}$ on U_i. Then, on U_{ij} we have

$$\beta_{ij} - \beta'_{ij} = (\alpha_i - \alpha_j) - (\mu_i - \mu_j) = \xi_i - \xi_j,$$

thus the cocycles $\{\beta_{ij}\}$ and $\{\beta'_{ij}\}$ give the same cohomology class in $\check{H}^1(\mathcal{U}, \mathcal{F} \otimes \mathcal{T})$. This last group is isomorphic to $H^1(X, \mathcal{F} \otimes T)$ by (4.5). The converse is easy. This finishes the proof. □

4.11.