Robin Hartshorne's Algebraic Geometry Solutions

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Chapter III Section 10 Smooth morphisms

10.1. Over a nonperfect field, smooth and regular are not equivalent. For example, let k_0 be a field of characteristic $p > 0$, let $k = k_0(t)$, and let $X \subset \mathbb{A}^2_k$ be the curve defined by $y^2 = x^p - t$. Show that every local ring of X is a regular local ring, but X is not smooth over k.

Proof. We need to suppose that char(k$_0$) = p > 2. Let $f = y^2 - x^p + t \in k[x, y]$. Then $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 2y$ so that $\mathrm{rk}\begin{pmatrix}\frac{\partial f}{\partial x}\\ \frac{\partial f}{\partial y}\end{pmatrix} = \mathrm{rk}\begin{pmatrix}0\\ 2y\end{pmatrix} = 1$ everywhere on X because on X $y \neq 0$. Indeed, if $y = 0$, then $x^p = t$ over $k = k_0(t)$, which is not possible. Hence X is regular everywhere and every local ring of X is a regular local ring.

Let's now prove that $X \to \mathrm{Spec}(k)$ is not smooth. Toward contradiction, suppose that it is smooth. Then by base change via $\mathrm{Spec}\left(\overline{k}\right) \to \mathrm{Spec}\left(k\right)$, the morphism $X_{\overline{k}} \to \mathrm{Spec}\left(\overline{k}\right)$ is smooth. But, this is not true: $X_{\overline{k}} \subset \mathbb{A}^2_{\overline{k}}$ is defined by the equation $y^2 = x^p - t = \left(x - t^{\frac{1}{p}}\right)^p$ over $\overline{k}$ and the point $(x, y) = \left(t^{\frac{1}{p}}, 0\right)$ on $X_{\overline{k}}$ has multiplicity 2 so that it is not regular at this point. Contradiction. Hence $X \to \mathrm{Spec}(k)$ is not smooth. □

10.2. Let $f : X \to Y$ be a proper, flat morphism of varieties over k. Suppose for some point $y \in Y$ that the fibre X_y is smooth over $k(y)$. Then show that there is an open neighborhood U of y in Y such that $f : f^{-1}(U) \to U$ is smooth.

Proof. Let n be the relative dimension of the flat morphism $f : X \to Y$. Since $X_y \to \mathrm{Spec}\left(k(y)\right)$ is smooth, $\Omega_{X_y/k(y)}$ is a locally free coherent sheaf of rank n on X_y. That is, for each $x \in X_y$, $\dim_{k(x)}\left(\Omega_{X/Y} \otimes k(x)\right) = \dim_{k(x)}\left(\Omega_{X_y/k(y)} \otimes k(x)\right) = n$. But, $\Omega_{X/Y} \otimes k(x) = \left(\Omega_{X/Y}\right)_x / \mathfrak{m}_x \left(\Omega_{X/Y}\right)_x$ so that by Nakayama's lemma, there exist sections $s_1, \cdots, s_n$ of $\Omega_{X/Y}$ over a neighborhood U_x of x whose images in $\Omega_{X/Y} \otimes k(x) = \left(\Omega_{X/Y}\right)_x / \mathfrak{m}_x \left(\Omega_{X/Y}\right)_x$ form a $k(x)$-basis and they generate $\Omega_{X/Y}$ over U_x. This implies that for all $z \in U_x$, $\dim_{k(z)}\left(\Omega_{X/Y} \otimes k(z)\right) \leq n$. But, by Theorem II-8.6A, $\dim_{k(z)}\left(\Omega_{X/Y} \otimes k(z)\right) \geq n$ so that $\dim_{k(z)}\left(\Omega_{X/Y} \otimes k(z)\right) = n$ for all $z \in U_x$, i.e. $\Omega_{X/Y}|_{U_x}$ is locally free of rank n. Since $x \in X_y$ was arbitrary, by collecting all such U_x, we see that $\{U_x\}_{x \in X_y}$ is a cover of X_y such that $\Omega_{X/Y}$ is locally free on $\bigcup_{x \in X_y} U_x$.

The remaining point is to find a kind of tubular neighborhood of X_y. Since $f : X \to Y$ is proper, by base change $\mathrm{Spec}\left(k(y)\right) \to Y$, $X_y \to \mathrm{Spec}\left(k(y)\right)$ is also proper. Thus, in particular, X_y is quasi-compact and there are finitely many points $x_1, \cdots, x_m \in X_y$ such that $U_{x_1}, \cdots, U_{x_m}$ cover X_y. Since f is flat, it is an open map so that $f(U_{x_i} \subset Y$ is open containing y. Let $U = \bigcap_{i=1}^m f(U_{x_i})$. This is then an open subset of Y containing y. Obviously, $f^{-1}(U) \subset \bigcup_{x \in X_y} U_x$ and thus $\Omega_{X/Y}|_{f^{-1}(U)}$ is locally free on $f^{-1}(U)$. Because flatness is stable under base change, that $\Omega_{X/Y}|_{f^{-1}(U)}$ is locally free of rank n on $f^{-1}(U)$ is equivalent to that $f^{-1}(U) \to U$ is smooth. This finishes the proof. □

10.3. A morphism $f : X \to Y$ of schemes of finite type over k is *étale* if it is smooth of relative dimension 0. It is *unramified* if for every $x \in X$, letting $y = f(x)$, we have $\mathfrak{m}_y \cdot \mathcal{O}_x = \mathfrak{m}_x$, and $k(x)$ is a separable algebraic extension of $k(y)$. Show that the following conditions are equivalent:

(i) f is étale;

(ii) f **is flat, and** $\Omega_{X/Y} = 0$**;**
(iii) f **is flat and unramified.**

Proof. (i) $\Leftrightarrow$ (ii) is obvious by definition. (ii) $\Leftrightarrow$(iii) is a direct consequence of Theorem II-8.6A. $\square$

10.4. Show that a morphism $f : X \to Y$ **of schemes of finite type over** k **is étale if and only if the following condition is satisfied: for each** $x \in X$**, let** $y = f(x)$**. Let** $\widehat{\mathcal{O}}_x$ **and** $\widehat{\mathcal{O}}_y$ **be the completions of the local rings at** x **and** y**. Choose fields of representatives (II, 8.25A)** $k(x) \subset \widehat{\mathcal{O}}_x$ **and** $k(y) \subset \widehat{\mathcal{O}}_y$ **so that** $k(y) \subset k(x)$ **via the natural map** $\widehat{\mathcal{O}}_y \to \widehat{\mathcal{O}}_x$**. Then our condition is that for every** $x \in X$**,** $k(x)$ **is a separable algebraic extension of** $k(y)$**, and the natural map**

$$\widehat{\mathcal{O}}_y \otimes_{k(y)} k(x) \to \widehat{\mathcal{O}}_x$$

is an isomorphism.

Proof. By definition, $f : X \to Y$ is unramified if and only if for all $x \in X$ with $y = f(x)$, $k(x)$ is separable over $k(y)$ and $\mathfrak{m}_y \cdot \mathcal{O}_x = \mathfrak{m}_x$. Also, Ex III-10.3 shows that f is étale if and only if f is flat and unramified. Thus, it is enough to show that f is flat if and only if for all $x \in X$ with $y = f(x)$, $\widehat{\mathcal{O}}_y \otimes_{k(y)} k(x) \xrightarrow{\simeq} \widehat{\mathcal{O}}_x$.

Since flatness is a local condition, it follows from the following three statements. (All rings are supposed to be noetherian.)

Claim (1)**.** *A is a ring and $I \subset A$ is an ideal. Then, the I-adic completion $A \to \widehat{A}$ is faithfully flat if and only if $I \subset \sqrt{A}$. (It is always flat.)*

Proof. See SGA 1, IV-Cor 3.2 $\square$

Claim (2)**.** *Let $(A, \mathfrak{m})$, $(B, \mathfrak{n})$ be local rings with a local homomorphism $A \to B$. Then,*

$$grB \simeq grA \otimes_A B \Leftrightarrow \widehat{B} \simeq \widehat{A} \otimes_A B.$$

Proof. Easy. $\square$

Claim (3)**.** *Let $(A, \mathfrak{m}), (B, \mathfrak{n})$, $A \to B$ be as above. Then, $A \to B$ is flat if and only if $\widehat{B} \simeq \widehat{A} \otimes_A B$.*

Proof. ($\Rightarrow$) Since $0 \to \mathfrak{m}^{r+1} \to \mathfrak{m}^r \to \mathfrak{m}^r/\mathfrak{m}^{r+1} \to 0$ is exact and $A \to B$ is flat, $0 \to \mathfrak{m}^{r+1} \otimes_A B \to \mathfrak{m}^r \otimes_A B \to \mathfrak{m}^r/\mathfrak{m}^{r+1} \otimes_A B \to 0$ is exact. Since $\mathfrak{m}B = \mathfrak{n}$ and $\mathfrak{m}^r/\mathfrak{m}^{r+1} \otimes_A B = \mathfrak{m}^r B/\mathfrak{m}^{r+1}B = (\mathfrak{m}B)^r/(\mathfrak{m}B)^{r+1}$, we immediately obtain that $grB \simeq grA \otimes_A B$ which implies that $\widehat{B} \simeq \widehat{A} \otimes_A B$ by Claim (2).

($\Leftarrow$) Let $M \to N$ be an injective A-module homomorphism. We want to show that $M \otimes_A B \to N \otimes_A B$ is an injection. Since $A \to \widehat{A}$ is flat, $B \otimes_A \widehat{A}$ is also a flat A-module. Thus, $M \otimes_A \left(B \otimes_A \widehat{A}\right) \to N \otimes_A \left(B \otimes_A \widehat{A}\right)$ is an injection But, $- \otimes_A \left(B \otimes_A \widehat{A}\right) \simeq (- \otimes_A B) \otimes_A \widehat{A}$ is an injection and since $A \to \widehat{A}$ is faithfully flat by Claim (1), $M \otimes_A B \to N \otimes_A B$ is injective as desired. This finishes the proof. $\square$

Thus, taking $A = \mathcal{O}_y$, $B = \mathcal{O}_x$ gives the desired result because, when $L = k(y)$, $k = k(x)$, we have $L = B \otimes_A k$ so taht $- \otimes_k L = - \otimes_k k \otimes_A B = - \otimes_A B$. $\square$

10.5. If x is a point of a scheme X, we define an *étale neighborhood* of x to be an étale morphism $f : U \to X$, together with a point $x' \in U$ such that $f(x') = x$. As an example of the use of étale neighborhoods, prove the following: if $\mathcal{F}$ is a coherent sheaf on X, and if every point of X has an étale neighborhood $f : U \to X$ for which $f^*\mathcal{F}$ is a free $\mathcal{O}_U$-module, then $\mathcal{F}$ is locally free on X.

Proof. The question being local, we may suppose that both U and X are affine. Furthermore, by localizing them at x and x', we reduce the problem to show the following:

$A \to B$ is an étale local homomorphism of local rings and M is an A-module such that $M \otimes_A B \simeq B^n$. Then, $M \simeq A^n$.

But, this is easy: $M \otimes_A B \simeq B^n \simeq A^n \otimes_A B$ and since $B \neq 0$ and B is A-flat, B is a faithfully flat A-module (SGA1 IV-cor 2.2) so that $M \simeq A^n$. This finishes the proof. $\square$