Robin Hartshorne's Algebraic Geometry Solutions

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Chapter II Section 8, Differentials

8.1.

(a).

(b).

(c).

(d).

8.2.

8.3.

(a).

(b).

(c).

8.4.

(a).

(b).

(c).

(d).

(e).

(f).

(g).

8.5.

(a).

(b).

8.6.

(a). Here, we assume that there exists at least one lefting $g : A \to B'$. We prove all the required propositions.

Claim. *I has a natural structure of B-module.*

Let $b \in B$, $x \in I$. Let $b' \in B'$ be a lifting of b under the given surjection $p : B' \to B$. Define $b \cdot x = b'x \in I$. If $b'' \in B'$ is another lifting of b, then $p(b'' - b') = 0$ implies $b'' - b' \in I$. Hence, $b''x - b'x = (b'' - b')x \in I^2 = 0$, i.e. $b \cdot x$ is well defined. It proves the claim.

Since we have a k-algebra homomorphism $f : A \to B$ and $g : A \to B'$ is a lifting, in fact, $b \cdot x = g(b)x$ by above claim for any lifting g.

If $g' : A \to B'$ is another such lifting, then obviously the image of $\theta = g - g'$ lies in I.

Claim. *$\theta : A \to I$ is a k-derivation.*

Obviously, it is additive because g, g' are. For $a \in k$, since $g(1) = g'(1)$, $\theta(a) = g(a) - g'(a) = ag(1) - ag'(1) = 0$. We now need to prove that for $a, b \in A$, $\theta(ab) = a\theta(b) + b\theta(a)$, i.e.

$$g(ab) - g'(ab) = a(g(b) - g'(b)) + b(g(a) - g'(a)).$$

Recall how the action of A was defined on I. Hence,

$$\begin{aligned} RHS &= g(a)(g(b) - g'(b)) + g'(b)(g(a) - g'(a)) = g(ab) - g(a)g'(b) + g'(b)g(a) - g'(ab) \\ &= g(ab) - g'(ab) = LHS \end{aligned}$$

so that θ is a k-derivation, i.e. $\theta \in Der_k(A, I) = \mathrm{Hom}_A(\Omega_{A/k}, I)$. It proves the claim.

Now, conversely, let $\theta \in \mathrm{Hom}_A(\Omega_{A/k}, I) = Der_k(A, I)$.

Claim. *$g' := g + \theta$ is another lifting of f.*

Since θ is additive, so is g'. Now,

$$\begin{aligned} g'(ab) &= g(ab) + \theta(ab) = g(ab) + a\theta(b) + b\theta(a) \\ &= g(a)g(b) + g(a)\theta(b) + g(b)\theta(a) + \theta(a)\theta(b) \\ &= (g(a) + \theta(a))(g(b) + \theta(b)) = g'(a)g'(b) \end{aligned}$$

so that g' is multiplicative.

If $a \in k$, then θ is a k-derivation so that $\theta(a) = 0$. Hence $g'(a) = g(a) = ag(1) = a$. Hence g' is a k-algebra homomorphism. Now, $(p \circ g')(a) = p(g(a) + \theta(a)) = p \circ g(a) + p(\theta(a)) = f(a)$ because $\theta(a) \in I$ and $p(I) = 0$. Hence g' is another lifting of f.

(b). For each i, choose $b_i \in B'$ such that $p(b_i) = f(\bar{x_i})$. Define $h : P = k[x_1, \cdots, x_n] \to B'$ be the k-algebra homomorphism determined by $h(x_i) := b_i$. Obviously, the diagram commutes by construction.

Let $q : P \to A$ be the given surjection. If $j \in J$, then since $q(j) = 0$, we have $f(q(j)) = p(h(j)) = 0$ i.e. $h(j) \in I$. Hence we have $h|_J : J \to I$. But $I^2 = 0$ implies that we have a k-homomorphism $\bar{h} : J/J^2 \to I$.

Claim. *This map is even A-linear.*

First, we note that J/J^2 has a canonical A-action. Let $a \in A$, $[j] \in J/J^2$. Choose any lifting $a' \in P$ of a and define $a \cdot [j] = [a'j]$. If we have another lifting a'' of a, then $a'' - a' \in J$ so that $(a'' - a')j \in J^2$, i.e. $[a'j] = [a''j]$ so, this action is well-defined.

In part (a), we noted that the action of A on I is well-defined. To show that $\bar{h} : J/J^2 \to I$ is A-equivariant, it is enough to show that the action of A is preserved. This is easy: Let $a \in A$ and choose a lifting $a' \in P$. Then by the commutativity of the diagram, $h(a')$ is a lifting of $f(a)$ so that for $[j] \in J/J^2$,

$$\bar{h}(a \cdot [j]) = \bar{h}([a'j]) = h(a'j) = h(a')h(j) = a \cdot h(j) = a \cdot \bar{j}([j]).$$

It proves the required A-linearity.

(c). By the hypothesis, $\mathrm{Spec}A \hookrightarrow \mathbb{A}^n_k$ is a nonsingular subvariety. Hence by (8.17), we have an exact sequence

$$0 \to J/J^2 \to \Omega_{P/k} \otimes A \to \Omega_{A/k} \to 0.$$

A being nonsingular, $\Omega_{A/k}$ is projective (because the sheaf $\Omega_{\mathrm{Spec}A/k}$ is locally free). Hence, above sequence splits and so by applying $\mathrm{Hom}_A(-I)$, we obtain

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_A(\Omega_{A/k}, I) & \longrightarrow & \mathrm{Hom}_A(\Omega_{P/k} \otimes A, I) & \longrightarrow & \mathrm{Hom}_A(J/J^2, I) & \longrightarrow & 0\,. \\ & & & & \downarrow \simeq & & & & \\ & & & & \mathrm{Hom}_P(\Omega_{P/k}, I) & \xrightarrow{=} & Der_k(P, I) & & \end{array}$$

Let $\theta \in \mathrm{Hom}_P(\Omega_{P/k}, I)$ be an element mapped to $\bar{h} \in \mathrm{Hom}_A(J/J^2, I)$ defined in part (b). Regard θ as a k-derivation of P to $B' \supset I$. Let $h' = h - \theta$.

Claim. *$h' : P \to B'$ is a k-homomorphism such that $h'(J) = 0$.*

Obviously, θ being a k-derivation, $h'(a) = a$ for $a \in k$ and h' is additive. If $a, b \in P$, then

$$h'(ab) = h(ab) - \theta(ab) = h(ab) - b\theta(a) - a\theta(b) + \theta(a)\theta(b)$$

$$= (h(a) - \theta(a))(h(b) - \theta(b)) = h'(a)h'(b).$$

If $j \in J$, $\theta(j) = \bar{h}(j) = h(j)$ so that $h(j) = h(j) - \theta(j) = 0$. Hence h' gives a rise to a k-homomorphism $g : A \to B'$. Since h was a lifting of f from P to B', obviously, g is indeed a required lifting.

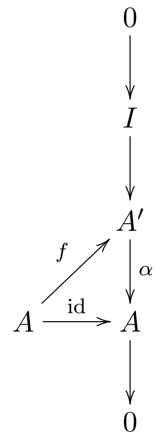
8.7. As an application of the infinitesimal lifting property, we consider the following general problem. Let X be a scheme of finite type over k, and let $\mathcal{F}$ be a coherent sheaf on X. We seek to classify schemes X' over k, which have a sheaf of ideals $\mathcal{I}$ such that $\mathcal{I}^2 = 0$ and $(X', \mathcal{O}_{X'}/\mathcal{I}) \simeq (X, \mathcal{O}_X)$, and such that $\mathcal{I}$ with its resulting structure of $\mathcal{O}_X$-module is isomorphic to the given sheaf $\mathcal{F}$. Such a pair $X', \mathcal{F}$ we call an *infinitesimal extension* of the scheme X bye the sheaf $\mathcal{F}$. One such extension, the *trivial* one, is obtained as follows. Take $\mathcal{O}_{X'} = \mathcal{O}_X \oplus \mathcal{F}$ as sheaves of abelian groups, and define multiplication by $(a \oplus f) \cdot (a' \oplus f') = aa' \oplus (af' + a'f)$. Then the topological space X with the sheaf of rings $\mathcal{O}_{X'}$ is an infinitesimal extension of X by $\mathcal{F}$.

The general problem of classifying extensions of X by $\mathcal{F}$ can be quite complicated. So for now, just prove the following special case: if X is affine and nonsingular, then any extension of X by a coherent sheaf $\mathcal{F}$ is isomorphic to the trivial one. See (III, Ex. 4.10) for another case.

Proof. Suppose that we have an infinitesimal extension

$$0 \to I \to A' \xrightarrow{\alpha} A \to 0$$

defined by a ring A' and its square-zero ideal I with $I^2 = 0$. By the infinitesimal lifting property we have a lifting f, that is a k-algebra homomorphism, of the identity map of A:



and it gives a splitting of $A' \simeq A \oplus I$ as k-modules. We show that it is in fact an isomorphism of k-algebras, where $A \oplus I$ is seen as given the structure of the trivial extension as in the statement of the problem.

For each $x, y \in A'$, we have $x - f(\alpha(x)), y - f(\alpha(y)) \in I$. Since $I^2 = 0$ we have

$$(x - f(\alpha(x)))(y - f(\alpha(y))) = 0$$

that gives $xy = -f(\alpha(x))f(\alpha(y)) + xf(\alpha(y)) + f(\alpha(x))y$. Thus,

$$\begin{aligned} xy - f(\alpha(xy)) &= xy - f(\alpha(x))f(\alpha(y)) = -2f(\alpha(x))f(\alpha(y)) + xf(\alpha(y)) + f(\alpha(x))y \\ &= (x - f(\alpha(x)))f(\alpha(y)) + f(\alpha(x))(x - f(\alpha(y))). \end{aligned}$$

This immediately implies that, when we identify $x \in A'$ with the pair $(f(\alpha(x)), x - f(\alpha(x)))$ of $A \oplus I$, the product structure of A' is identical to that of $A \oplus I$, as desired. Thus there is only one extension up to isomorphism. $\square$

8.8.