QUADRISECANTS OF KNOTS WITH SMALL CROSSING NUMBER

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Abstract. It is known that every nontrivial knot has at least two quadrisecants. In this paper we illustrate quadrisecants for several configurations of the knots of crossing number up to five. Given a knot, we mark each intersection point of each of its quadrisecants; replacing the subarcs between marked points by straight segments gives a modified knot called the quadrisecant approximation. We conjecture that this always has the same knot type as the original.

1. Introduction

A *knot* is a locally flat simple closed curve in the Euclidean space $\mathbb{R}^3$. A *quadrisecant* of a knot K is a straight line L such that $K \cap L$ is a set of four points.[3, 5, 6] The *knottedness* of a knot K is the minimal number of singular points on the boundary of locally flat singular spanning disks of K.[2, 6]

Theorem 1. *A generic polygonal knot with knottedness k has at least $k^2/2$ quadrisecants.*[6]

Corollary 2. *Every nontrivial generic polygonal knot has at least two quadrisecants.*

A polygonal knot is said to be in *general position* if no quadruple of vertices are coplanar, no triple of edges are linearly dependent, and no quadruple of edges are on a single hyperboloid of one sheet.

Proposition 3. *Every polygonal knot in general position has only finitely many quadrisecants.*

Budney, Conant, Scannell and Sinha have the same result.[1] Since every smooth knot is arbitrarily close to a polygonal knot, we expect that smooth knots in general position have finitely many quadrisecants. Quadrisecants of knots in general position are studied by Kuperberg.[3]

Lemma 4. *If four edges are mutually disjoint, then there are at most two quadrisecants to them.*

Proof: Three of the four edges belong to the same family of generators of a unique hyperboloid of one sheet. Then the fourth edge cannot be any generator of the hyperboloid. Therefore it intersects the hyperboloid in at most two points. A generator of the hyperboloid which passes through the intersection point and belongs to the other family is a quadrisecant only if it meets each of the first three edges. □

Proof of Proposition 3 : For any quadrisecant, the four secant points must belong to four distinct edges. If the four edges are mutually disjoint, then they have at most two quadrisecants, by Lemma 4. If two edges are adjacent, they determine a plane. By the general position condition, the extension of each of the other two edges intersects the plane. The line determined by the two intersection points is a quadrisecant if it meets all the four edges in distinct points. □

In this work we have located quadrisecants for some knots of crossing number not greater than five. We used Maple 8.00 for all the computations and graphics. For smooth knots, all solutions are numerical. For polygonal knots whose vertices have rational coordinates, the coefficients of the equations of the quadrisecants are rational numbers if any of the two edges involved are adjacent and roots of integral quadratic equations if all four edges are mutually disjoint. Therefore solutions to polygonal knots are exact if the vertices have rational coordinates. All of our examples meet the conjecture of Morton and Mond that there are at least $n(n-1)/2$ quadrisecants where n is the minimal crossing number.[5]

Let K be a knot which has finitely many quadrisecants. Then they cut K into finitely many subarcs. Straightening each of the subarcs with the end points fixed, we obtain a polygonal knot $\widehat{K}$ which may have self-intersections. We call $\widehat{K}$ the *quadrisecant approximation* of K. It is interesting that our examples show that taking quadrisecant approximation does not change the knot type.

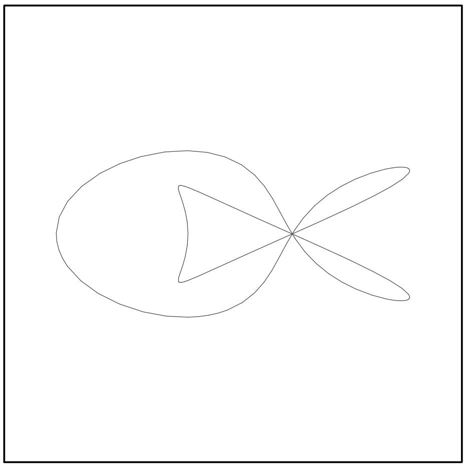
Conjecture 1. *If K has finitely many quadrisecants, then the quadrisecant approximation $\widehat{K}$ has the knot type of K. Furthermore K and $\widehat{K}$ have the same set of quadrisecants.*

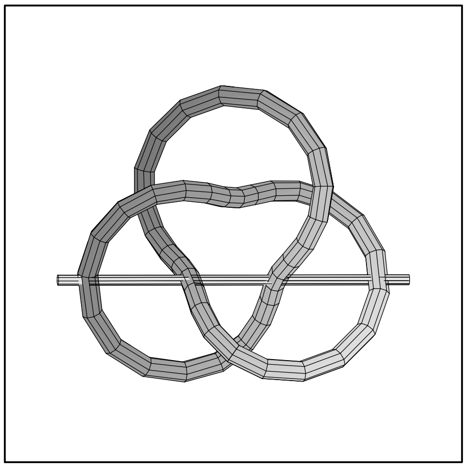
2. The knot $\mathbf{3_1}$ — the trefoil knot

We first used the following parameterization of the trefoil knot T_1.

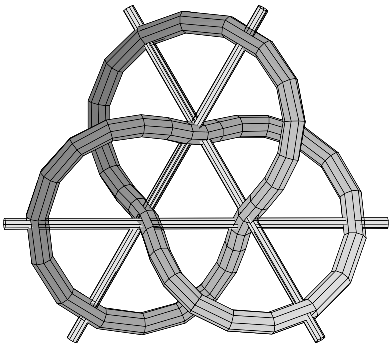
$$\begin{aligned} x(t) &= (2+\sin 3t)\cos 2t \\ y(t) &= (2+\sin 3t)\sin 2t \\ z(t) &= \cos 3t \end{aligned}$$

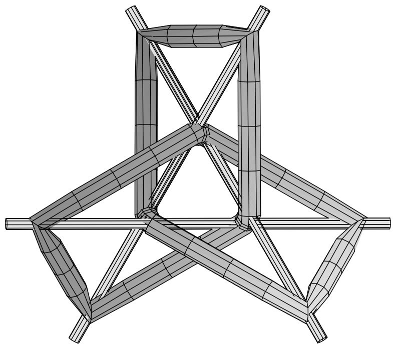
As this knot has the $2\pi/3$-rotational symmetry, it has a multiple of three quadrisecants. It has only three quadrisecants. The left figure below is the projection of T_1 into a plane perpendicular to the quadrisecant shown in the right figure below.



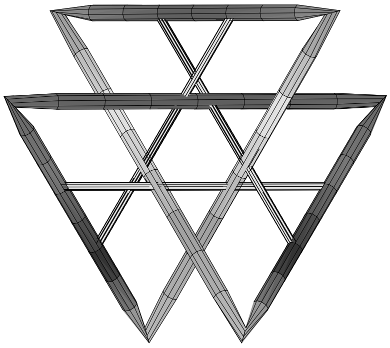


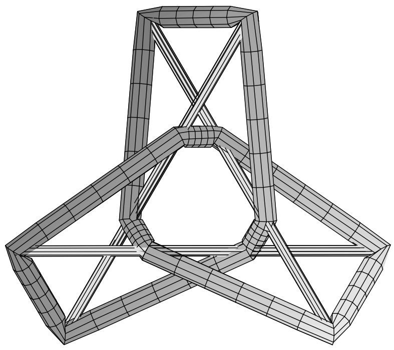
The left figure below is T_1 with all of its three quadrisecants. The right figure below is the quadrisecant approximation $\widehat{T_1}$.





The polygonal knot T_2 with vertices cyclically located at $(-6, 23, -9)$, $(-6, -23, 9)$, $(23, -6, -9)$, $(-17, 17, 9)$, $(-17, -17, -9)$, $(23, 6, 9)$ is also a trefoil knot. It has only three quadrisecants. All the secant points have rational coordinates. The left figure below is T_2 with all of its quadrisecants. The right figure below is the quadrisecant approximation $\widehat{T_2}$. There are no other quadrisecants of $\widehat{T_2}$ than those three of T_2.

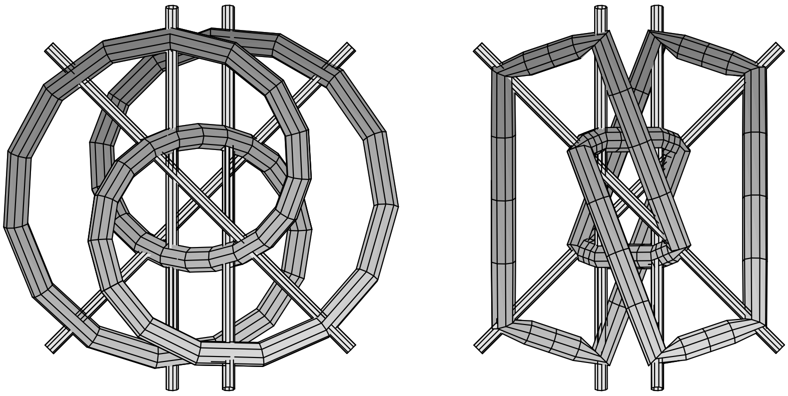




The knot T_3 given by the parameterization

$$\begin{aligned} x(t) &= (2 + \sin 2t) \cos 3t \\ y(t) &= (2 + \sin 2t) \sin 3t \\ z(t) &= \cos 2t \end{aligned}$$

is also a trefoil knot and has a π-rotational symmetry. It has four quadrisecants as shown below left. On the right is the quadrisecant approximation $\widehat{T_3}$.

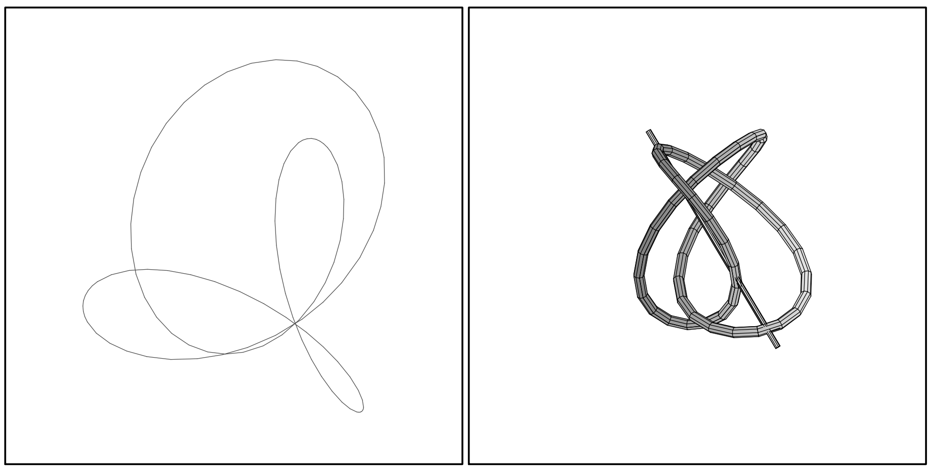


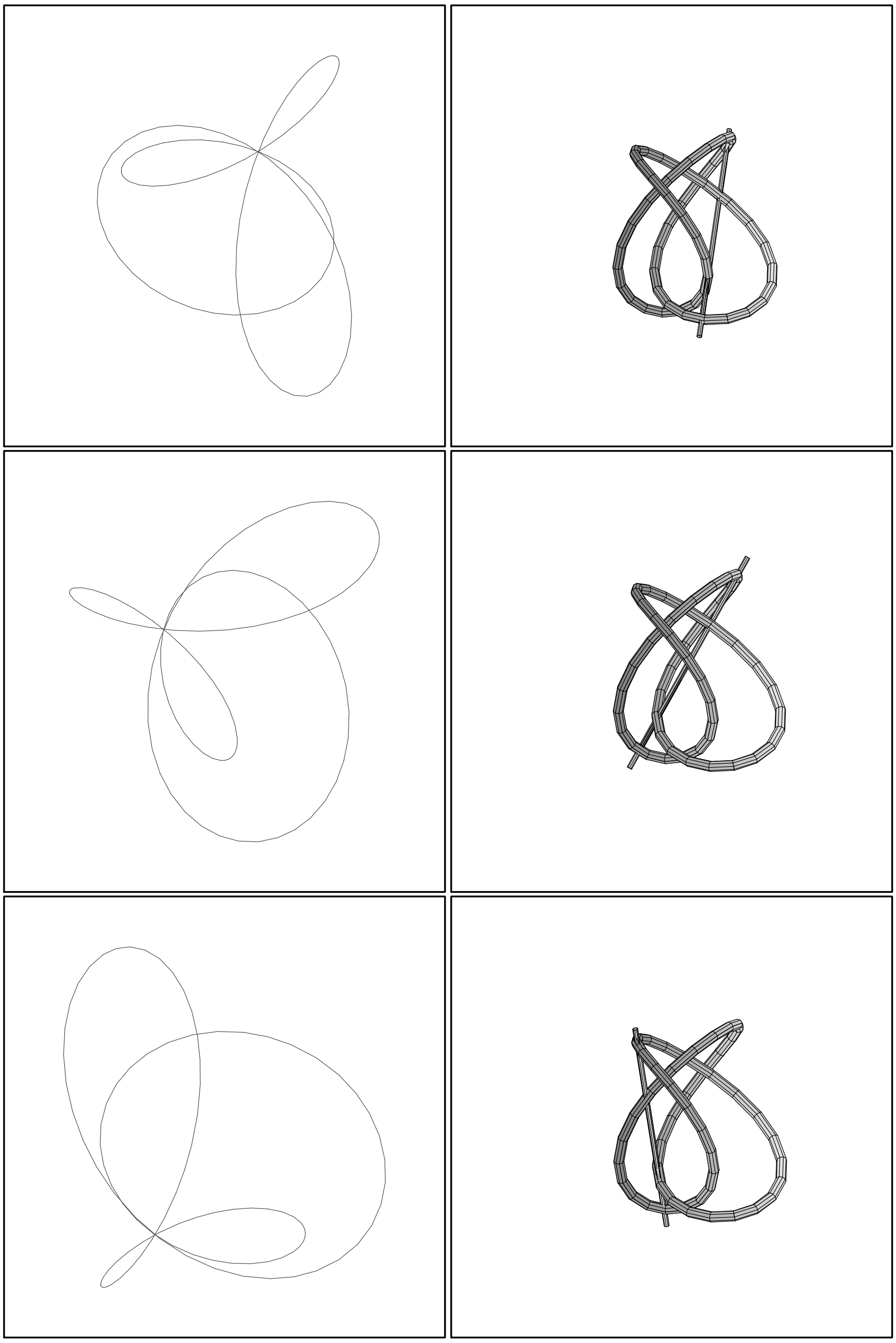
3. The knot $\mathbf{4_1}$ — the figure eight knot

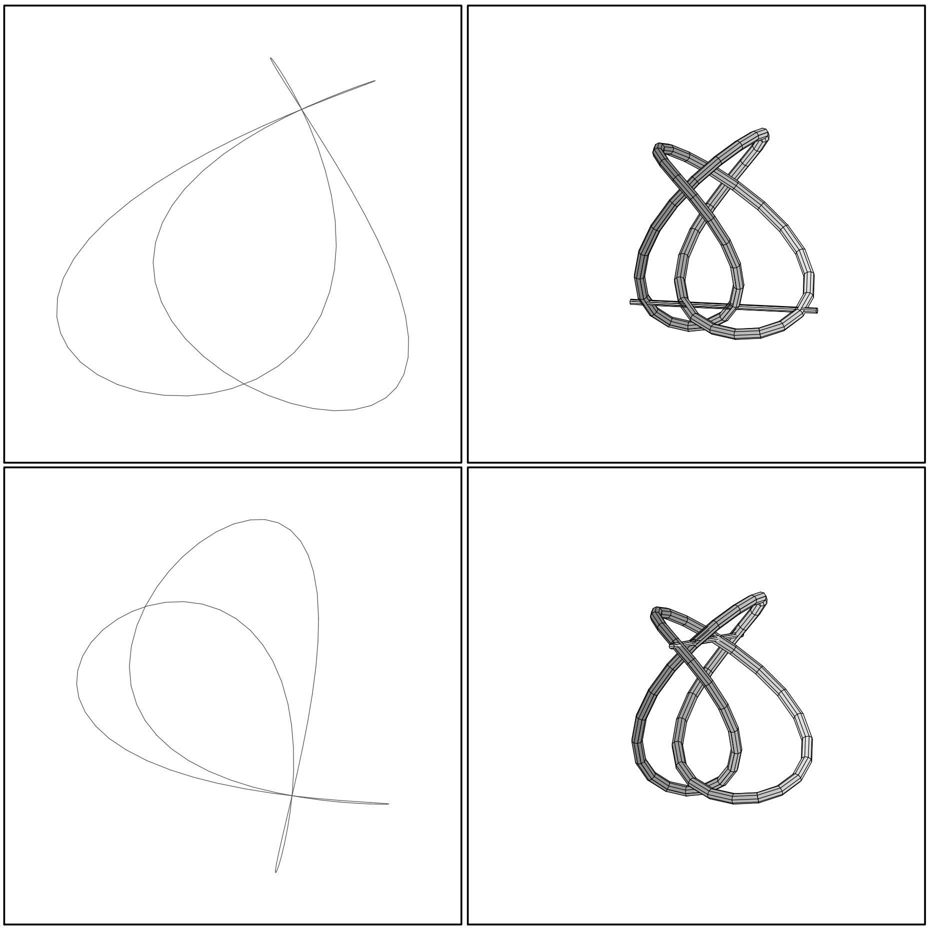
The knot F_1 with the following parameterization is a figure eight knot.

$$\begin{aligned} x(t) &= 32\cos t - 51\sin t - 104\cos 2t - 34\sin 2t + 104\cos 3t - 91\sin 3t \\ y(t) &= 94\cos t + 41\sin t + 113\cos 2t - 68\cos 3t - 124\sin 3t \\ z(t) &= 16\cos t + 73\sin t - 211\cos 2t - 39\sin 2t - 99\cos 3t - 21\sin 3t \end{aligned}$$

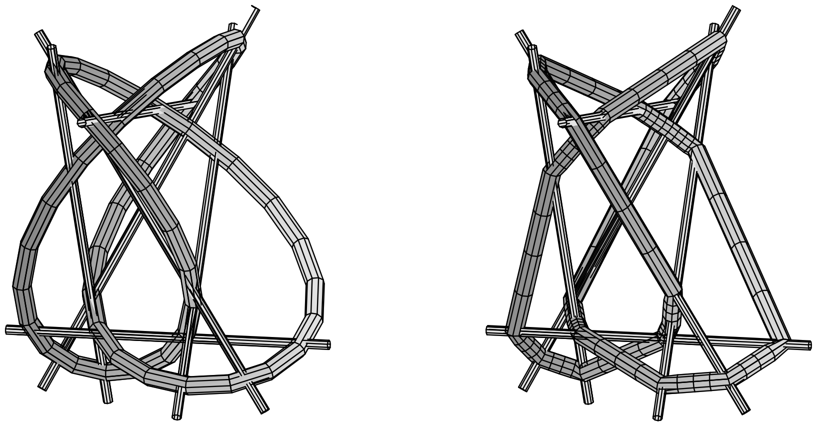
It has the property that the coordinate functions are polynomials in $\sin t$ and $\cos t$ of degree three which is minimal for all figure eight knots. This minimal degree is an invariant called the *harmonic index*.[8] F_1 has six quadrisecants. In each row of the figures below, the left figure is the projection of F_1 into a plane perpendicular to the quadrisecant shown in the right figure.



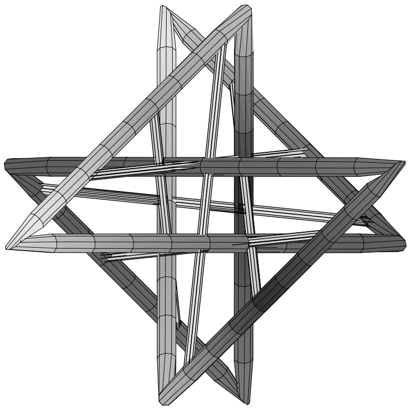


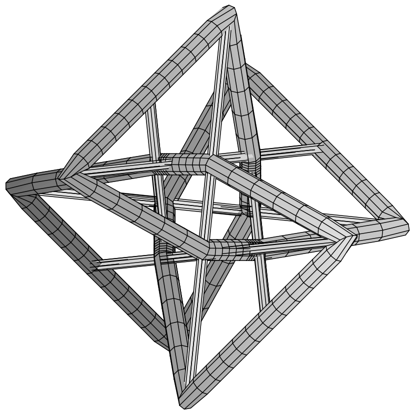


The left figure below is F_1 together with all of its six quadrisecants. The right figure below is the quadrisecant approximation $\widehat{F}_1$.



The polygonal knot F_2 whose vertices are cyclically located at $(10, 10, 2)$, $(-3, -10, 2)$, $(-10, 2, -10)$, $(3, 2, 10)$, $(10, -10, -2)$, $(-3, 10, -2)$, $(-10, -2, 10)$, $(3, -2, -10)$ is also a figure eight knot. It has six quadrisecants. All the secant points have rational coordinates. The left figure below is F_2 together with the six quadrisecants. The right figure below is the quadrisecant approximation $\widehat{F}_2$. There are no other quadrisecants of $\widehat{F}_2$ than those six of F_2. Each of F_2 and $\widehat{F}_2$ has a π-rotational symmetry and the $\pi/2$-rotation about the axis of symmetry gives the mirror image.[4]



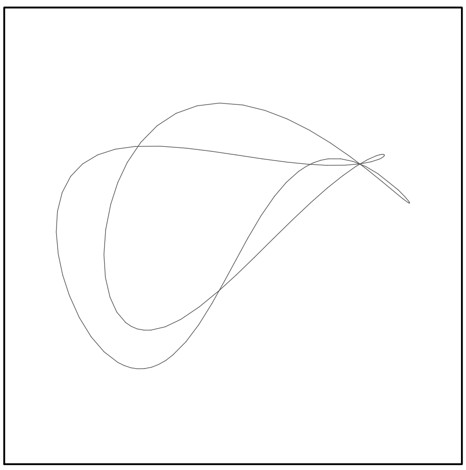


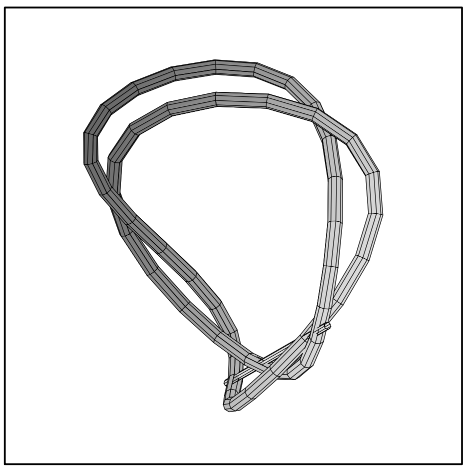
4. The knot $\mathbf{5_1}$ — the torus knot of type (5,2)

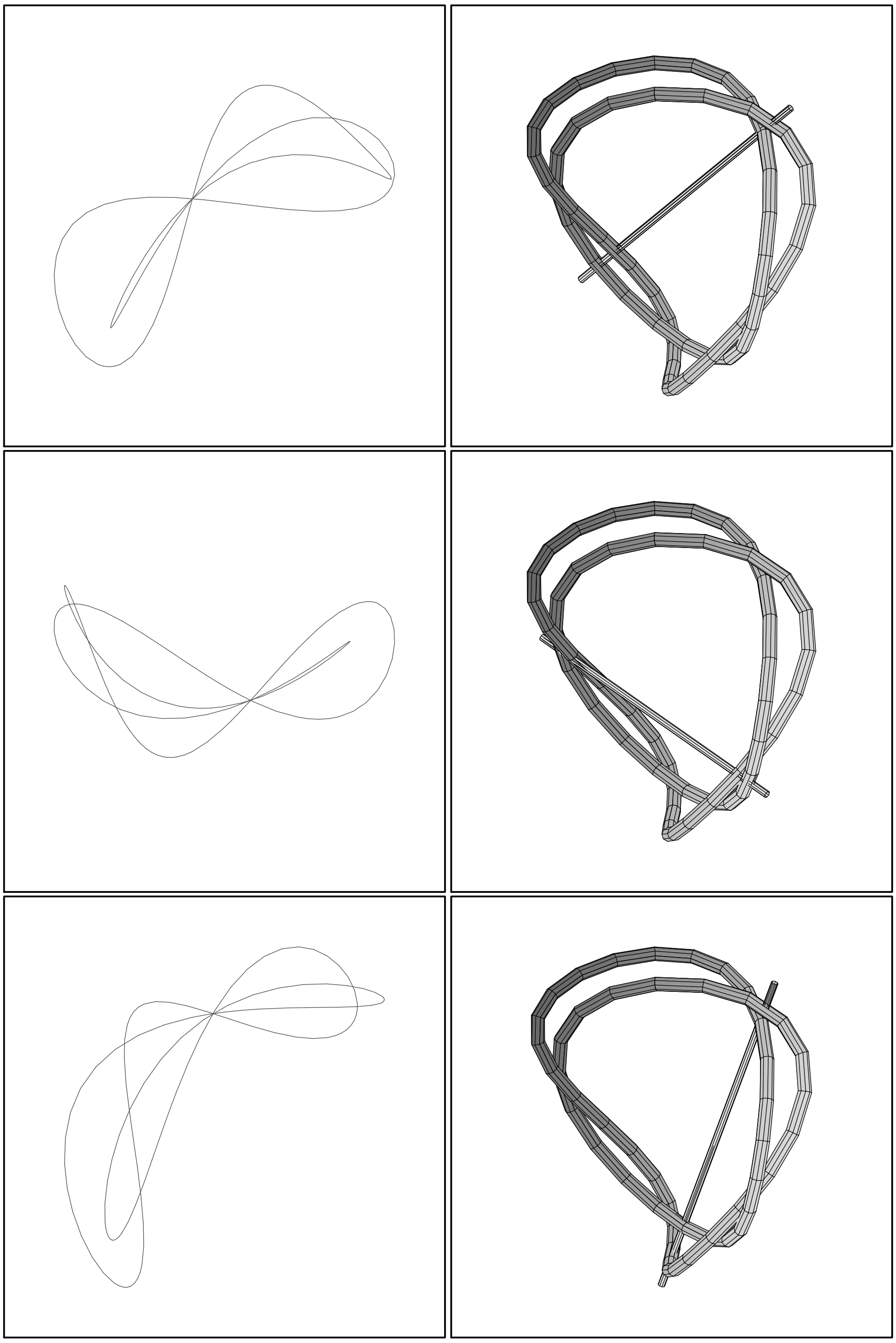
The knot C with the following parameterization is a type $(5,2)$ torus knot.

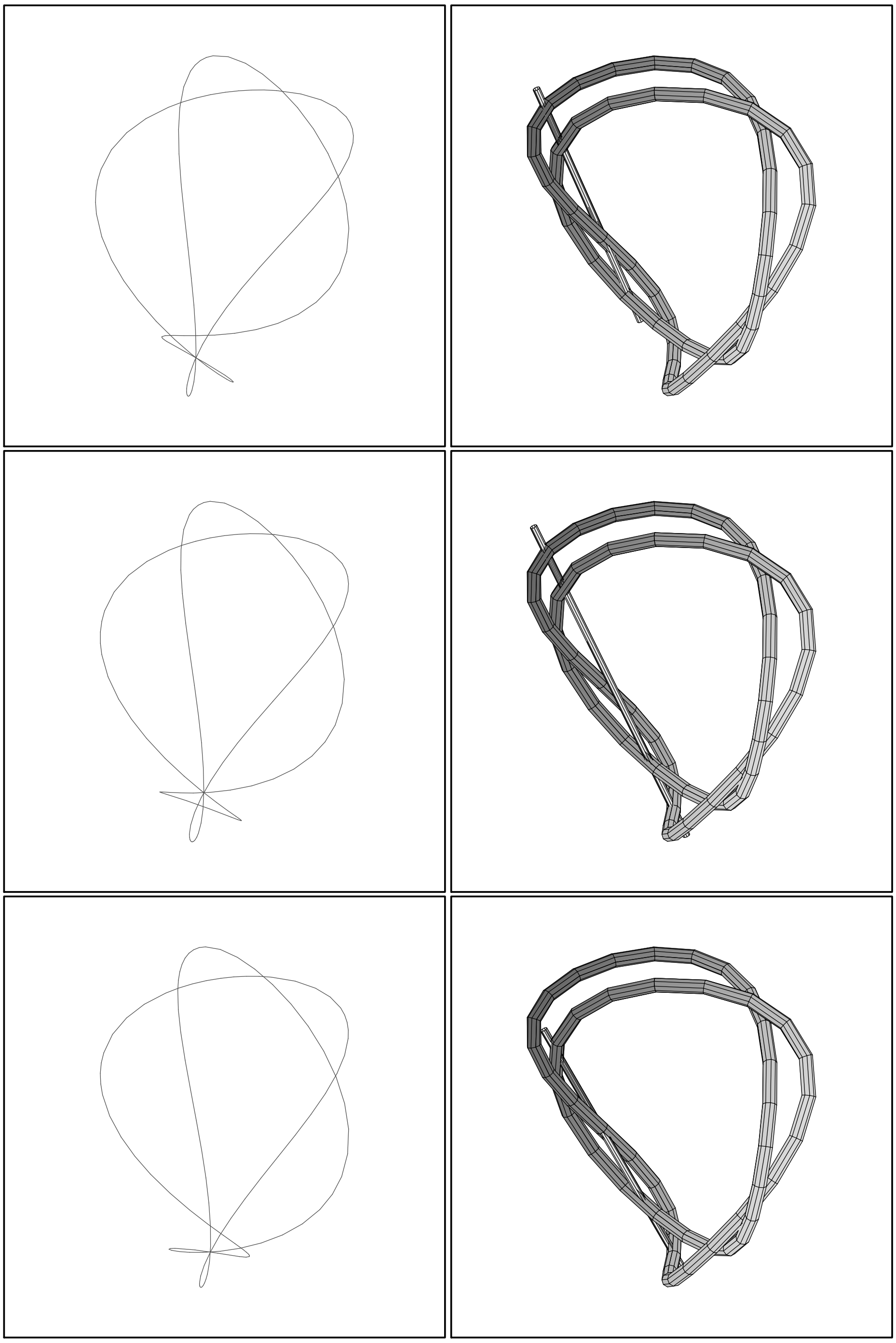
$$\begin{aligned}
x(t) &= 88\cos t + 115\sin t - 475\cos 2t - 127\sin 2t - 87\cos 3t \\
&\qquad +36\sin 3t + 11\cos 4t - 19\sin 4t \\
y(t) &= 89\cos t - 32\sin t - 172\cos 2t + 294\sin 2t + 76\cos 3t \\
&\qquad +102\sin 3t - 61\cos 4t + 113\sin 4t \\
z(t) &= 44\cos t - 69\sin t + 34\cos 2t + 223\sin 2t + 16\cos 3t \\
&\qquad +120\sin 3t + 42\cos 4t - 125\sin 4t
\end{aligned}$$

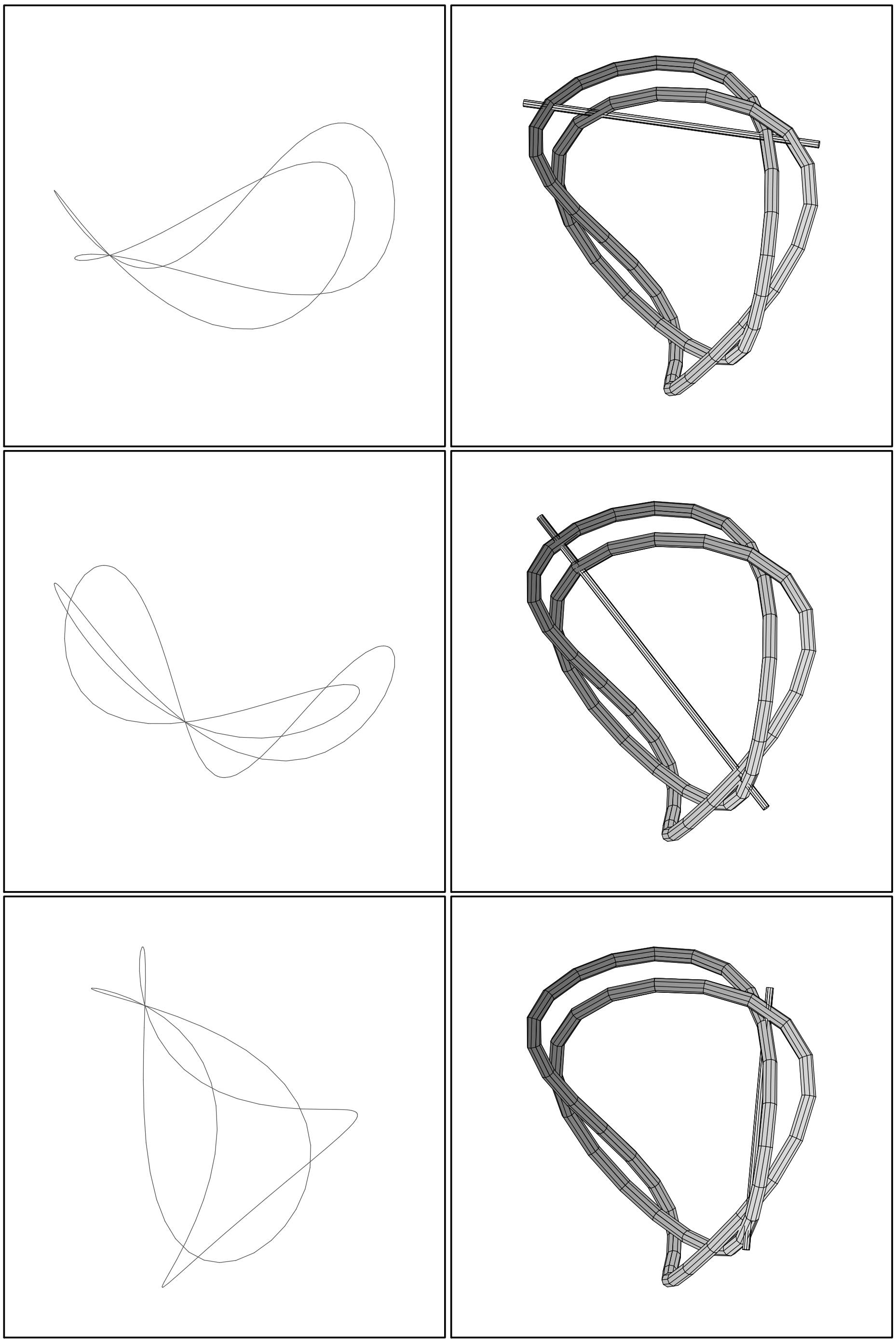
The coordinate functions are polynomials in $\sin t$ and $\cos t$ of degree four which is minimal for all torus knots of type $(5,2)$.[8] C has ten quadrisecants. In each row of the figures below, the left figure is the projection of C into a plane perpendicular to the quadrisecant shown in the right figure.



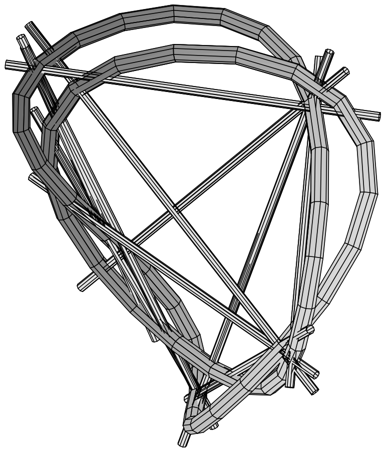


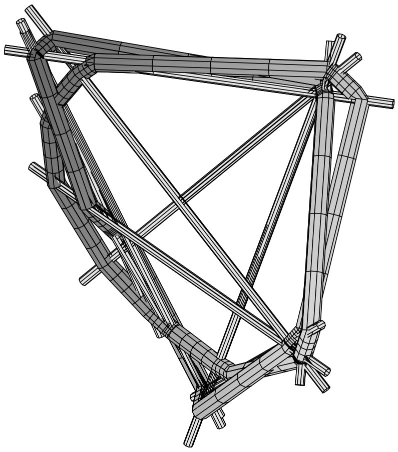






The left figure below is C together with all of its ten quadrisecants. The right figure below is the quadrisecant approximation $\widehat{C}$.

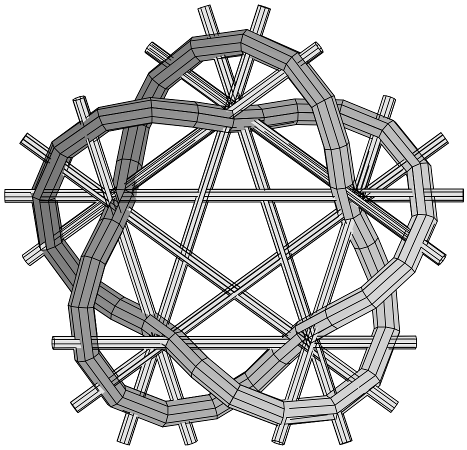


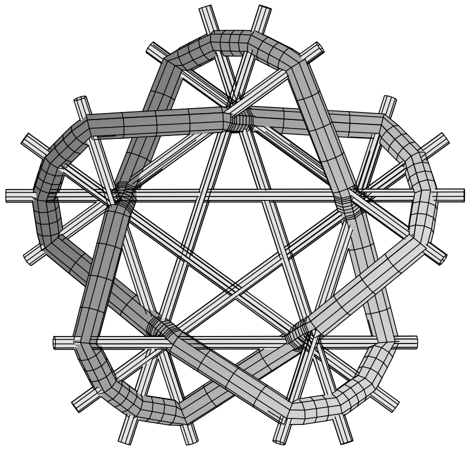


The torus knot of type (5,2) on a standard torus parameterized by

$$\begin{aligned} x(t) &= (4+\sin 5t)\cos 2t \\ y(t) &= (4+\sin 5t)\sin 2t \\ z(t) &= \cos 5t \end{aligned}$$

is shown below left with its ten quadrisecants. On the right is its quadrisecant approximation shown with the ten quadrisecants.



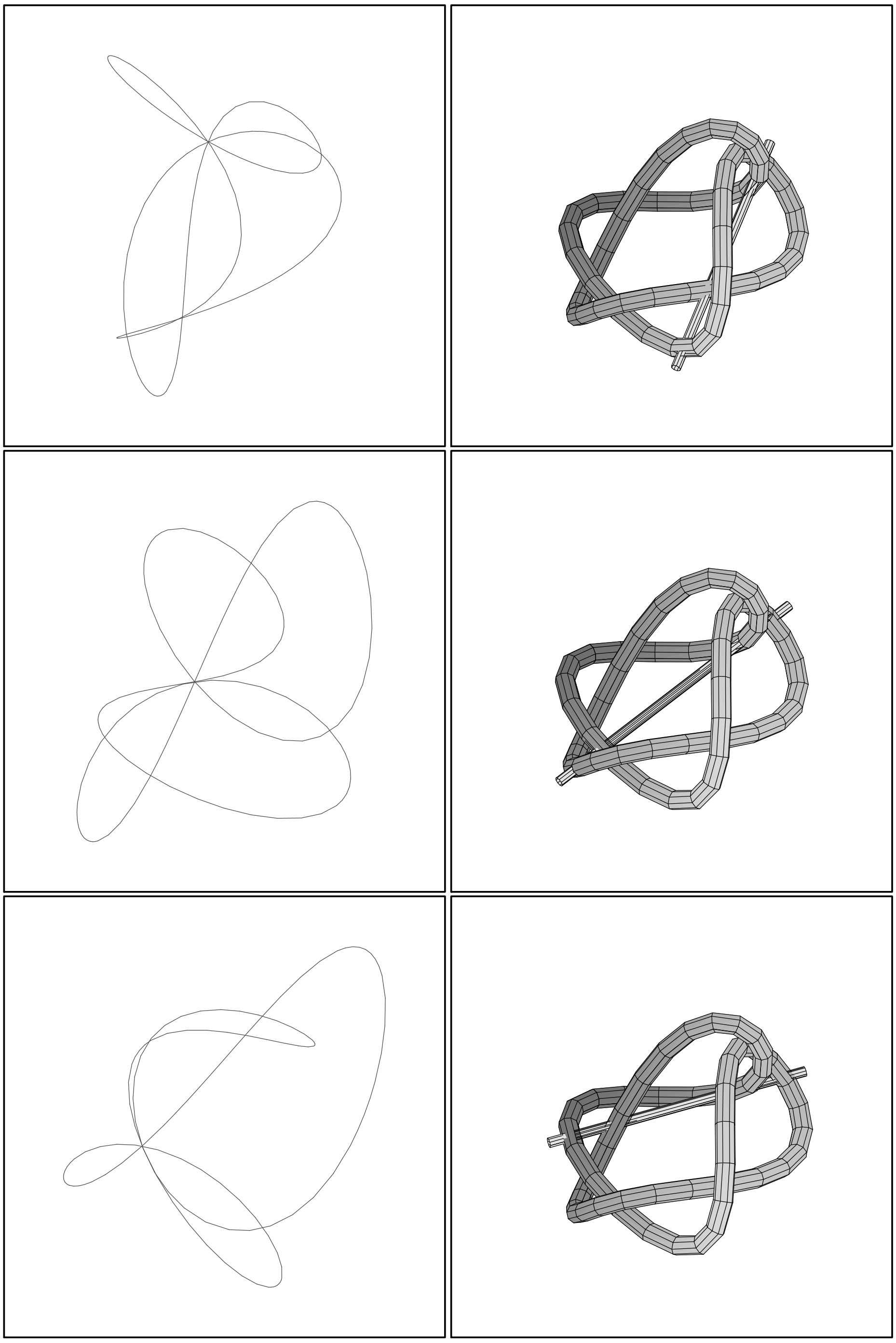


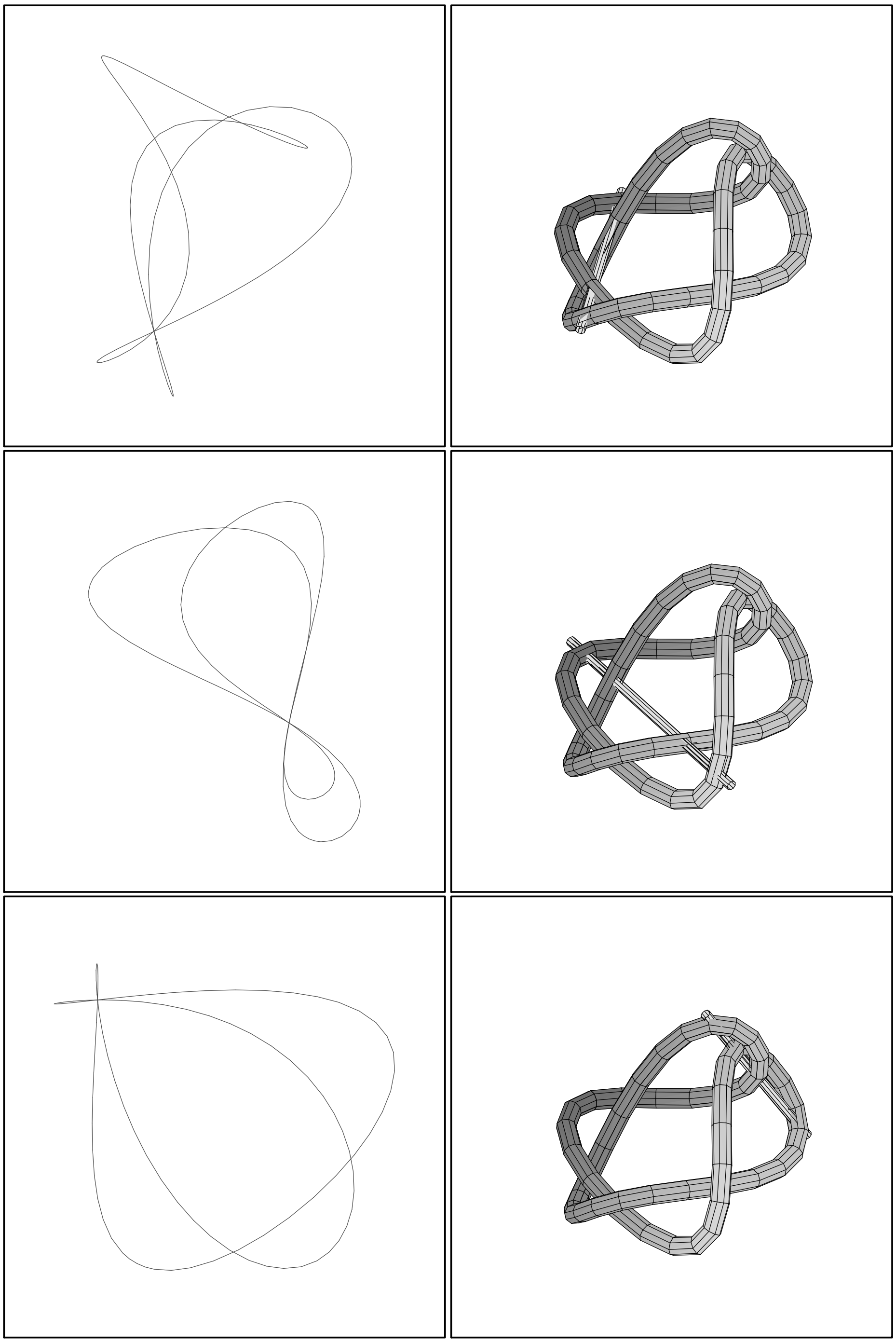
5. The knot $\mathbf{5_2}$

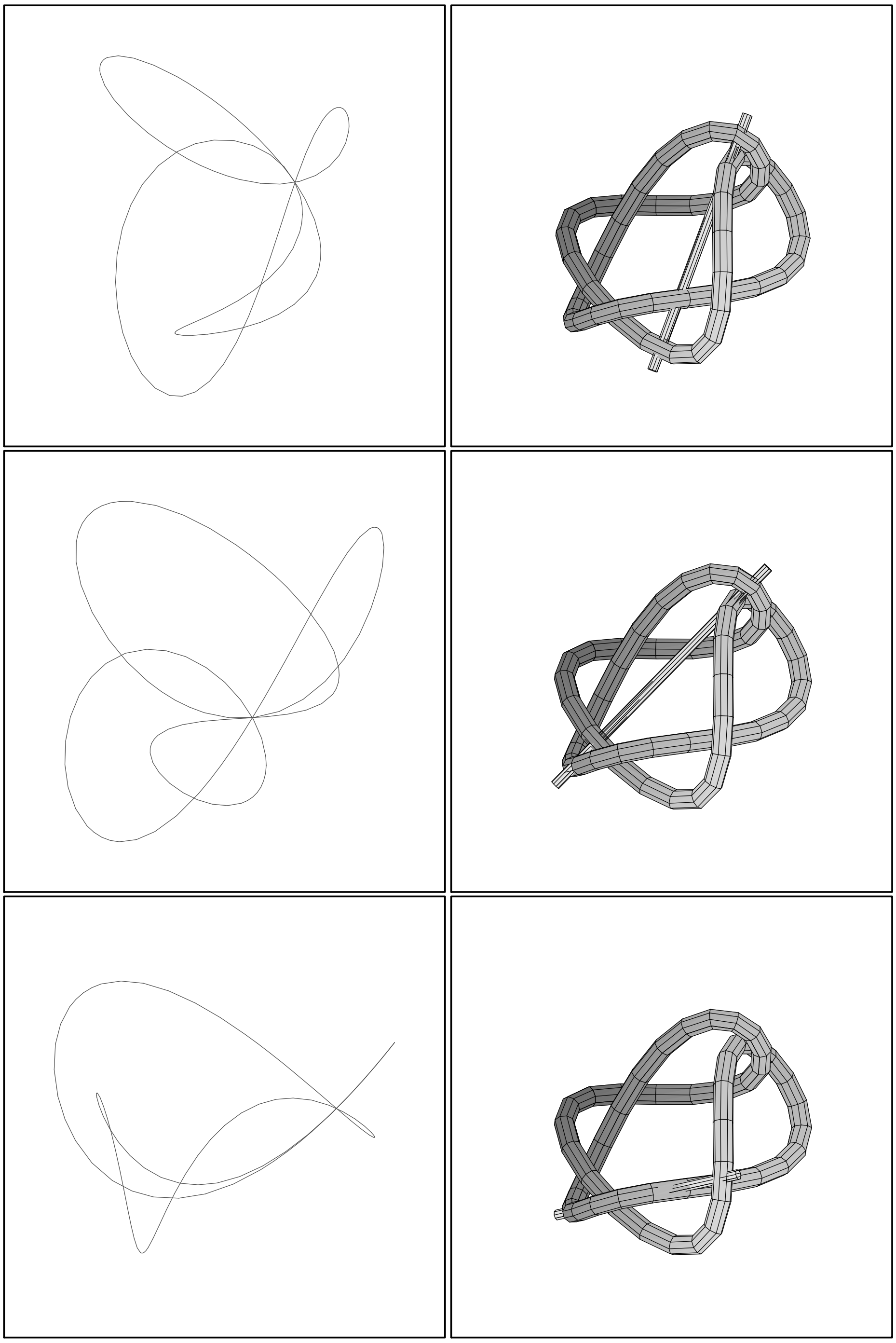
The knot with the following parameterization is a knot of type 5_2.

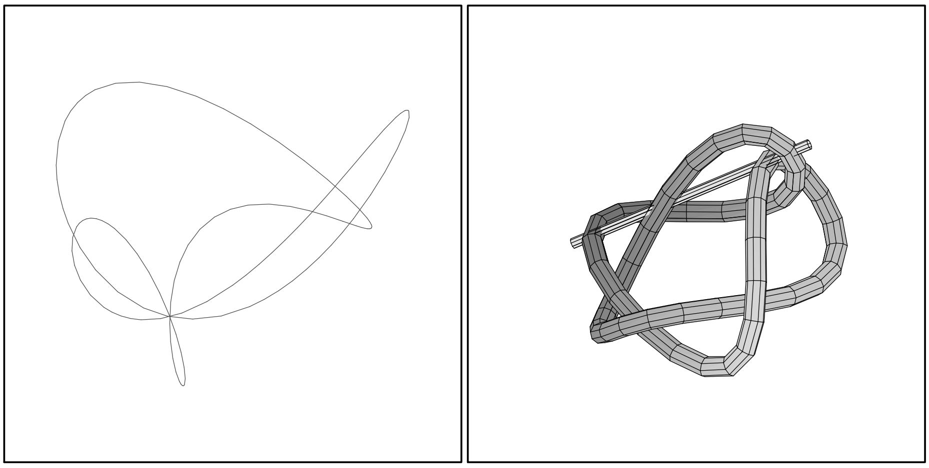
$$\begin{aligned} x(t) &= -33\cos t + 43\sin t + 214\sin 2t - 101\cos 3t - 47\sin 3t + 11\sin 4t \\ y(t) &= -57\cos t + 99\sin t - 54\cos 2t - 159\sin 2t - 117\cos 3t \\ &\qquad -5\sin 3t - 31\cos 4t - 45\sin 4t \\ z(t) &= 34\cos t - 21\sin t - 100\cos 2t - 93\sin 2t - 27\cos 3t \\ &\qquad -16\sin 3t + 52\cos 4t + 84\sin 4t \end{aligned}$$

It has ten quadrisecants. In each row of the figures below, the left figure is the projection of the knot into a plane perpendicular to the quadrisecant shown in the right figure.

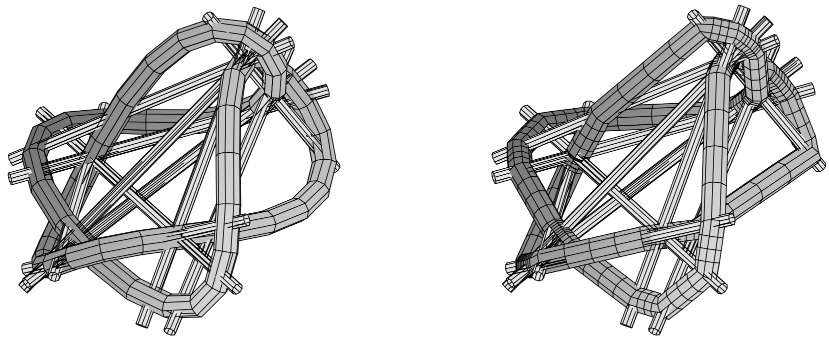








The left figure below shows the knot together with all of its ten quadrisecants. The right figure below is the quadrisecant approximation.



Acknowledgments

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