Superbridge Index of Knots

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1 Introduction

A *knot* is a piecewise smooth simple closed curve embedded in the three dimensional euclidean space $\mathbb{R}^3$. Two knots are *equivalent* if there is a piecewise smooth autohomeomorphism of $\mathbb{R}^3$ mapping one knot onto the other. The equivalence class of a knot K will be called the *knot type* of K and denoted by $[K]$. Two knots are *isotopic* if one can be continuously deformed to the other through a 1-parameter family of knots. If two knots are isotopic then they are equivalent. Conversely, if two knots are equivalent then they are isotopic up to reflection through a plane.

Given a knot K and a unit vector $\vec{v}$ in $\mathbb{R}^3$, we define $b_{\vec{v}}(K)$ as the number of connected components of the preimage of the set of local maximum values of the orthogonal projection $K \to \mathbb{R}\vec{v}$. We call $b_{\vec{v}}(K)$ the *crookedness* of K with respect to $\vec{v}$. Figure 1 illustrates an example. The *superbridge number* and *superbridge index* of K are defined by the formulae

$$s(K) = \max_{\|\vec{v}\|=1} b_{\vec{v}}(K) \quad \text{and} \quad s[K] = \min_{K' \in [K]} s(K') = \min_{K' \in [K]} \max_{\|\vec{v}\|=1} b_{\vec{v}}(K'),$$

respectively.

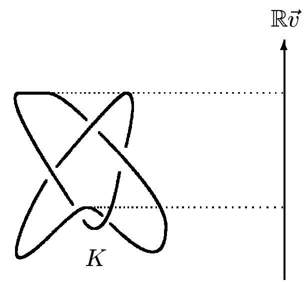


Figure 1: $b_{\vec{v}}(K) = 3$

2 Superbridge index and other invariants

Braid index

A *closed braid* is a knot which wraps around an axis in a way that every half plane bounded by the axis meets the knot in a fixed number of points transversely. By Alexander, every knot is isotopic to a closed braid [A]. If a closed braid meets every half plane bounded by its axis in n points, we say that its *braid number* is n and it is a *closed n-braid*. The minimal braid number among all closed braids isotopic to a knot K, denoted by $\beta(K)$, is called the *braid index* of K.

Proposition 2.1 (Kuiper). *For every knot K, the following inequality holds.*

$$s[K] \leq 2\,\beta(K)$$

Sketch of Proof. Let η be the curve parametrized by $t \mapsto (\cos t, \sin t, \cos^2 t)$, $t \in [0, 2\pi]$ and let $\vec{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ be a unit vector. Then $b_{\vec{v}}(\eta)$ counts the number

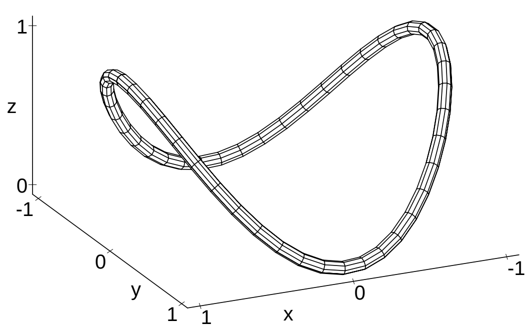


Figure 2: A tube along the curve $\eta : t \mapsto (\cos t, \sin t, \cos^2 t)$.

of local maximum points of the function

$$t \mapsto v_1 \cos t + v_2 \sin t + v_3 \cos^2 t$$

over the interval $[0, 2\pi]$ with the two end points 0 and 2π identified. Therefore $b_{\vec{v}}(\eta) \leq 2$. The equality holds if $1/\sqrt{2} < |v_3| \leq 1$.

Suppose $\beta(K) = n$. Then we can find a closed braid which is isotopic to K inside a thin tubular neighborhood of η. More precisely, there exist smooth functions λ and μ with period 2π, satisfying $\lambda(t)^2 + \mu(t)^2 \leq 1$ such that the curve K_ϵ given by the parametrization

$$K_\epsilon(t) = ((1 + \epsilon\lambda(t)) \cos nt, (1 + \epsilon\lambda(t)) \sin nt, \epsilon\mu(t) + \cos^2 nt),$$

$t \in [0, 2\pi]$, is equivalent to the knot K for each sufficiently small positive number ϵ. Notice that K_0 is an n-fold covering of η, by identifying the endpoints of $[0, 2\pi]$. For every unit vector $\vec{v}$ and for sufficiently small positive ϵ, we have $b_{\vec{v}}(K_\epsilon) \leq 2n$. Therefore $s[K] \leq 2n$. □

Bridge index

In a diagram of a non-trivial knot, maximal overpasses are called *bridges*. The minimal number of bridges among all diagrams of knots isotopic to a given non-trivial knot K, denoted by $b[K]$, is called the *bridge index* of K. The bridge index of a trivial knot is defined to be 1.

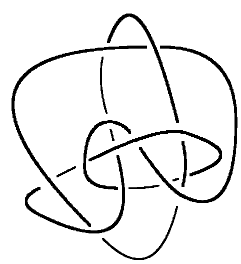


Figure 3: A diagram with seven bridges

Kuiper's superbridge index was inspired by the following alternative definition of bridge index. The *bridge number* of a knot K, denoted by $b(K)$, is the minimum of its crookednesses with respect to all directions, i.e.,

$$b(K) = \min_{\|\vec{v}\|=1} b_{\vec{v}}(K).$$

Then the bridge index of K is equal to the minimal bridge number among all knots equivalent to K, i.e.,

$$b[K] = \min_{K' \in [K]} b(K') = \min_{K' \in [K]} \min_{\|\vec{v}\|=1} b_{\vec{v}}(K').$$

A knot K is referred to as an *n-bridge knot* if $b[K] = n$.

Proposition 2.2 (Kuiper). *For every non-trivial knot K the following inequality holds.*

$$b[K] < s[K]$$

Sketch of Proof. Kuiper's proof utilizes Milnor's total curvature τ for closed curves. $\tau(K)$ measures the mean value of the sum of the number of local maxima and the number of local minima of the projection $K \to \mathbb{R}\vec{v}$ for $\vec{v} \in S^2$. Therefore the inequality $2\,b(K) \leq \tau(K) \leq 2\,s(K)$ holds. Let $\vec{v}_0$ be a unit vector such that $b(K) = b_{\vec{v}_0}(K)$ and, for $\lambda > 0$, let E_λ be the linear transformation of $\mathbb{R}^3$ scaling $1 : \lambda$ in the direction of $\vec{v}_0$. Then

$$\lim_{\lambda \to \infty} \tau(E_\lambda(K)) = 2\,b(K).$$

Taking the infimum over the equivalence class of K, we obtain

$$\inf_{K' \in [K]} \tau(K') = 2\,b[K]$$

which is shown, by Milnor, to be strictly smaller than any $\tau(K')$ if K is a non-trivial knot. Therefore we have

$$2\,b[K] < \tau(K') \leq 2\,s[K]$$

whenever $s(K') = s[K]$. $\square$

Theorem 2.3 (Furstenberg-Li-Schneider). *For every knot K, the following inequality holds.*

$$s[K] \leq 5\,b[K] - 3$$

Proof. Every n-bridge knot can be presented as a $2n$-plat diagram of a $(2n-1)$-braid with an extra straight strand. We construct a closed singular $(2n-1)$-braid by deforming this $2n$-plat diagram as in Figure 4. This closed singular braid has superbridge index not bigger than $4n-2$, since we can apply Proposition 2.1 to closed singular braids. Removing the $n-1$ singular points in a way to recover the given n-bridge knot, we may increase the superbridge number no more than $n-1$. $\square$

Corollary 2.4. *Every 2-bridge knot K satisfies the inequality $3 \leq s[K] \leq 7$.*

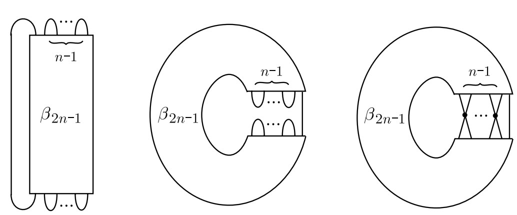


Figure 4:

Polygon index

The *polygon index* of a knot K, denoted by $\mathfrak{p}(K)$, is the minimal number of straight edges to form a knot equivalent to K.

Given a polygonal knot K with n straight edges, no interior point of an edge of K attains a local extremum of the projection $K \to \mathbb{R}\vec{v}$ unless the whole edge does. Therefore we have

$$n \geq b_{\vec{v}}(K) + b_{-\vec{v}}(K) = 2\, b_{\vec{v}}(K).$$

This proves the following proposition.

Proposition 2.5. *For every knot K, the following inequality holds.*

$$2\, s[K] \leq \mathfrak{p}(K)$$

We will use symbols in the knot tables of [BZ, R] where the trefoil knots and the figure eight knot are denoted by 3_1 and 4_1, respectively. Figure 5 shows a hexagonal trefoil with vertices at $(7,-7,-1), (-3,10,1), (-10,-3,-1), (10,-3,1), (3,10,-1), (-7,-7,1)$ and a heptagonal figure eight knot with vertices at $(3,6,4), (3,0,7), (0,4,15), (6,5,1), (4,0,15), (2,2,0), (4,4,8)$, both projected into the xy-plane. Therefore Proposition 2.2 and Proposition 2.5 imply the following corollary.

Corollary 2.6. $s[3_1] = s[4_1] = 3$.

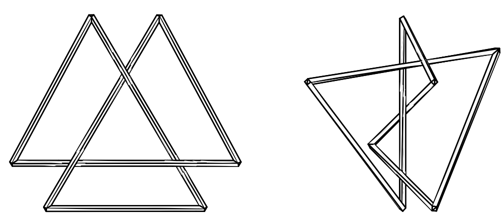


Figure 5: A hexagonal 3_1 and a septagonal 4_1.

Harmonic index

The *harmonic index* of a knot K, denoted $\mathfrak{h}(K)$, is the minimum integer N such that there exist polynomials $f(x,y)$, $g(x,y)$ and $h(x,y)$ whose maximal total degree is N and the parametrized curve

$$t \mapsto (f(\sin t, \cos t), g(\sin t, \cos t), h(\sin t, \cos t)) \tag{1}$$

for $t \in [0, 2\pi]$ represents the knot type of K.

Proposition 2.7. *For any knot K, the following inequality holds.*

$$s[K] \leq \mathfrak{h}(K)$$

Proof. Suppose $f(x,y)$, $g(x,y)$ and $h(x,y)$ are polynomials whose maximal total degree is d. Then the projection into an axis of the parametrized curve (1) is a linear combination

$$a\, f(\sin t, \cos t) + b\, g(\sin t, \cos t) + c\, h(\sin t, \cos t)$$

with $a^2 + b^2 + c^2 = 1$, hence a polynomial in $\sin t$ and $\cos t$ with total degree not exceeding d. Therefore the projection has at most d local maxima for $t \in [0, 2\pi]$ with the two endpoints 0 and 2π identified. $\square$

3 Torus knots

Given a pair of relatively prime positive integers p and q, let $T_{p,q}$ denote the knot determined by the parametrization

$$\begin{cases} x(t) = (2 - \cos qt) \cos pt \\ y(t) = (2 - \cos qt) \sin pt \\ z(t) = \sin qt \end{cases}, \qquad t \in [0, 2\pi].$$

We call $T_{p,q}$ a *standard torus knot of type* (p,q). A *torus knot of type* (p,q) is a knot which is isotopic to $T_{p,q}$.

Lemma 3.1. *Suppose p and q are relatively prime positive integers satisfying $2 \leq p < q$. Then*

$$\mathfrak{p}(T_{p,q}) \leq 2q.$$

Proof. Let α satisfy $\pi p/q < \alpha < \min\{\pi, 2\pi p/q\}$ and let $A = (1, 0, -1)$, $B = (\cos\alpha, \sin\alpha, 1)$ and $C = (\cos\beta, -\sin\beta, 1)$ where $\beta = 2\pi p/q - \alpha$. Then the segments AB and AC lie on the circular hyperboloids $x^2 + y^2 - z^2 \sin^2(\alpha/2) = \cos^2(\alpha/2)$ and $x^2 + y^2 - z^2 \sin^2(\beta/2) = \cos^2(\beta/2)$, respectively. Let T denote the linear

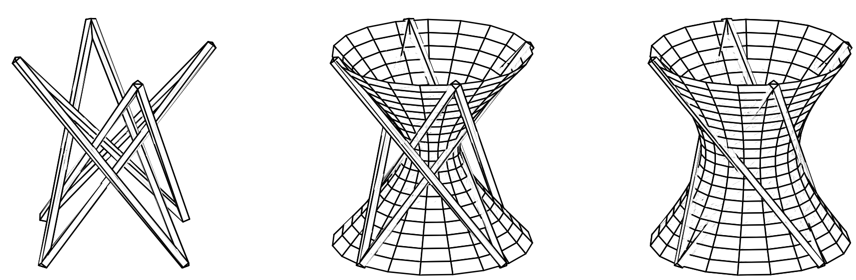


Figure 6: A polygonal torus knot of type $(3, 4)$

transformation of $\mathbb{R}^3$ given by a rotation in $2\pi p/q$ about the z-axis. Then

$$\bigcup_{i=0}^{q-1} T^i(AB \cup AC)$$

is a polygonal knot with $2q$ edges, which is isotopic to $T_{p,q}$. See Figure 6 for a polygonal torus knot of type $(3, 4)$ constructed as described above, with $\alpha = 33\pi/40$. $\square$

Theorem 3.2 (Kuiper). *Suppose p and q are relatively prime positive integers satisfying $2 \le p < q$. Then*

$$s[T_{p,q}] = \min\{2p, q\}$$

Sketch of Proof. By Proposition 2.1, we have $s[T_{p,q}] \le 2p$, since $\beta(T_{p,q}) = p$. By Proposition 2.5 and Lemma 3.1, we have $s[T_{p,q}] \le q$. Therefore $s[T_{p,q}] \le \min\{2p, q\}$.

To show $s[T_{p,q}] \ge \min\{2p, q\}$, Kuiper examined a triangulation of the pair (T, A) where T is a torus embedded isotopically to a standard torus and A an annulus whose core is a torus knot of type (p, q). See [Kui] for detail. $\square$

By Proposition 2.5, Lemma 3.1 and Theorem 3.2, we have the following theorem.

Theorem 3.3. *For each pair of relatively prime integers p and q satisfying $2 \le p < q < 2p$, we have*

$$\mathfrak{p}(T_{p,q}) = 2q.$$

4 Deformations not increasing superbridges

In this section we introduce two moves on knots, one local and the other global, which do not increase the superbridge numbers. Although they are crucial tools to prove major results such as Lemma 5.1, Theorem 6.3 and Theorem 6.4, it may not be clear where they are used in the proofs because we only sketch the proofs here. Readers with deep interests may find the details in [JJ1, J2].

Local straightening

Lemma 4.1. *Given a knot K, let $\bar{K}$ be a knot obtained by replacing a subarc of K with a straight line segment joining the end points of the subarc. Then $s(K) \ge s(\bar{K})$. The equality holds if $s(K) = s[K]$ and if $\bar{K}$ is isotopic to K.*

Proof. Given a unit vector $\vec{v}$, let $g\colon (-1, 2) \to \mathbb{R}\vec{v}$ be a parametrization of the orthogonal projection of an open neighborhood of the subarc into $\mathbb{R}\vec{v}$, where the subarc corresponds to the closed interval $[0,1]$. Then the projection of a neighborhood of the straight line segment in $\bar{K}$ can be parametrized by

$$\bar{g}(t) = \begin{cases} (1-t)g(0) + tg(1) & \text{if } t \in [0,1] \\ g(t) & \text{if } t \in (-1,0] \cup [1,2). \end{cases}$$

Since $\bar{g}$ has no more local maxima than g, we have $b_{\vec{v}}(K) \geq b_{\vec{v}}(\bar{K})$ for any $\vec{v}$. Therefore $s(K) \geq s(\bar{K})$. If $s(K) = s[K]$ and if $\bar{K}$ is isotopic to K, then we have

$$s(K) \geq s(\bar{K}) \geq s[\bar{K}] = s[K] = s(K).$$

This completes the proof. $\square$

Linear transformations

Proposition 4.2. *Given a knot K and a nonsingular linear transformation ϕ of $\mathbb{R}^3$, we have $s(\phi(K)) = s(K)$. In particular, if a knot K and a unit vector $\vec{v}$ satisfy $b_{\vec{v}}(K) = s(K) = s[K]$, then $b_{\vec{v}^\phi}(\phi(K)) = s(\phi(K)) = s[K]$.*

Proposition 4.2 is an easy consequence of Lemma 4.3 below. For a unit vector $\vec{v}$ and a non-singular linear transformation $\phi\colon \mathbb{R}^3 \to \mathbb{R}^3$, let $\vec{v}^\phi$ denote the unit vector contained in the one-dimensional subspace $(\phi(\vec{v}^\perp))^\perp$ satisfying $\phi(\vec{v})\cdot\vec{v}^\phi > 0$.

Lemma 4.3. *Given a unit vector $\vec{v} \in \mathbb{R}^3$ and a nonsingular linear transformation ϕ of $\mathbb{R}^3$, the equality*

$$b_{\vec{v}^\phi}(\phi(K)) = b_{\vec{v}}(K)$$

holds for any knot K.

Proof. At each local maximum point P of the projection $K \to \mathbb{R}\vec{v}$, there is an open disk d_P perpendicular to $\vec{v}$ and tangent to K at P. Then $\phi(d_P)$ is tangent to $\phi(K)$ at $\phi(P)$ and is perpendicular to $\vec{v}^\phi$. By the definition of $\vec{v}^\phi$, $\phi(P)$ is a local maximum point of the projection $\phi(K) \to \mathbb{R}\vec{v}^\phi$ and hence $b_{\vec{v}}(K) \leq b_{\vec{v}^\phi}(\phi(K))$. Since $(\vec{v}^\phi)^{\phi^{-1}} = \vec{v}^{(\phi^{-1}\phi)} = \vec{v}$, we also get

$$b_{\vec{v}}(K) = b_{(\vec{v}^\phi)^{\phi^{-1}}}(\phi^{-1}(\phi(K))) \geq b_{\vec{v}^\phi}(\phi(K)).$$

This proves the lemma. $\square$

5 3-superbridge knots

By Theorem 3.2, we know that the torus knot of type $(p, pr+1)$ has superbridge index $2p$, for $p \geq 2$ and $r \geq 2$. Therefore, for any even number $n \geq 4$, there are infinitely many n-superbridge knots. On the other hand, it is not known if there exist any odd number $n > 4$ such that there are infinitely many n-superbridge knots. We do know that there are finitely many 3-superbridge knots [JJ1, JJ2].

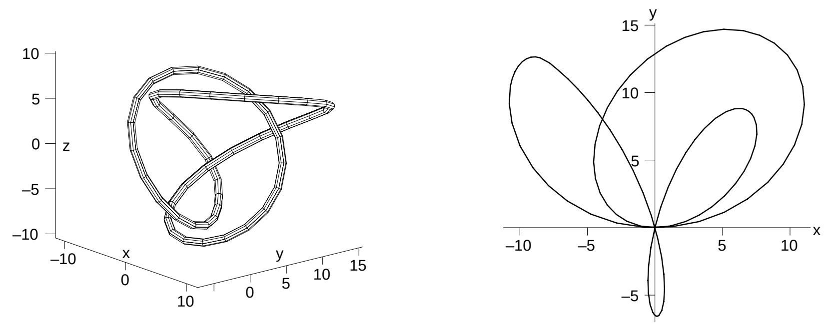
K	$b[K]$	$s[K]$	$\mathfrak{p}(K)$	K	$b[K]$	$s[K]$	$\mathfrak{p}(K)$	K	$b[K]$	$s[K]$	$\mathfrak{p}(K)$
3_1	2	3	6	8_{15}	3	4–6	9–12	9_{22}	3	4–7	9–14
4_1	2	3	7	8_{16}	3	4	9	9_{23}	2	4–7	9–14
5_1	2	4	8	8_{17}	3	4	9	9_{24}	3	4–6	9–12
5_2	2	3–4	8	8_{18}	3	4	8–9	9_{25}	3	4–7	9–15
6_1	2	3–4	8	8_{19}	3	4	8	9_{26}	2	4–6	9–12
6_2	2	3–4	8	8_{20}	3	4	8	9_{27}	2	4–6	9–12
6_3	2	3–4	8	8_{21}	3	4	9	9_{28}	3	4–6	9–12
7_1	2	4	9	9_1	2	4	9–13	9_{29}	3	4–7	9–15
7_2	2	3–4	9	9_2	2	4–7	9–14	9_{30}	3	4–6	9–13
7_3	2	3–4	9	9_3	2	4–6	9–12	9_{31}	2	4–6	9–13
7_4	2	3–4	9	9_4	2	4–7	9–14	9_{32}	3	4–6	9–12
7_5	2	4	9	9_5	2	4–6	9–13	9_{33}	3	4–6	9–12
7_6	2	4	9	9_6	2	4–6	9–13	9_{34}	3	4–6	9–12
7_7	2	4	9	9_7	2	4–6	9–12	9_{35}	3	4–6	9–13
8_1	2	4–5	9–10	9_8	2	4–6	9–13	9_{36}	3	4–7	9–14
8_2	2	4–5	9–11	9_9	2	4–6	9–13	9_{37}	3	4–7	9–14
8_3	2	4–6	9–12	9_{10}	2	4–6	9–13	9_{38}	3	4–7	9–15
8_4	2	3–5	9–10	9_{11}	2	4–6	9–13	9_{39}	3	4–6	9–13
8_5	3	4–6	9–12	9_{12}	2	4–6	9–12	9_{40}	3	4	9
8_6	2	4–6	9–12	9_{13}	2	4–6	9–13	9_{41}	3	4	9
8_7	2	3–6	9–12	9_{14}	2	4–7	9–14	9_{42}	3	4	9
8_8	2	4–5	9–11	9_{15}	2	4–5	9–11	9_{43}	3	4–5	9–10
8_9	2	3–6	9–12	9_{16}	3	4–7	9–14	9_{44}	3	4–5	9–10
8_{10}	3	4–6	9–12	9_{17}	2	4–7	9–14	9_{45}	3	4–5	9–10
8_{11}	2	4–5	9–10	9_{18}	2	4–6	9–13	9_{46}	3	4	9
8_{12}	2	4–6	9–12	9_{19}	2	4–6	9–13	9_{47}	3	4–6	9–12
8_{13}	2	4–5	9–11	9_{20}	2	4–6	9–13	9_{48}	3	4–6	9–12
8_{14}	2	4–5	9–11	9_{21}	2	4–7	9–14	9_{49}	3	4–5	9–11

Table 1: Prime knots up to 9 crossings

Lemma 5.1. *Every* 3*-superbridge knot is a* 2*-bridge knot with at most nine crossings.*

Sketch of Proof. Let K be a knot satisfying $s(K) = s[K] = 3$. By Proposition 2.2, we know that K is a 2-bridge knot. To show that K has a projection which has no more than nine crossings, we need to consider its quadrisecant.

A *quadrisecant* of a knot is a straight line meeting the knot exactly in four points. Figure 7 shows a figure eight knot having the z-axis as one of its quadrisecant. By [P, MM], every non-trivial knot has a quadrisecant [Kup, Theorem 1]. Let Q be a quadrisecant of K. We may assume that the projection $\pi(K)$ of K into $Q^\perp$, a plane perpendicular to Q, has a quadruple point and finitely many transverse double points as the only singular points. Inside a tubular neighborhood of Q, K can be isotoped to have only transverse double points near the quadruple point $\pi(Q)$. There are eighteen different crossing patterns possible as shown



$$
\begin{aligned}
x(t) =&(2t-1)(4t-1)(10t-1)(25t-16)(25t-21)(50t-9)\times\\
&(386t^6-708t^5-201t^4+945t^3-383t^2-\frac{42224361}{1146679}t-\frac{2701080}{1146679})\\
y(t) =&-70(2t-1)^2(4t-1)(10t-1)(25t-21)^2\times\\
&(229t^6-776t^5+806t^4-197t^3-56t^2-\frac{1667040}{277477}t-\frac{104544}{277477})\\
z(t) =&(20t-3)(25t-9)(25t-16)(25t-23)(1233t^8-5985t^7+11394t^6\\
&-10375t^5+4167t^4-243t^3-179t^2-\frac{2145804}{166595}t-\frac{712368}{832975})
\end{aligned}
$$

for $0 \le t \le 1$.

Figure 7: A figure eight knot having the z-axis as a quadrisecant

in Figure 8.

The assumption $s(K) = 3$ forces that no line in $Q^{\perp}$ can cross $\pi(K)$ more than six times. This ensures that $\pi(K)$ has to be as shown in Figure 9, with the boxes containing braids with two strings and at most three crossings. Every combination of one of the projection and a crossing pattern, which has more than nine crossings, can be deformed easily to a non-alternating diagram having no more than ten crossings. Since 2-bridge knots are alternating, the minimal crossing number of such combination is not bigger than nine. See [JJ1] for detail. □

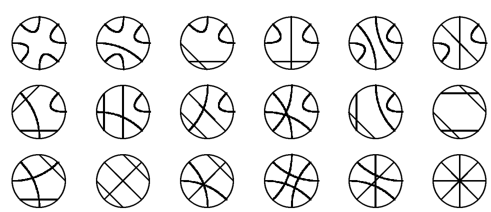


Figure 8: Crossing patterns near the quadrisecant

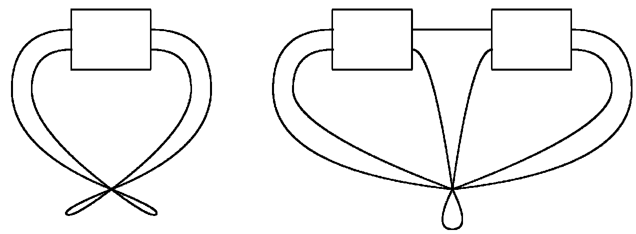


Figure 9: Two possible projections

In the article [JJ2], the author and Jeon examined in detail, all possible diagrams obtained by combining the projections of Figure 9 and crossing patterns of Figure 8, and concluded that

Theorem 5.2 (Jeon-Jin). *All 3-superbridge knots are among the twelve knots:*

$$3_1,\ 4_1,\ 5_2,\ 6_1,\ 6_2,\ 6_3,\ 7_2,\ 7_3,\ 7_4,\ 8_4,\ 8_7,\ 8_9.$$

6 Composite knots

Two knots K and L can be added in the following manner. By an isotopy, we may place K and L so that there is a plane $\mathcal{P}$ such that $K \cap \mathcal{P} = L \cap \mathcal{P} = K \cap L = \gamma$ is a simple arc. Then $(K \cup L) \setminus \operatorname{int} \gamma$ is a knot called a *connected sum* of K and L, denoted $K \sharp L$. Depending on the types of two knots and the way they are matched over a common arc, up to two distinct knot types of connected sums are possible.

Theorem 6.1 (Schubert). *Every connected sum $K_1 \sharp K_2$ of two knots K_1 and K_2 satisfies the equality*

$$b[K_1 \sharp K_2] = b[K_1] + b[K_2] - 1.$$

Corollary 6.2. *If K_1 and K_2 are non-trivial knots, any connected sum $K_1 \sharp K_2$ satisfies the inequality*

$$s[K_1 \sharp K_2] \geq 4.$$

Theorem 6.3. *Every connected sum $K_1 \sharp K_2$ of two knots K_1 and K_2 satisfies the inequality*

$$s[K_1 \sharp K_2] \leq \max\{2\beta(K_1) + \beta(K_2), \beta(K_1) + 2\beta(K_2)\} - 1.$$

Sketch of Proof. Let η_1 and η_2 be the trivial knots parametrized by

$$t \mapsto (\cos t, \sin t, \frac{1}{2}\cos^2 t) \qquad \text{and} \qquad t \mapsto (-\frac{3}{2} + \frac{1}{2}\cos^2 t, -\sin t, \frac{3}{2} + \cos t)$$

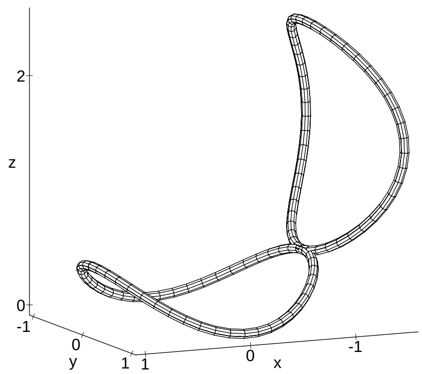


Figure 10: $\eta_1 \cup \eta_2$

respectively, for $t \in [0, 2\pi]$. They meet at $(-1, 0, \frac{1}{2})$. This is a point contributing 1 to each of the numbers $s(\eta_1) = b_{\mathbf{k}}(\eta_1) = 2\beta(\eta_1) = 2$, $b(\eta_1) = b_{-\mathbf{i}}(\eta_1) = \beta(\eta_1) = 1$, $s(\eta_2) = b_{\mathbf{i}}(\eta_2) = 2\beta(\eta_3) = 2$ and $b(\eta_2) = b_{-\mathbf{k}}(\eta_2) = \beta(\eta_2) = 1$. For sufficiently small distinct positive numbers, ϵ and δ, we consider the subarc $\eta_i^{\epsilon,\delta}$ of η_i, restricted over the interval $[\pi - \epsilon, \pi + \delta]$, for $i = 1, 2$. Let ℓ_ϵ and ℓ_δ be the segments joining $\eta_1(\pi - \epsilon)$ to $\eta_2(\pi - \epsilon)$ and $\eta_1(\pi + \delta)$ to $\eta_2(\pi + \delta)$, respectively. Then the knot obtained from $\eta_1 \cup \eta_2$ by replacing the part $\eta_1^{\epsilon,\delta} \cup \eta_2^{\epsilon,\delta}$ with $\ell_\epsilon \cup \ell_\delta$ can be considered as a connected sum $\eta_1 \sharp \eta_2$. In this process, we were able to eliminated the local maximum point $(-1, 0, \frac{1}{2})$ of η_1 since it is also a local minimum point of η_2 in the direction of $\mathbf{k}$. A similar statement is true in the direction of $\mathbf{i}$. Therefore we have

$$\begin{aligned} s(\eta_1 \sharp \eta_2) &= b_{\mathbf{k}}(\eta_1 \sharp \eta_2) = b_{\mathbf{k}}(\eta_1) + b_{\mathbf{k}}(\eta_2) - 1 = 2\beta(\eta_1) + \beta(\eta_2) - 1 \\ &= b_{\mathbf{i}}(\eta_1 \sharp \eta_2) = b_{\mathbf{i}}(\eta_1) + b_{\mathbf{i}}(\eta_2) - 1 = \beta(\eta_1) + 2\beta(\eta_2) - 1. \end{aligned}$$

This implies that the following inequality holds for any unit vector $\vec{v}$.

$$b_{\vec{v}}(\eta_1 \sharp \eta_2) \leq \max\{2\beta(\eta_1) + \beta(\eta_2), \beta(\eta_1) + 2\beta(\eta_2)\} - 1$$

Given two non-trivial knots K_1 and K_2, we may assume that K_i is contained in a tubular neighborhood of η_i, as a closed $\beta(K_i)$-braid, for $i = 1, 2$, and that in a neighborhood of $(-1, 0, \frac{1}{2})$, their union is identical with $\eta_1^{\epsilon,\delta} \cup \eta_2^{\epsilon,\delta}$. Applying the same process, we obtain

$$b_{\vec{v}}(K_1 \sharp K_2) \leq \max\{2\beta(K_1) + \beta(K_2), \beta(K_1) + 2\beta(K_2)\} - 1.$$

A rigorous argument can be found in [J2]. $\square$

Theorem 6.4. *For any torus knots K_1 and K_2, the inequality holds.*

$$s[K_1 \sharp K_2] \leq \max\{s[K_1] + b[K_2], b[K_1] + s[K_2]\} - 1$$

K	$s[K]$	lower bound	upper bound
$3_1 \sharp 3_1$	4	$b[K]=3$	Corollary 6.5
$3_1 \sharp 4_1$	4	$b[K]=3$	$\mathfrak{p}(K)=9$
$3_1 \sharp 5_1$	$5\star$	$s[5_1]=4$	Theorem 6.4
$3_1 \sharp 7_1$	$5\star$	$s[7_1]=4$	Theorem 6.4
$3_1 \sharp 7_5$	$5\star$	$s[7_5]=4$	$\mathfrak{p}(K)\le 11$
$3_1 \sharp 7_6$	$5\star$	$s[7_6]=4$	$\mathfrak{p}(K)\le 11$
$3_1 \sharp 7_7$	$5\star$	$s[7_7]=4$	$\mathfrak{p}(K)\le 11$
$3_1 \sharp 8_{16}$	5	$b[K]=4$	$\mathfrak{p}(K)\le 11$
$3_1 \sharp 8_{17}$	5	$b[K]=4$	$\mathfrak{p}(K)\le 11$
$3_1 \sharp 8_{18}$	5	$b[K]=4$	$\mathfrak{p}(K)\le 11$
$3_1 \sharp 8_{19}$	5	$b[K]=4$	Corollary 6.5
$3_1 \sharp 8_{20}$	5	$b[K]=4$	$\mathfrak{p}(K)\le 10$
$3_1 \sharp 8_{21}$	5	$b[K]=4$	$\mathfrak{p}(K)\le 11$
$3_1 \sharp 9_1$	$5\star$	$s[9_1]=4$	Theorem 6.4
$3_1 \sharp 9_{40}$	5	$b[K]=4$	$\mathfrak{p}(K)\le 11$
$3_1 \sharp 9_{41}$	5	$b[K]=4$	$\mathfrak{p}(K)\le 11$
$3_1 \sharp 9_{44}$	5	$b[K]=4$	$\mathfrak{p}(K)\le 11$
$3_1 \sharp 9_{46}$	5	$b[K]=4$	$\mathfrak{p}(K)\le 11$
$4_1 \sharp 5_1$	$5\star$	$s[5_1]=4$	$\mathfrak{p}(K)\le 11$
$4_1 \sharp 8_{19}$	5	$b[K]=4$	$\mathfrak{p}(K)\le 11$
$4_1 \sharp 8_{20}$	5	$b[K]=4$	$\mathfrak{p}(K)\le 11$
$5_1 \sharp 5_1$	$5\star$	$s[5_1]=4$	Theorem 6.4
$5_1 \sharp 7_1$	$5\star$	$s[7_1]=4$	Theorem 6.4
$7_1 \sharp 7_1$	$5\star$	$s[7_1]=4$	Theorem 6.4
$8_{19} \sharp 8_{19}$	6	$b[K]=5$	Corollary 6.5
$3_1 \sharp 3_1 \sharp 3_1$	5	$b[K]=4$	$\mathfrak{p}(K)\le 10$
$3_1 \sharp 3_1 \sharp 4_1$	5	$b[K]=4$	$\mathfrak{p}(K)\le 11$

$\star$: valid if Conjecture 6 holds.

Table 2: Some composites knots and their superbridge index

Sketch of Proof. Let K_i be a torus knot of type (p_i, q_i) with $2 < p_i < q_i$ and $\gcd(p_i, q_i) = 1$, for $i = 1, 2$. If $q_i > 2p_i$, then $s[K_i] = 2p_i = 2\beta(K_i)$ and $b[K_i] = p_i = \beta(K_i)$, for each i. Therefore, if $q_i > 2p_i$, for $i = 1, 2$, the inequality of this theorem is a special case of Theorem 6.3.

If $q_i < 2p_i$, we use the $2q_i$-edged polygonal torus knot constructed in the proof of Lemma 3.1 instead of the one embedded in η_i, and obtain $s[K_i] = 2q_i$ and $b(K_i) = p_i$. A construction for the connected sum $K_1 \sharp K_2$ similar to that of Theorem 6.3 leads us to derive the inequality. □

Corollary 6.5. *For any torus knots K_1 and K_2, the inequality holds.*

$$s[K_1 \sharp K_2] \le s[K_1] + s[K_2] - 2$$

The next corollary shows that the equalities in Theorem 6.4 and Corollary 6.5 hold in infinitely many cases.

Corollary 6.6. *Let $p_i \geq 2$ and let K_i be the torus knot of type (p_i, p_i+1), for $i = 1, 2$. Then*

$$s[K_1 \sharp K_2] = p_1 + p_2.$$

Proof. By Theorem 3.2, $s[K_i] = p_i + 1$. Since $b[K_i] = p_i$, Theorem 6.1, Proposition 2.2 and Theorem 6.4, imply $p_1 + p_2 - 1 < s[K_1 \sharp K_2] \leq p_1 + p_2$. $\square$

The inequality in Theorem 6.4 is equivalent to

$$s[K_1] + s[K_2] - s[K_1 \sharp K_2] \geq \min\{s[K_1] - b[K_1], s[K_2] - b[K_2]\} + 1.$$

If K_i is a torus knot of type (p_i, q_i) with $2 \leq p_i < q_i$, the right hand side of the above inequality is equal to $\min\{p_1, p_2, q_1 - p_1, q_2 - p_2\} + 1$, which can be arbitrarily large. Therefore we have

Corollary 6.7. *The difference $s[K_1] + s[K_2] - s[K_1 \sharp K_2]$ can be arbitrarily large.*

7 Conjectures

As we have discussed at the beginning of Section 5, there are infinitely many n-superbridge knots for any even number $n \geq 4$. We expect the same is true for all odd number $n \geq 5$.

Conjecture 1. *There are infinitely many n-superbridge knots for any positive integer $n \geq 4$.*

Theorem 6.4 implies that a connected sum of any torus knot K with a trefoil knot T satisfies the inequality $s[K \sharp T] \leq s[K] + 1$. As it is generally expected that the superbridge index of a composite knot would be bigger than that of any of the factor knots, which is true for bridge index, we conjecture that the equality holds for any non-trivial knot K. By Corollary 6.6, this conjecture holds for torus knots of type $(p, p+1)$, such as 3_1, 8_{19}, etc. It also holds for 4_1, 8_{16}, 8_{17}, 8_{18}, 8_{20}, 8_{21}, 9_{40}, 9_{41}, 9_{46}, $3_1 \sharp 3_1$ and $3_1 \sharp 4_1$.

Conjecture 2. *Every non-trivial knot K satisfies*

$$s[K \sharp T] = s[K] + 1$$

where T is a trefoil knot.

By Theorem 3.2, we know that the torus knot of type $(n, 2n-1)$ is a $(2n-1)$-superbridge knot. This knot is the closure of the n-braid $(\sigma_1 \sigma_2 \cdots \sigma_{n-1})^{2n-1}$. On the other hand, the torus knot of type $(2, 2k-1)$ is the closure of the 2-braid σ_1^{2k-1} and has superbridge index 4 if $k \geq 3$. As the insertion of a power of the full twist σ_1^2 to these closed 2-braids does not change the superbridge index, we expect the same with the above closed $(2n-1)$-braids.

Conjecture 3. *For $n \geq 3$ and $k \geq 0$, the closure of the n-braid*

$$\sigma_1^{2k}(\sigma_1\sigma_2\cdots\sigma_{n-1})^{2n-1} \tag{2}$$

is a $(2n-1)$-superbridge knot.

A theorem of Stallings [St] implies that the closure of the braid in (2) for $k \geq 0$ is a fibred knot with the fibre surface obtained by Seifert's algorithm on the closed braid diagram. This surface is the one with minimal genus, which is $(n-1)^2 + k$. Therefore for each n, such knots are all distinct. Notice that the braid (2) is positive and the diagram of its closure is visually prime. According to Cromwell [Cr], they are all prime knots. Both Conjecture 2 and Conjecture 3 imply Conjecture 1 but Conjecture 3 is more interesting because of the primeness of these knots.

We expect that Theorem 6.4 and Corollary 6.5 hold for any knots.

Conjecture 4. *Any connected sum of two knots K_1 and K_2 satisfies the inequality*

$$s[K_1\sharp K_2] \leq \max\{s[K_1]+b[K_2], b[K_1]+s[K_2]\} - 1.$$

Conjecture 5. *If K_1 and K_2 are non-trivial knots, any connected sum $K_1\sharp K_2$ satisfies the inequality*

$$s[K_1\sharp K_2] \leq s[K_1]+s[K_2] - 2.$$

By Proposition 2.2, the bridge index is a strict lower bound for superbridge index. No other lower bound is known. For any composite knot, we expect its superbridge index is larger that that of its factor knots.

Conjecture 6. *If K_1 and K_2 are non-trivial knots, any connected sum $K_1\sharp K_2$ satisfies the inequality*

$$s[K_1\sharp K_2] > \max\{s[K_1], s[K_2]\}.$$

We have a list of implications among the above conjectures.

- Conjecture 2 $\Rightarrow$ Conjecture 1.
- Conjecture 3 $\Rightarrow$ Conjecture 1.
- Conjecture 4 $\Rightarrow$ Conjecture 5.
- Conjecture 5 and Conjecture 6 $\Rightarrow$ Conjecture 2.
- Conjecture 6 for torus knots $\Rightarrow$ Conjecture 2.

The trefoil knots and the figure eight knot are the only known 3-superbridge knots. Because 3-superbridged embedding of a knot must have an extremely simple geometric shape, we expect these are the only 3-superbridge knots.

Conjecture 7. *If $s[K] = 3$ then K is either a trefoil knot or the figure eight knot.*

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