On amphicheiral knots with symmetric union presentations

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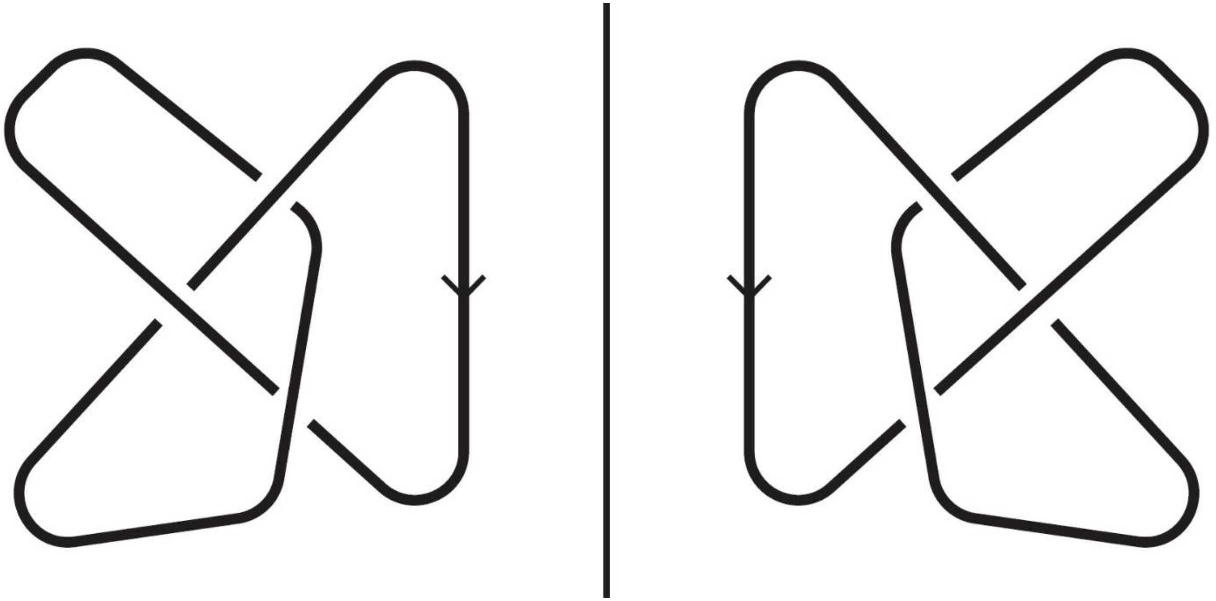
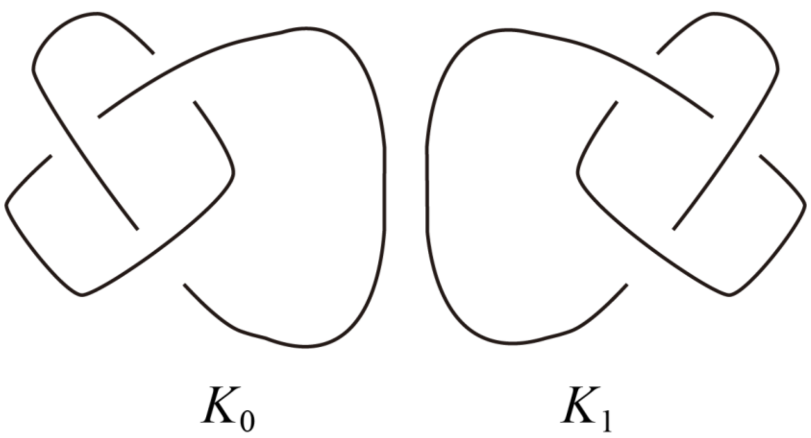
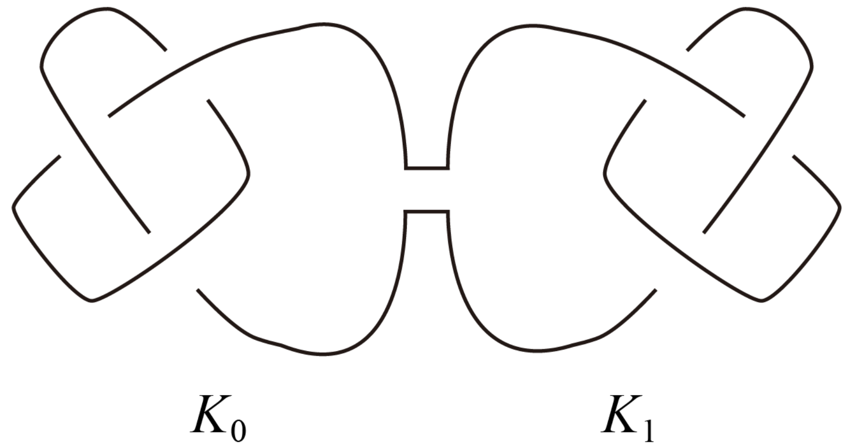
Def. A knot is a closed connected oriented 1-manifold smoothly embedded in S^3.

Def. The mirror image of a knot is obtained by reflecting it in a plane in $R^3 \subset S^3$.



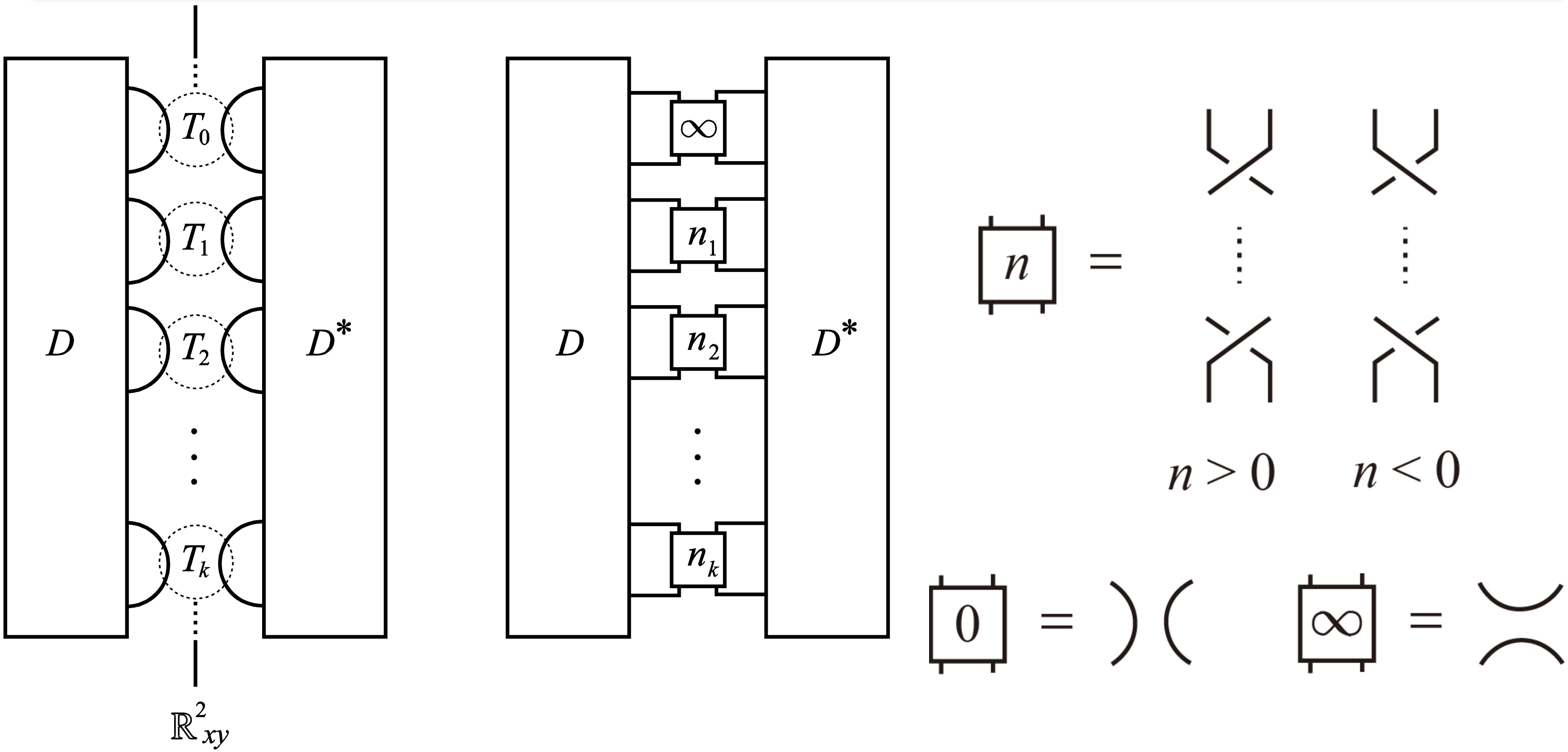
Def. Let K_0 and K_1 be unoriented knots. Then the connected sum $K_0 \# K_1$ is defined as follows.



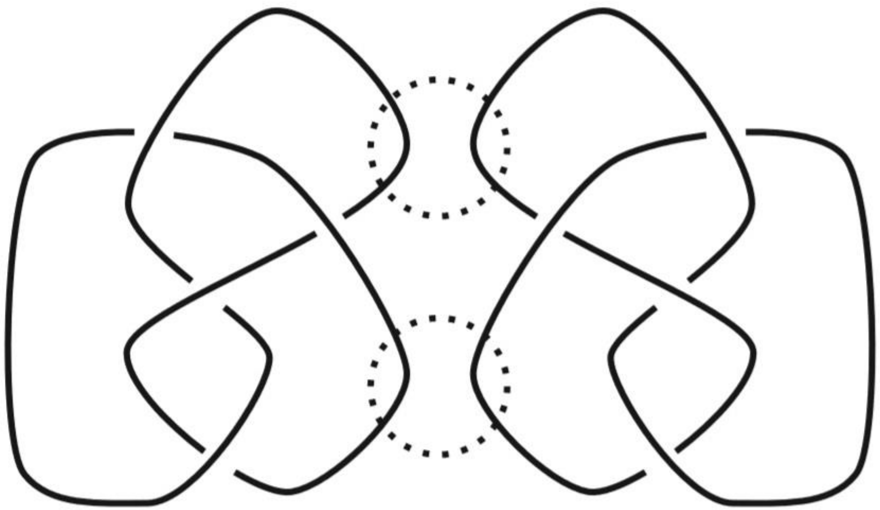


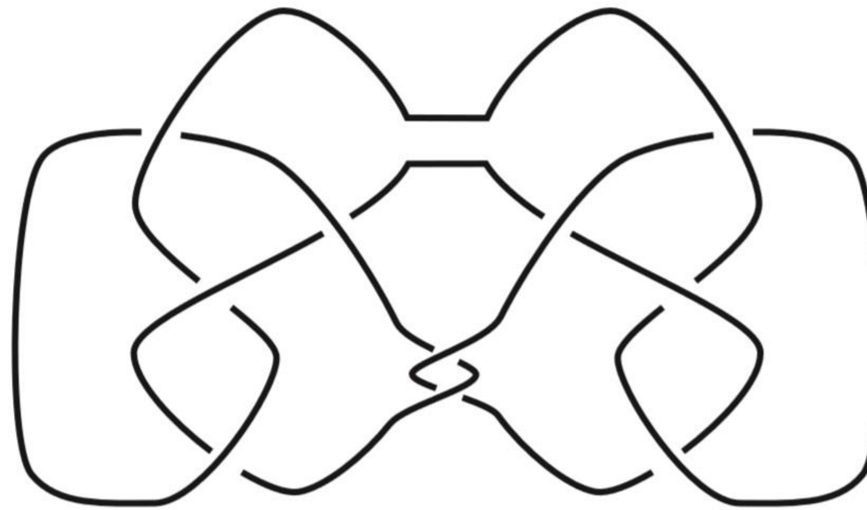
Symmetric union

A *symmetric union* $D \cup D^*(n_1, \cdots, n_k)$ $(\mathbb{Z} \ni n_i \neq \infty)$
is defined by the following diagram :



Symmetric union





Tangles

Property

Fact.
(1) $D \cup D^*(n)$ (S. Kinoshita-H. Terasaka (OMJ 1957))
(2) $D \cup D^*(n_1, \cdots, n_k)$ (C. Lamm (OJM 2000))

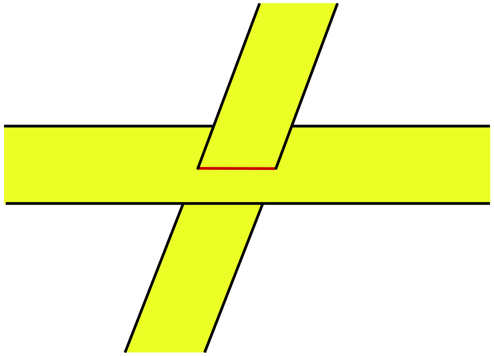
Fact (Lamm).
(1) Every symmetric union is a ribbon knot.

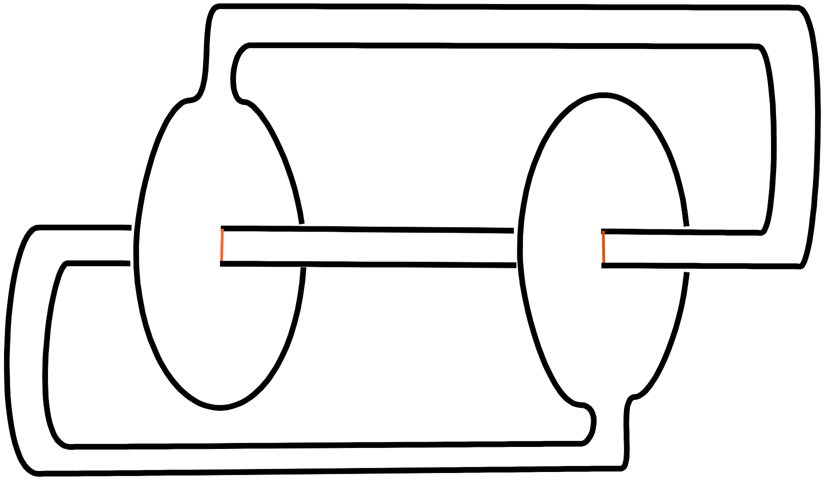
(2) $\triangle(D \cup D^*(n_1, \cdots, n_k)) = \triangle(D \cup D^*(n'_1, \cdots, n'_k))$
if $n_i \equiv n_i'$ (mod 2) for all i.
($\triangle$: *Alexander polynomial.*)

(3) $\det(D \cup D^*(n_1, \cdots, n_k)) = \det(D)^2$.

Ribbon knot

Def. A ribbon knot is a knot that bounds a self-intersecting disk with only ribbon singularities.





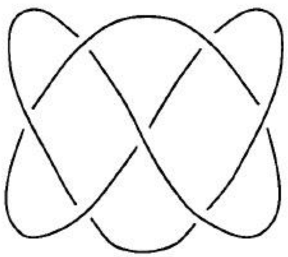
Ribbon singularity

Open problem

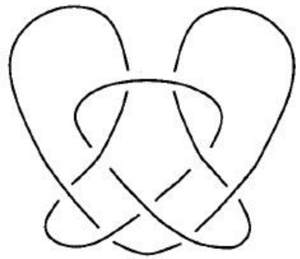
Problem (Lamm OJM (2000)).
Is every ribbon knot, symmetric union?

(1) Every ribbon knot with crossing number $\leqq$ 10 is a symmetric union.

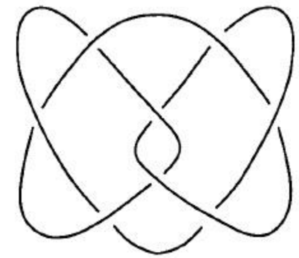
(2) Every 2-bridge ribbon knot is a symmetric union.



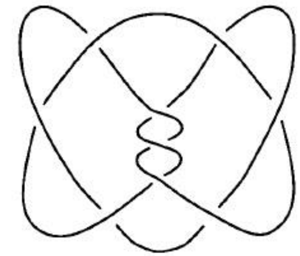
6_1



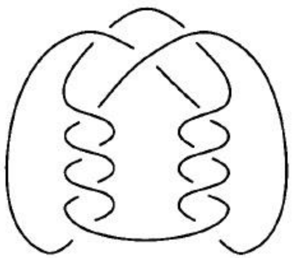
8_8



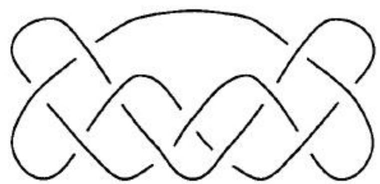
8_{20}



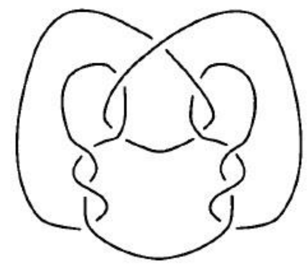
9_{46}



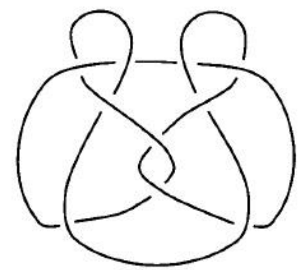
10_3



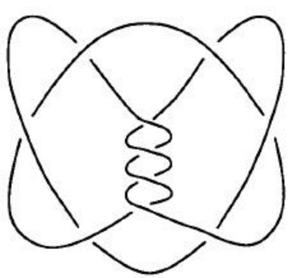
10_{22}



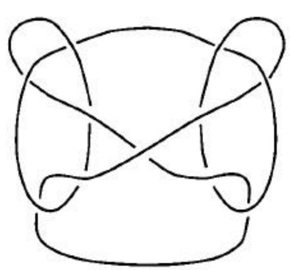
10_{35}



10_{137}



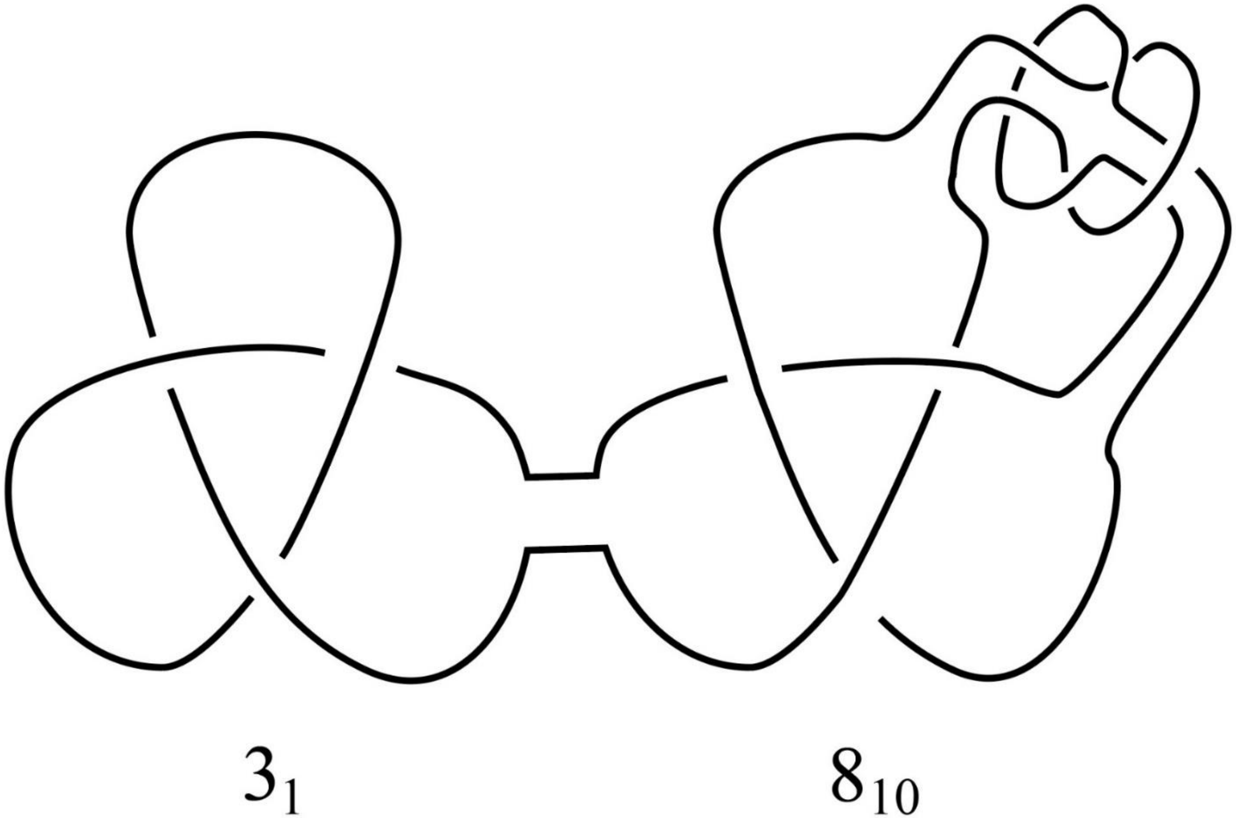
10_{140}



10_{153}

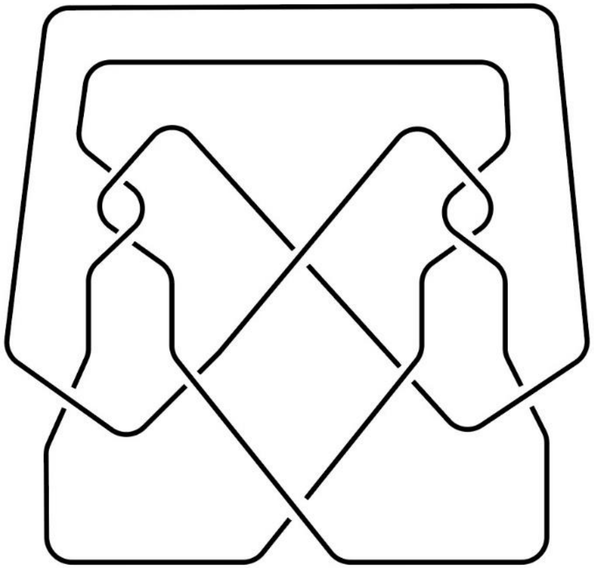
Potential counterexample

C. Lamm, *The search for nonsymmetric ribbon knots. Exp. Math. 30 (2021), no. 3, 349–363*.



Amphicheiral knot

Def. A knot K is negative amphicheiral or simply amphicheiral if K is equivalent to $-K^*$, where $-K^*$ is the mirror image of K with the reversed orientation.



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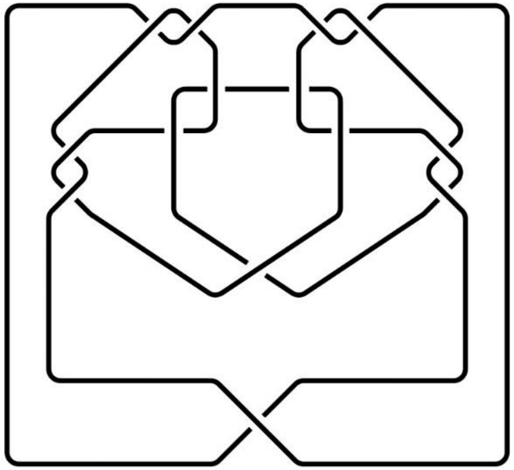


10_{99}

Amphicheiral knot

Theorem (T, "*The Jones polynomial of knots with symmetric union presentations*", J. Korean Math. Soc. 52 (2015)).
Let K be a knot with a symmetric union presentation of the Form $D \cup D^*(n)$.
Then we have

$$t^{m} V_K(t) + (-1)^m V_K(t^{-1}) = (t^{m}+(-1)^m) V_D(t) V_{D^*}(t).$$

In particular, if K is amphicheiral, then

$$V_K(t) = V_D(t) V_{D^*}(t).$$

Question

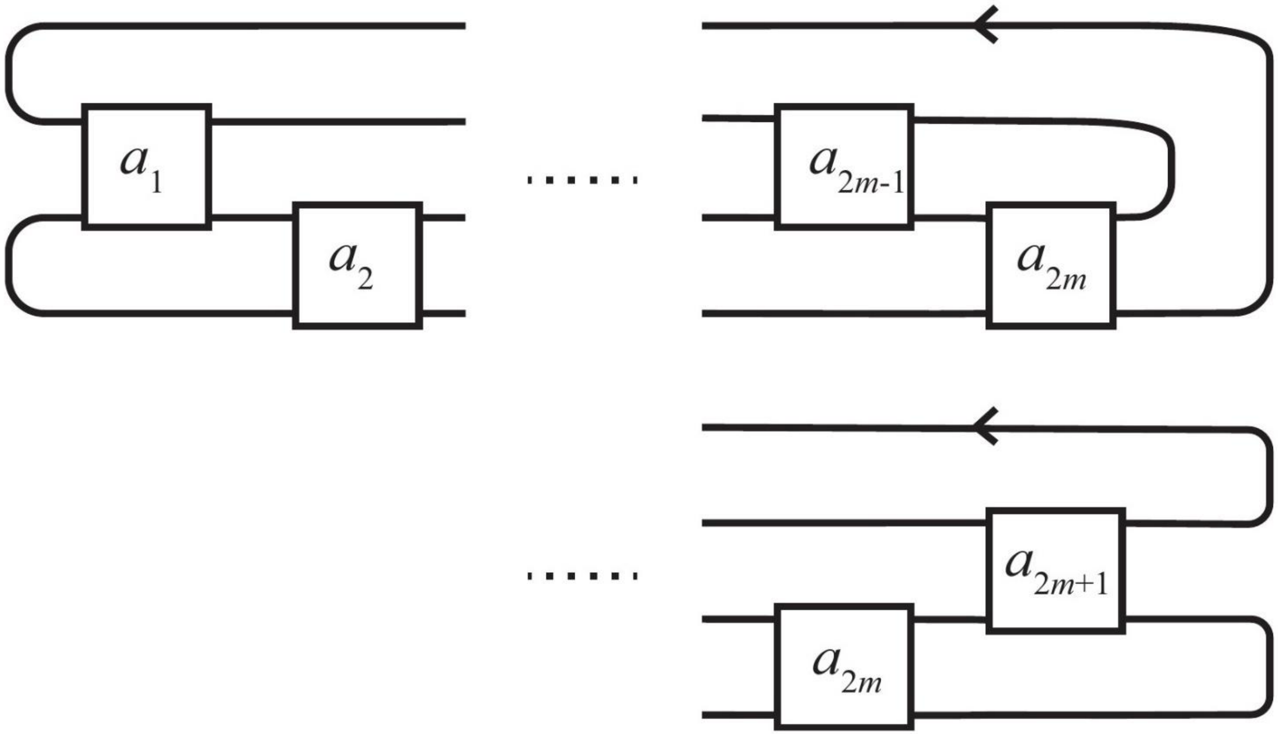
Theorem (F. C. Kose "On amphicheiral symmetric unions"), arXiv:2111.08765).
Let K be a prime amphicheiral knot. Then K does not admit a symmetric union presentation of the form $D \cup D^*(n)$.

- Every 2-bridge ribbon knot is a prime symmetric union.

Question.
What types of symmetric unions appear in the set of amphicheiral 2-bridge knots?

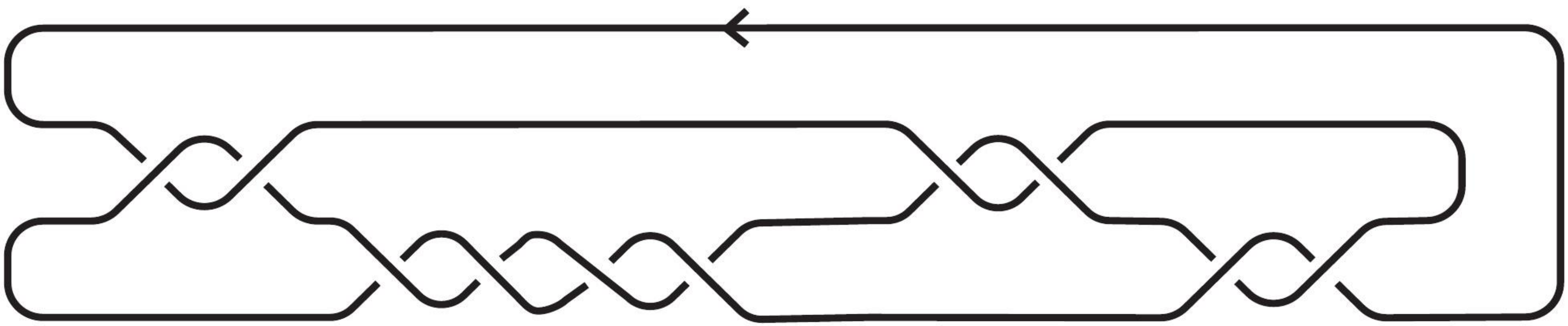
2-bridge knot

Def. We define the 2-bridge knot $K(a_1, \ldots, a_n)$ via the diagram $C(a_1, \ldots, a_n)$ with n twist regions as follows.



2-bridge knot

Def. We define the 2-bridge knot $K(a_1, \dots, a_n)$ via the diagram $C(a_1, \dots, a_n)$ with n twist regions as follows.



$C(2, 4, -2, -2)$

A classification of 2-bridge knots

Fact.
(1) Any 2-bridge knot admits the representation $K(2a_1, \dots, 2a_{2m})$ which is unique, up to the symmetry

$$K(2a_1, \dots, 2a_{2m}) = K(-2a_{2m}, \dots, -2a_1) .$$

(2) If $K(2a_1, \dots, 2a_{2m})$ is amphicheiral, then we have

$$a_i = a_{2m-i+1} \; (i = 1, 2, \dots, 2m).$$

・ S. Baader, A. Kjuchukova, L. Lewark, F. Misev and A. Ray, *Average four-genus of two-bridge knots*, to appear in Proc. Amer. Math. Soc., DOI: https://doi.org/10.1090/proc/14784.

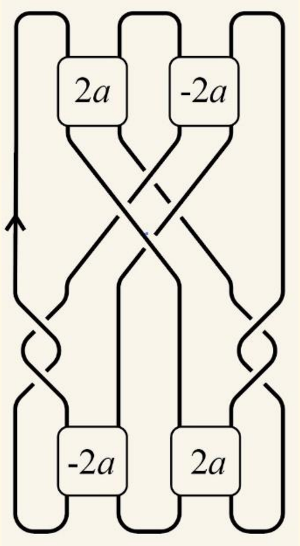
・ A. Kawauchi, *A survey of knot theory*, Translated and revised from the 1990 Japanese original by the author. Birkhauser Verlag, Basel, 1996.

Theorem (T).

Let K be a 2-bridge symmetric union. Then the followings are equivalent.

(1) K is amphicheiral.

(2) K has the symmetric union presentation as follows.



Proposition (T).

Let K be a 2-bridge ribbon knot. Then the followings are equivalent.

(1) K is amphicheiral.

(2) K is equivalent to $K(2a, 2, 2a, 2a, 2, 2a)$.

Proof of Proposition (outline)

Fact (C. Lamm, *Symmetric union presentations for 2-bridge ribbon knots*.)

A 2-bridge ribbon knot K is one of the following three types:

(1) $K(2a, 2, 2b, -2, -2a, 2b)$ with $a, b \neq 0$.

(2) $K(2a, 2, 2b, 2a, 2, 2b)$ with $a, b \neq 0$.

(3) $K(a, b, \dots, w, x, x+2, w, \dots, b, a)$ with parameters > 0

Remark. Note that the mirror images are contained.

Proof of Proposition (outline)

Suppose that K is amphicheiral.

In the case of (1), we have $K(-2, 2, -2, -2, 2, -2)$ by using a classification of the 2-bridge knots.

In the case of (2), we have $K(2a, 2, 2a, 2a, 2, 2a)$ by using the same method.

In the case of (3), *we know that the 2-bridge knot is not amphicheiral by using the Jones polynomial.*

Thus, we have $K(2a, 2, 2a, 2a, 2, 2a)$ with $a \neq 0$.

Proof of Proposition (outline)

In the case of their mirror images, we have

(1) $K(-2a, -2, -2b, 2, 2a, -2b)$ with $a, b \neq 0$.

(2) $K(-2a, -2, -2b, -2a, -2, -2b)$ with $a, b \neq 0$.

(3) $K(a, b, \ldots, w, x, x - 2, w, \ldots, b, a)$ with parameters < 0.

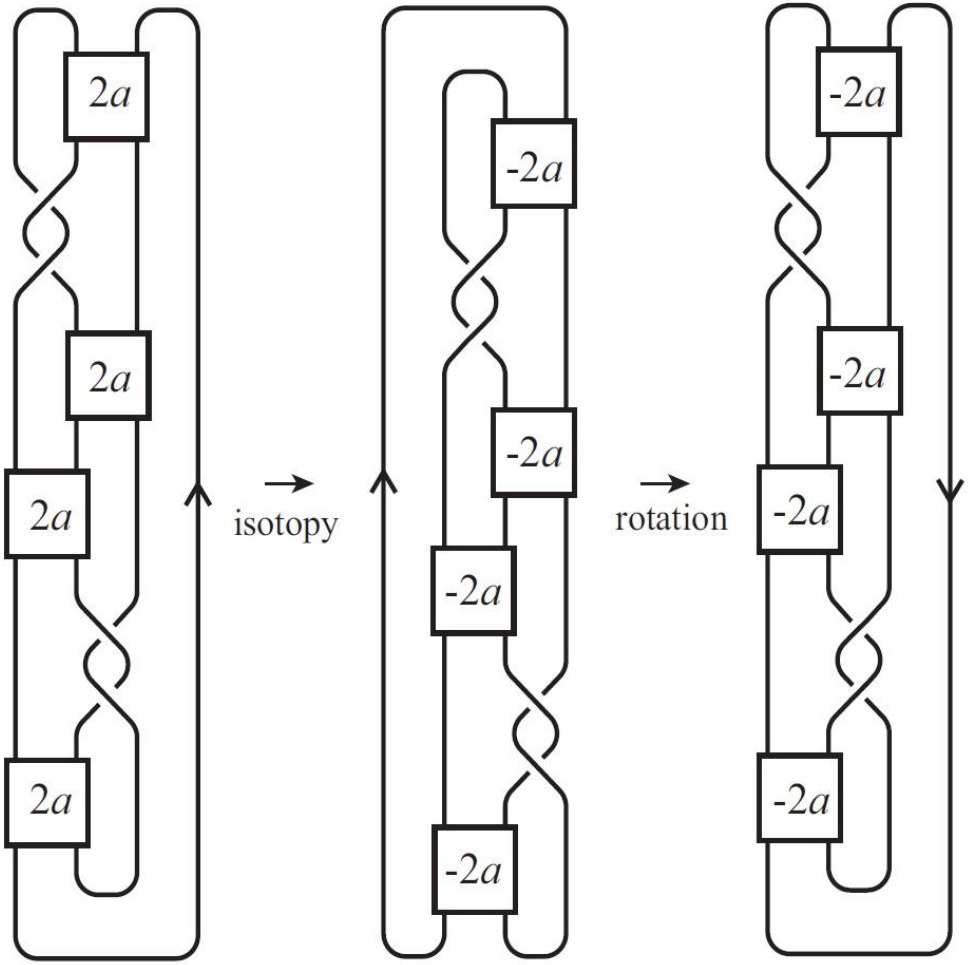
then we also have the following:

$K(-2a, -2, -2b, 2, 2a, -2b) \Rightarrow K(2, -2, 2, 2, -2, 2) = K(-2, 2, -2, -2, 2, -2)$

and $K(-2a, -2, -2a, -2a, -2, -2a) = K(2a, 2, 2a, 2a, 2, 2a)$ by the symmetry.

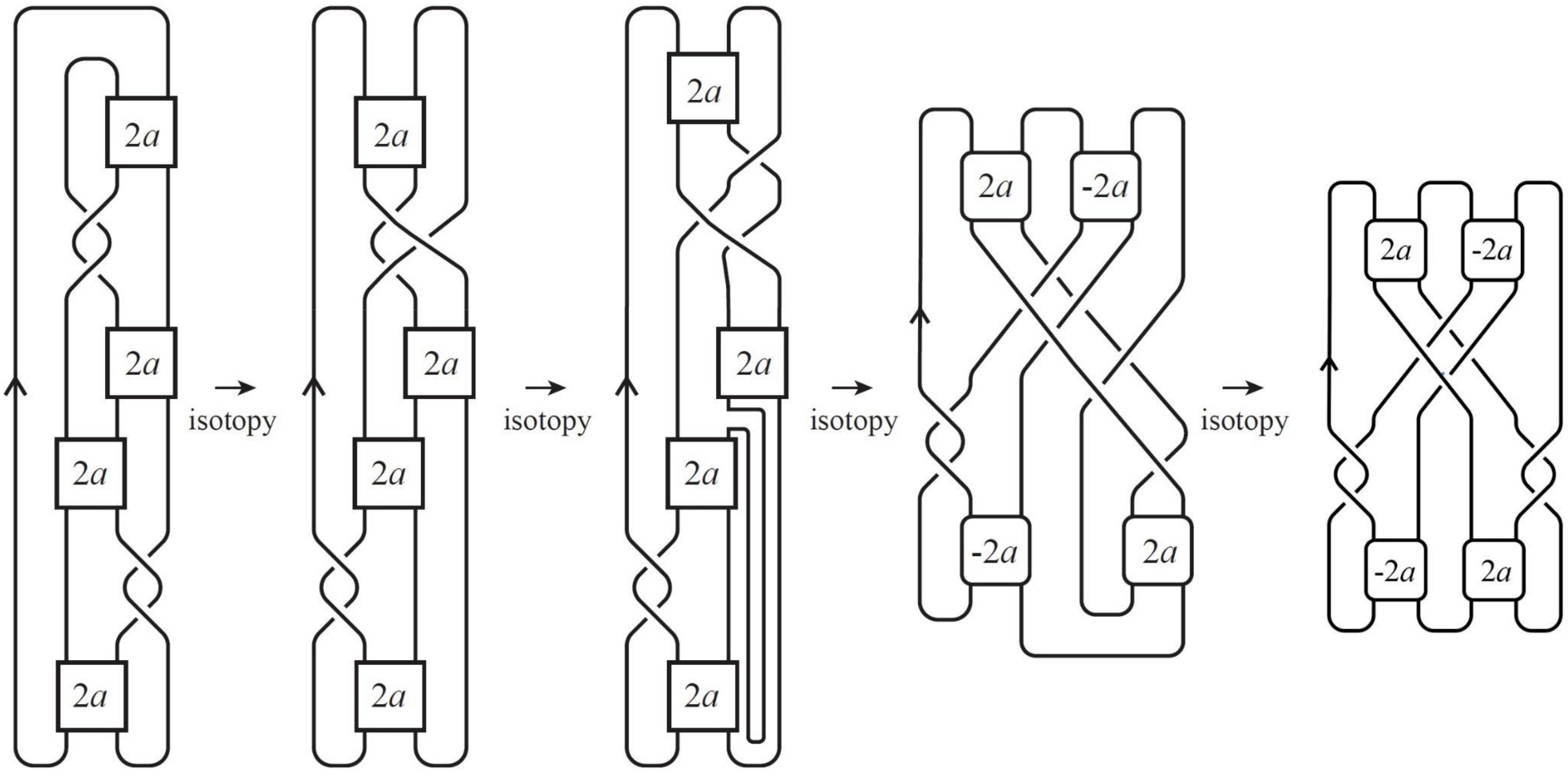
Proof of Proposition (outline)

Conversely, suppose that $K = K(2a, 2, 2a, 2a, 2, 2a)$ with $a \neq 0$. Then we know that K is amphicheiral as follows.



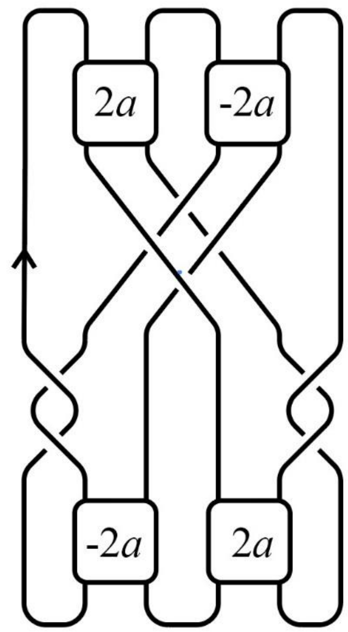
Proof of Theorem

Suppose that K is amphicheiral. Then $K = K(2a, 2, 2a, 2a, 2, 2a)$ by Proposition. Then we know that K is a symmetric union by using a result of C. Lamm as follows.

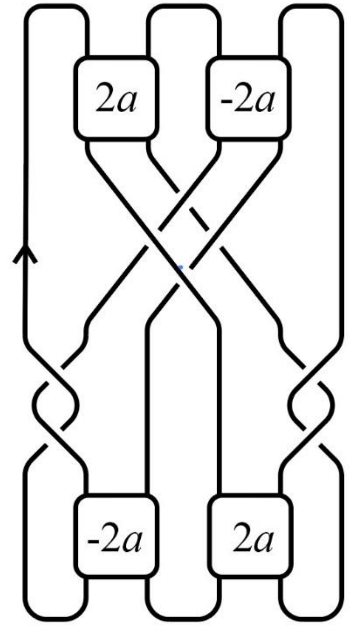


Proof of Theorem

Conversely, suppose that *K* is a symmetric union as follows. Then we know that it is amphicheiral since it is equivalent to *K*(2*a*, 2, 2*a*, 2*a*, 2, 2*a*).

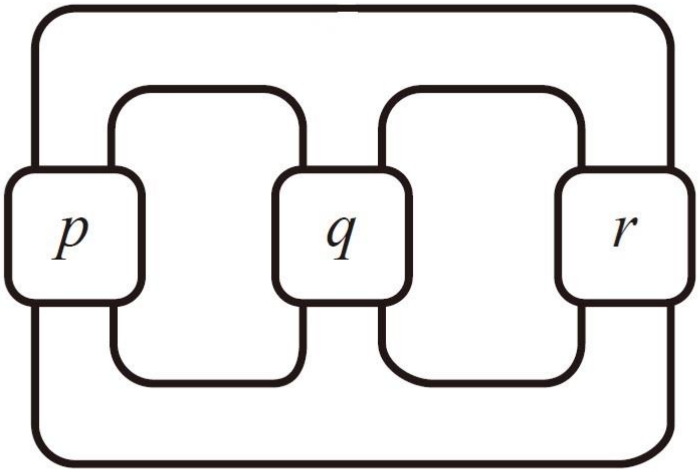


Corollary (T).
There exist infinitely many amphicheiral 2-bridge knots such that they have symmetric union presentations with *two twist regions*. In fact, any amphicheiral 2-bridge symmetric union has such a symmetric union presentation.



3-stranded pretzel knot

Def. We define the pretzel knot $P(p, q, r)$ by the diagram as follows.



Proposition (T).

If $P(p, q, r)$ has a symmetric union presentation, then it is not amphicheiral.

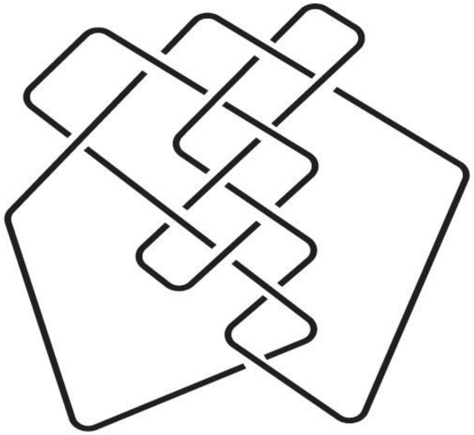
J. Greene and S. Jabuka, The slice-ribbon conjecture for 3-stranded pretzel knots, Amer. J. Math. 133 (2011), no. 3, 555-580.

A study for finding symmetric union presentation

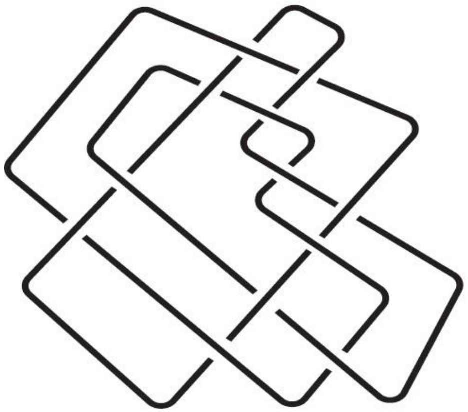
• In 2012, Seeliger studied symmetric union presentations for ribbon knots with crossing number up to 11 through 14 and showed that 122 ribbon knots have symmetric union presentations.

• In the list of all 137 prime ribbon knots with crossing number 11 or 12, there are two amphicheiral ribbon knots 12*a*990 and 12*a*1225 for which no symmetric union presentations were found.

• We can show that 12*a*990 and 12*a*1225 are 3-bridge knots by Proposition. By a result of Kose, they cannot be $D \cup D^*(n)$.



12*a*990



12*a*1225

Reference

1. C. Lamm, *Symmetric unions and ribbon knots*, Osaka J. Math., Vol. 37 (2000), 537-550.

2. T. Tanaka, *The Jones polynomial of knots with symmetric union presentations*, J. Korean Math. Soc. 52 (2015), no. 2, 389-402.

3. C. Lamm, *Symmetric union presentations for 2-bridge ribbon knots*, J. Knot Theory Ramifications 30 (2021), no. 12, 11 pp.

4. C. Lamm, *The search for non-symmetric ribbon knots*, arXiv:1710.06909, 2017.

5. F. C. Kose, *On amphicheiral symmetric unions*, arXiv:2111.08765, 2021.

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Thank you for your attention