S_m^N-crossing change and polynomial invariants of links

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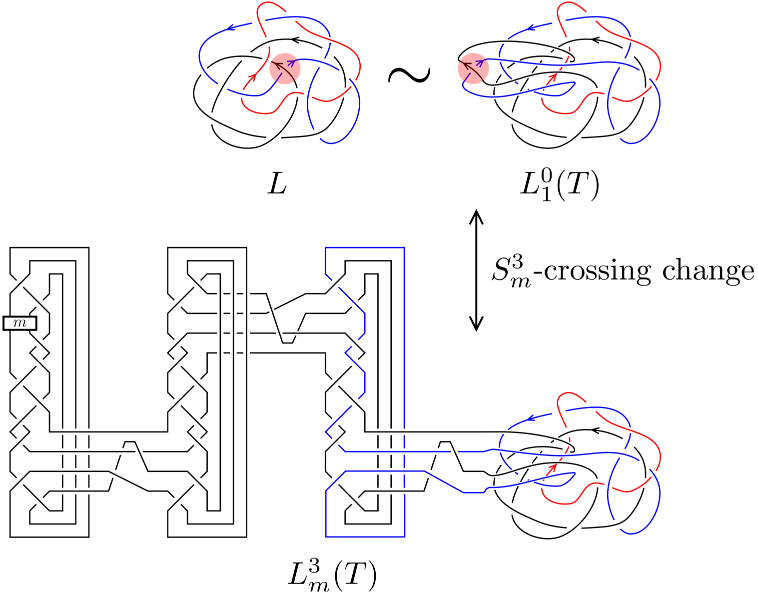
Knots and Spatial Graphs 2023

KAIST

June 17, 2023

We introduce a local change for links called an S_m^N-crossing change and show that there exists an infinite family of links with the coefficient polynomials from 0-th to s-th for any s of the HOMFLYPT and Kauffman polynomials of any link in each case.



Background of the S_m^N-crossing change: We focus on $12a1249$.

$\Gamma(12a1249; x) = 1,\ p_1(12a1249; v) = f_1(12a1249; a) = 0,\ \Gamma^{(2,1)}(12a1249; x) = 1.$

[Reference] Hideo Takioka, Infinitely many knots with the trivial $(2,1)$-cable Γ-polynomial, J. Knot Theory Ramifications 27 (2018), no. 2, 1850013, 18 pp.

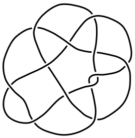
Here, $p_n(L; v) \in \mathbb{Z}[v^{\pm 1}]$, $f_n(L; a) \in \mathbb{Z}[a^{\pm 1}]$ are the n-th coefficient polynomials $(n \geq 0)$ of the HOMFLYPT and Kauffman polynomials, respectively.

In particular, we have $p_0(L; a^{-1}\sqrt{-1}) = f_0(L; a)$.

Moreover, since it is known that p_0 is a Laurent polynomial of the variable v^{-2}, we put $v^{-2} = x$ and call it the Γ-polynomial $\Gamma(L; x) \in \mathbb{Z}[x^{\pm 1}]$. Namely, we have

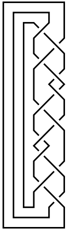
$$\Gamma(L; v^{-2}) = p_0(L; v),\ \Gamma(L; -a^2) = f_0(L; a).$$

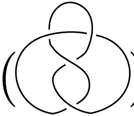
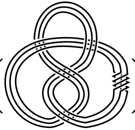
Let $I^{(p,q)}$ be the (p,q)-cabling of a knot invariant I for coprime integers $p(>0)$ and q.



$12a1249$

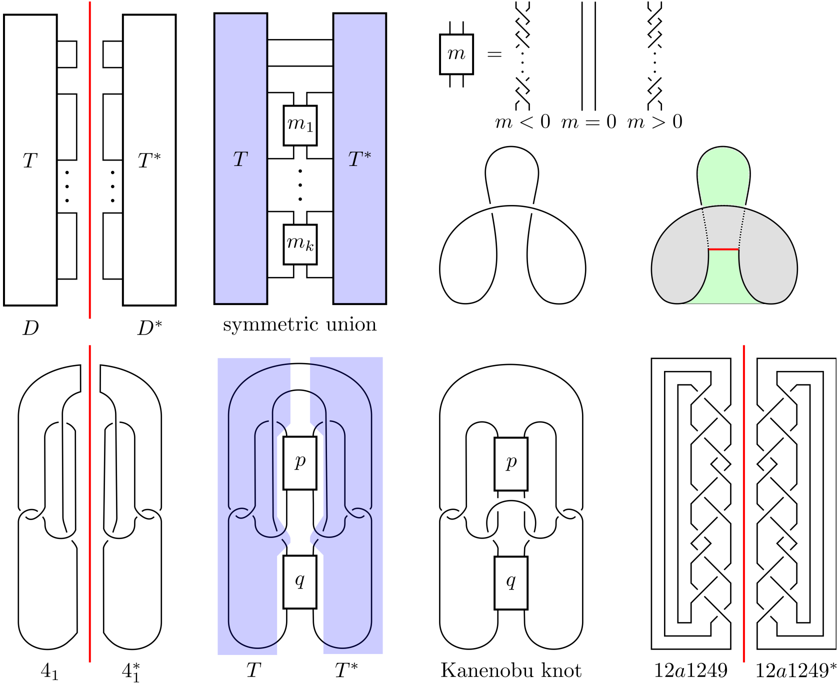
$\sim$



$I^{(3,4)}($  $) = I($  $).$

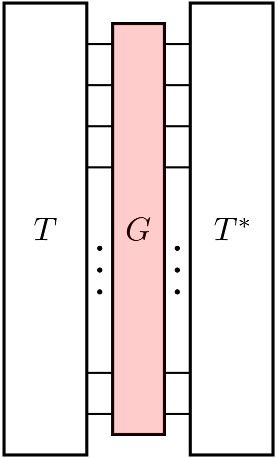
K $\qquad$ $K^{(3,4)}$

We consider a symmetric union of $12a1249$.

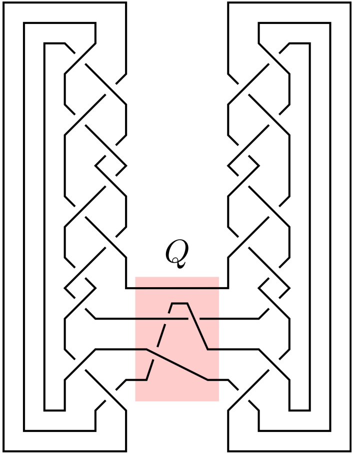


We generalize the symmetric union and discover a tangle Q.

$$\Gamma(K^2_{-1,1};x)=1,\ p_i(K^2_{-1,1};v)=f_j(K^2_{-1,1};a)=0\ \ (i=1,2,3,\ j=1,2),$$
$$\Gamma^{(p,1)}(K^2_{-1,1};x)=1\ \ (p=2,3,4).$$



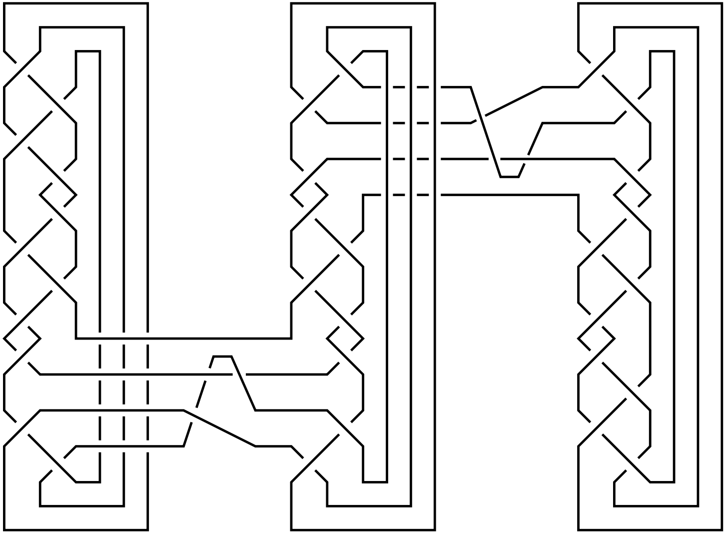
generalized
symmetric union



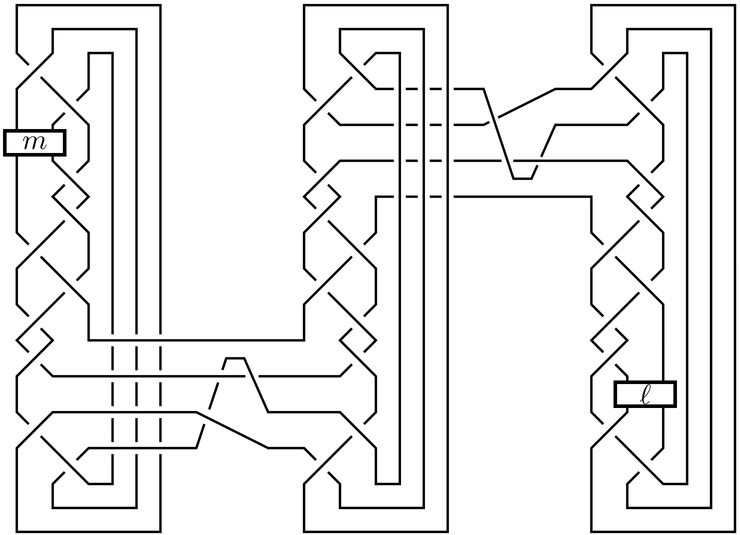
$K^2_{-1,1}$

We develop the generalized symmetric union.

We see that the coefficient polynomials from 0-th to s-th for any s are trivial.
(We conjecture that cablings of the Γ-polynomial are also trivial.)

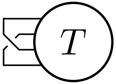


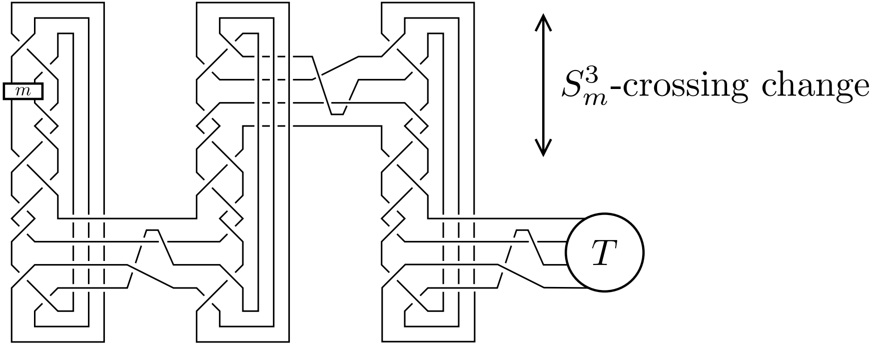
We can insert half-twists preserving the trivial coefficient polynomials.



We reach the S^N_m-crossing change.

We see that there exists an infinite family of links with the coefficient polynomials from 0-th to s-th for any s of the HOMFLYPT and Kauffman polynomials of any link in each case. (We conjecture that cablings of the Γ-polynomial also coincide.)

$L \sim L^0_1(T) =$ 



$L^3_m(T)$

Outline

1. **HOMFLYPT and Kauffman polynomials**
2. **Coefficient polynomials of the HOMFLYPT and Kauffman polynomials**
3. S_m^N**-crossing change**
4. **Link** $L_m^N(T)$ **and main result**
5. **Knot** $K_{m,\ell}^N$

1. HOMFLYPT and Kauffman polynomials

HOMFLYPT polynomial

The HOMFLYPT polynomial $P(L; v, z) \in \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$ is an invariant for oriented links, which is computed by the following skein relation.

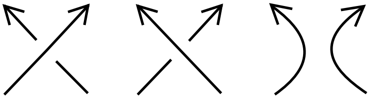
For the unknot U, we have $P(U; v, z) = 1$.

For a skein triple (L_+, L_-, L_0), the following holds.

Skein relation

$$v^{-1}P(L_+; v, z) - vP(L_-; v, z) = zP(L_0; v, z).$$

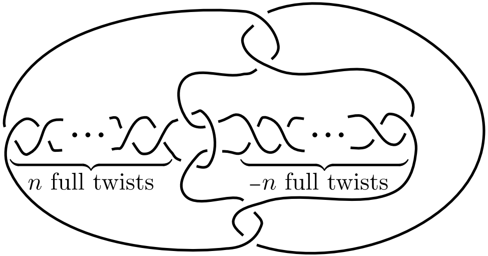
Here, a skein triple (L_+, L_-, L_0) is a triple of oriented links L_+, L_-, L_0 which are identical except near one point as in the figure below.



L_+ L_- L_0

Kanenobu knot

$$P(k(n); v, z) = \left((v^{-2} - 1 + v^2) - z^2\right)^2.$$



$k(n)$

[Reference] Taizo Kanenobu, Infinitely many knots with the same polynomial invariant, Proc. Amer. Math. Soc. 97 (1986), no. 1, 158–162.

Kauffman polynomial

The Kauffman polynomial $F(L;a,z) \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ is an invariant for oriented links, which is defined by $F(L;a,z) = a^{-w(D)}\Lambda(D;a,z)$.

Here, $w(D)$ is the writhe of a diagram D of an oriented link L and $\Lambda(D;a,z) \in \mathbb{Z}[a^{\pm 1}, z^{\pm 1}]$ is a regular isotopy invariant for unoriented link diagrams defined by the following.

For the unknot diagram $\bigcirc$ without crossings, we have $\Lambda(\bigcirc;a,z) = 1$.

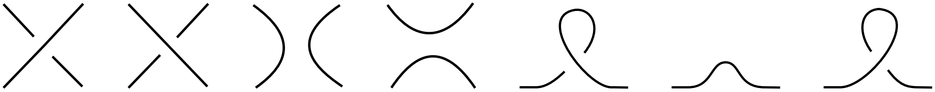
For a skein quadruple $(D_+, D_-, D_0, D_\infty)$, the following holds.

Skein relation

$$\Lambda(D_+;a,z) + \Lambda(D_-;a,z) = z\big(\Lambda(D_0;a,z) + \Lambda(D_\infty;a,z)\big).$$

For a Reidemeister move of type I as in the figure below, we have

$$\Lambda(C_+;a,z) = a\Lambda(C_0;a,z),\ \Lambda(C_-;a,z) = a^{-1}\Lambda(C_0;a,z).$$



D_+ D_- D_0 D_∞ C_+ C_0 C_-

Kanenobu's question

In Kirby's list, we have Kanenobu's question.

Are there infinitely many different knots with the same Kauffman polynomial?

As a recent Kanenobu's result, it is shown that there exist infinitely many different knots with the same HOMFLYPT and Q-polynomials.

Here, the Q-polynomial is an invariant for unoriented links, which is obtained from the Kauffman polynomial:

$$Q(L;x) = F(L;1,x).$$

2. Coefficient polynomials of the HOMFLYPT and Kauffman polynomials

Coefficient polynomials of the HOMFLYPT polynomial

The HOMFLYPT polynomial of an r-component link L is presented by the following:

$$P(L; v, z) = (-v^{-1}z)^{-r+1} \sum_{n \geq 0} p_n(L; v) z^{2n}.$$

Here, $p_n(L; v) \in \mathbb{Z}[v^{\pm 1}]$ and $p_0(L; v) \neq 0$.

We call $p_n(L; v)$ the n-th coefficient polynomial of the HOMFLYPT polynomial.

Let r, r' be the numbers of components of links L, L', respectively.
We see easily that if $r \neq r'$ then $P(L; v, z) \neq P(L'; v, z)$.

It is shown that there exists an infinite family of links with the coefficient polynomials from 0-th to s-th for any s of the HOMFLYPT polynomial of any link by Kawauchi and Miyazawa independently.

[References] Akio Kawauchi, Almost identical link imitations and the skein polynomial, Knots 90 (Osaka, 1990), 465–476, de Gruyter, Berlin, 1992.

Yasuyuki Miyazawa, A fake HOMFLY polynomial of a knot, Osaka J. Math. 50 (2013), no. 4, 1073–1096.

Coefficient polynomials of the Kauffman polynomial

The Kauffman polynomial of an r-component link L is presented by the following:

$$F(L;a,z) = (az)^{-r+1} \sum_{n\geq 0} f_n(L;a) z^n.$$

Here, $f_n(L;a) \in \mathbb{Z}[a^{\pm 1}]$ and $f_0(L;a) \neq 0$.

We call $f_n(L;a)$ the n-th coefficient polynomial of the Kauffman polynomial.

Let r, r' be the numbers of components of links L, L', respectively.
We see easily that if $r \neq r'$ then $F(L;a,z) \neq F(L';a,z)$.

In this talk, we show that there exists an infinite family of links with the coefficient polynomials from 0-th to s-th for any s of the HOMFLYPT and Kauffman polynomials of any link in each case by using the S_m^N-crossing change.

Connected sum for the HOMFLYPT polynomial

Let $L\#L'$ be a connected sum of links L and L'. Then we have

$$P(L\#L';v,z) = P(L;v,z)P(L';v,z).$$

Let K be a knot with $p_0(K;v)=1$, $p_i(K;v)=0$ $(1 \le i \le s)$ and $p_{s+1}(K;v) \ne 0$.

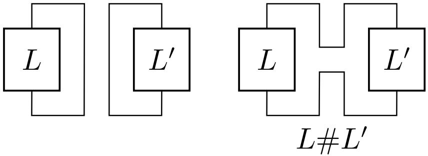
For any link L and the knot K, we have

$$P(L;v,z) = (-v^{-1}z)^{-r+1}\sum_{n\ge 0} p_n(L;v)z^{2n},\ \ P(K;v,z) = 1 + \sum_{i \ge s+1} p_i(K;v)z^{2i}.$$

Therefore, we have $P(L\#K;v,z) = P(L;v,z)P(K;v,z)$

$$= P(L;v,z) + (-v^{-1}z)^{-r+1} \sum_{k\ge s+1}\Big(\sum_{n+i=k} p_n(L;v)p_i(K;v)\Big)z^{2k}.$$

Therefore, we have $p_i(L\#K;v) = p_i(L;v)$ $(0 \le i \le s)$ and $p_{s+1}(L\#K;v) \ne p_{s+1}(L;v)$.



Connected sum for the Kauffman polynomial

Let $L\#L'$ be a connected sum of links L and L'. Then we have

$$F(L\#L';a,z) = F(L;a,z)F(L';a,z).$$

Let K be a knot with $f_0(K;a) = 1,\ f_i(K;a) = 0\ (1 \leq i \leq s)$ and $f_{s+1}(K;a) \neq 0$.

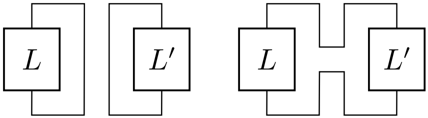
For any link L and the knot K, we have

$$F(L;a,z) = (az)^{-r+1}\sum_{n\geq 0} f_n(L;a)z^n,\ F(K;a,z) = 1 + \sum_{i\geq s+1} f_i(K;a)z^i.$$

Therefore, we have $F(L\#K;a,z) = F(L;a,z)F(K;a,z)$

$$= F(L;a,z) + (az)^{-r+1}\sum_{k\geq s+1}\Big(\sum_{n+i=k} f_n(L;a)f_i(K;a)\Big)z^k.$$

Therefore, we have $f_i(L\#K;a) = f_i(L;a)\ (0 \leq i \leq s)$ and $f_{s+1}(L\#K;a) \neq f_{s+1}(L;a)$.



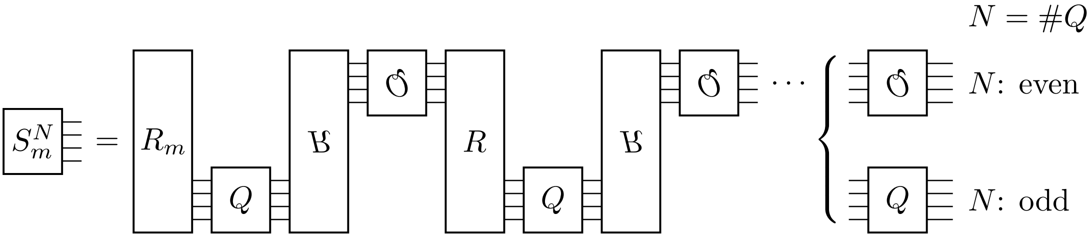
$L\#L'$

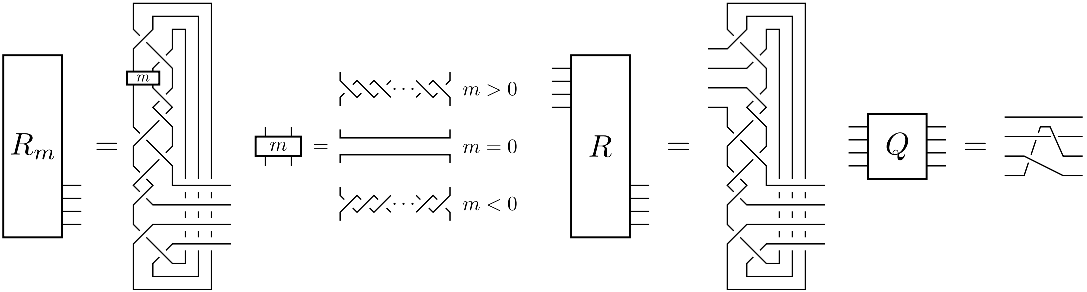
3. S_m^N-crossing change

From here, we always consider a link diagram and a tangle diagram.

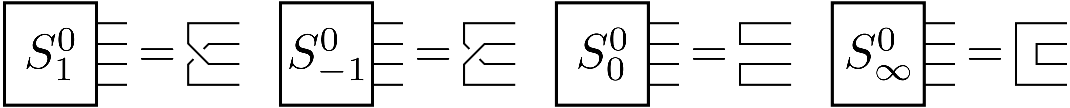
Therefore, we call them just a link and a tangle for simplicity.

Tangle S_m^N

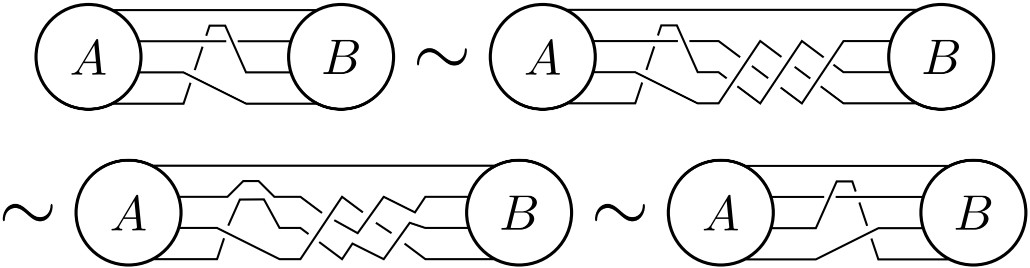




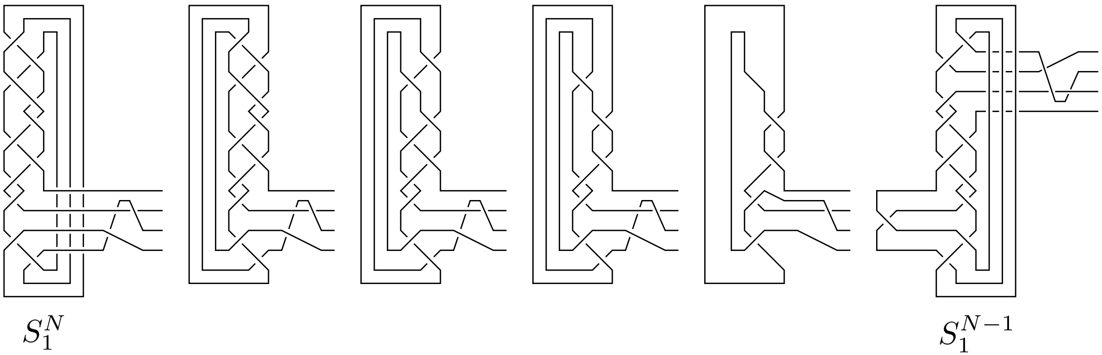
Tangles S^0_1, S^0_{-1}, S^0_0, S^0_∞



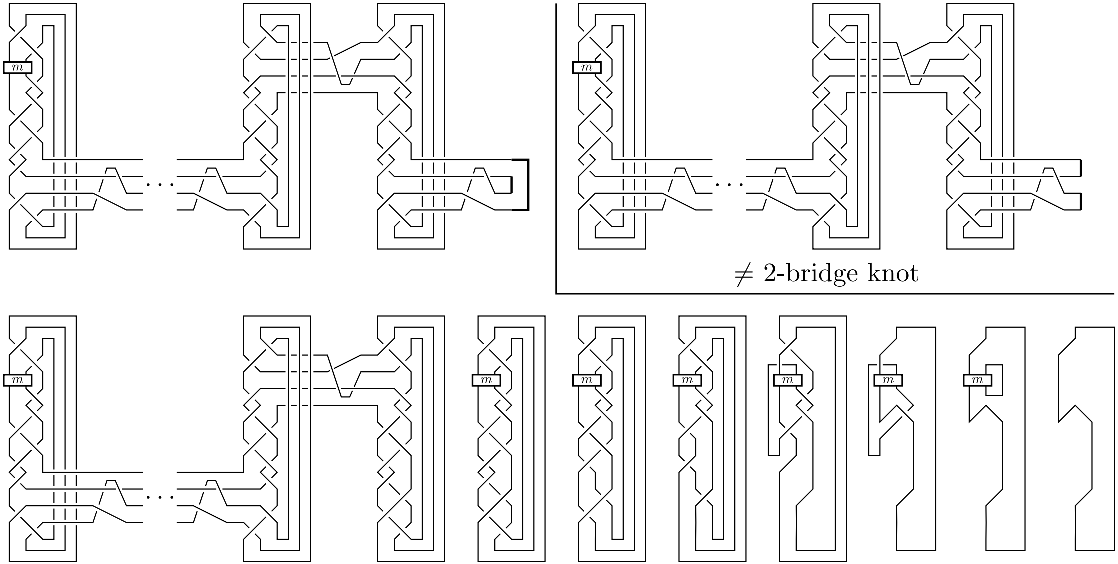
The tangle Q is equivalent to its mirror image.



The tangle S_1^N is equivalent to S_1^{N-1} for $N \geq 1$.

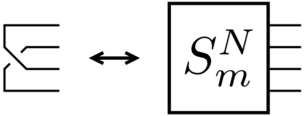


The tangle S_m^N is prime for $m \neq 1, N \geq 1$.



S_m^N-crossing change

We define an S_m^N-crossing change as a local change for links which changes a single crossing S_1^0 to the tangle S_m^N or vice versa as in the figure below.



In particular, an S^0_{-1}-crossing change is the so-called crossing change.

4. Link $L_m^N(T)$ and main result

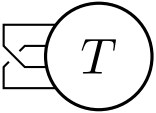
Link $L_m^N(T)$

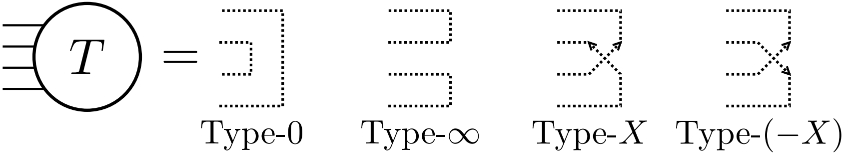
Let L be a link. We can deform the link L into a link with a single crossing and a tangle T denoted by $L_1^0(T)$ as in the figure below.

The tangle T consists of two arcs and link components.

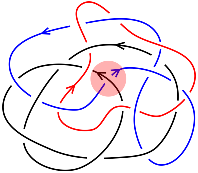
We consider connection types of the two arcs, which are called Type-0, Type-∞, Type-X, Type-$(-X)$.

We denote a link obtained by an S_m^N-crossing change at the single crossing by $L_m^N(T)$.

$$L \sim L_1^0(T) =$$ 

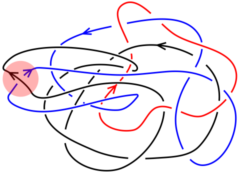


Example



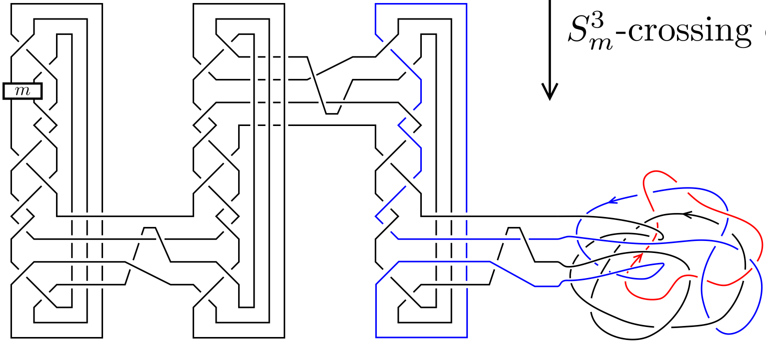
L

$\sim$



$L_1^0(T)$

S^3_m-crossing change



$L^3_m(T)$

Cases (I)–(VIII)

We consider all the pairs of the parity of an integer m and the type of a tangle T as in the table below.

In particular, we can check the numbers of components of the links $L_1^0(T)(\sim L)$, $L_0^0(T)$, $L_\infty^0(T)$, $L_m^N(T)$.

(m,T)	$\#L_1^0(T)(\sim L)$	$\#L_0^0(T)$	$\#L_\infty^0(T)$	$\#L_m^N(T)$
(I) $(\text{odd}, 0)$	r	r	$r+1$	r
(II) (odd, ∞)	r	$r+1$	r	r
(III) (odd, X)	r	$r-1$	$r-1$	r
(IV) $(\text{odd}, -X)$	r	$r-1$	$r-1$	r
(V) $(\text{even}, 0)$	r	r	$r+1$	r
(VI) (even, ∞)	r	$r+1$	r	r
(VII) (even, X)	r	$r-1$	$r-1$	r
(VIII) $(\text{even}, -X)$	r	$r-1$	$r-1$	r

The HOMFLYPT polynomial of $L_m^N(T)$

Let $\zeta = z^4(2+z^2)^2$.

(I, III) $P(L_m^N) = P(L) + (1-v^{-m+1})\zeta^N\left(\frac{z}{v^{-1}-v}P(L_\infty^0) - P(L)\right)$.

(II, IV) $P(L_m^N) = P(L) + \frac{z}{v^{-1}-v}(1-v^{-m+1})\zeta^{N-1}\Big(\big((-v^{-3}+v)z^3+(v^{-1}+v)z^5\big)P(L) + \big((-v^{-2}+1+v^2-v^4)z^2+(2+2v^2)z^4\big)P(L_0^0)\Big)$.

(V, VII) $P(L_m^N) = P(L) + \zeta^{N-1}\big(P(L_\infty^0) - (v^{-1}-v)z^{-1}P(L)\big)\Big(v^{-m}\big((v^{-1}-v^3)z^3+(v^{-1}+v)z^5\big) + \frac{z}{v^{-1}-v}(1-v^{-m})(v^{-1}+v)^2z^4\Big)$.

[(VI, VIII) $N=1$]

$P(L_m^1) = P(L) + \Big((-v^{-1}+v+vz^2)v^{-m} - (2z+z^3)\frac{z}{v^{-1}-v}(1-v^{-m})\Big)\Big(\big((2v^{-1}-3v+v^3)z^2+(v^{-1}-4v)z^4-vz^6\big)P(L) + \big((1-2v^2+v^4)z+(1-3v^2)z^3-v^2z^5\big)P(L_0^0)\Big)$.

[(VI, VIII) $N \geq 2$]

$P(L_m^N) = P(L) + \zeta^{N-2}\Big(\big((-v^{-3}+v)z^3+(v^{-1}+v)z^5\big)P(L) + \big((-v^{-2}+1+v^2-v^4)z^2+(2+2v^2)z^4\big)P(L_0^0)\Big)\Big(v^{-m}\big((v^{-1}-v^3)z^3+(v^{-1}+v)z^5\big) + \frac{z}{v^{-1}-v}(1-v^{-m})(v^{-1}+v)^2z^4\Big)$.

Main result: Coefficient polynomials of the HOMFLYPT polynomial of $L_m^N(T)$

Let L be a link. Let N be a positive integer. There exists an infinite family of links $\{L_m^N(T)\}_{m\in\mathbb{Z}}$ such that $p_i(L_m^N(T);v) = p_i(L;v)$ for $0 \le i \le s$,

where $s = \begin{cases} \text{(I)} & 2N-1, \\ \text{(II)} & 2N-2, \\ \text{(III)} & 2N-1, \\ \text{(IV)} & 2N-1, \\ \text{(V)} & 2N-2, \\ \text{(VI)} & 2N-3\ (N \neq 1), \\ \text{(VII)} & 2N-2, \\ \text{(VIII)} & 2N-2. \end{cases}$ In particular, $L_1^N(T)$ is equivalent to L.

Moreover, we have $p_{s+1}(L_m^N(T);v) \neq p_{s+1}(L_{m'}^N(T);v)$ for $m \neq m'$ except in the following case.

A link $L_1^0(T)$ with a tangle T of Type-0 satisfies

$$p_0(L_1^0(T);v) = p_0(L_{-1}^0(T);v). \quad \cdots (*)$$

Remark

We consider a link $L_1^0(T)$ with a tangle T of Type-0 which satisfies

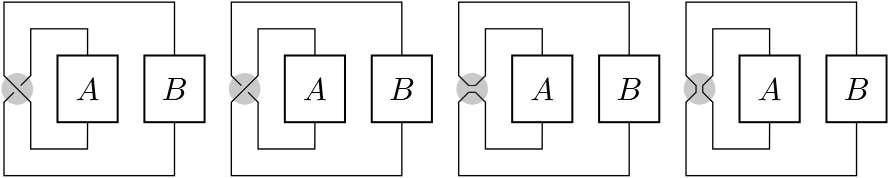
$$P(L_1^0(T); v, z) = P(L_{-1}^0(T); v, z). \quad \cdots (**)$$

Then, we see that the link $L_1^0(T)$ satisfies the condition $(*)$ and we have $P(L_m^N(T); v, z) = P(L; v, z)$ for any $m \in \mathbb{Z}$ and $N \in \mathbb{Z}_{>0}$.

A link $L_1^0(T)$ with a nugatory crossing and a tangle T of Type-0 as in the figure below satisfies the condition $(**)$. However, in this case, we see that the link $L_m^N(T)$ is equivalent to the link L.

Question: Is there a link $L_1^0(T)$ with a non-nugatory crossing and a tangle T of Type-0 which satisfies the condition $(**)$?

(We can find such a link $L_1^0(T)$ for the condition $(*)$ easily.)



$L_1^0(T)$ $\quad$ $L_{-1}^0(T)$ $\quad$ $L_0^0(T)$ $\quad$ $L_\infty^0(T)$

The Kauffman polynomial of $L_m^N(T)$

We calculate the Kauffman polynomial of the link $L_m^N(T)$ by using the Λ-polynomial.

We denote $L_m^N(T)$ by L_m^N for simplicity.

$F(L_m^N) = a^{-w(L_m^N)}\Lambda(L_m^N)$.

By Proposition 2.2 [Kanenobu 1989],

$\Lambda(L_m^N) = \sigma_m \Lambda(L_1^N) - \sigma_{m-1}\Lambda(L_0^N) + \tau_m \Lambda(L_\infty^N)$,

Here, $\sigma_m \in \mathbb{Z}[z]$, $\sigma_0 = 0$, $\sigma_1 = 1$, $\sigma_{m-1} + \sigma_{m+1} = z\sigma_m$,

$\tau_m \in \mathbb{Z}[a^{\pm 1}, z]$, $\tau_0 = \tau_1 = 0$, $\tau_{m-1} + \tau_{m+1} = z\tau_m + a^{-m}z$.

[Reference] Taizo Kanenobu, Examples on polynomial invariants of knots and links. II, Osaka J. Math. 26 (1989), no. 3, 465–482.

The Λ-polynomial of $L_m^N(T)$

$\boldsymbol{\Lambda(L_0^N)}$

$$= \Lambda(L_1^{N-1})\Big(1+(-3a^{-2}-6-3a^2)z^2+(a^{-3}-2a^{-1}-a+2a^3)z^3+(a^{-4}+10a^{-2}+16+10a^2)z^4+(a^{-3}+10a^{-1}+7a-3a^3)z^5+(-a^{-4}-8a^{-2}-13-11a^2)z^6+(-2a^{-3}-12a^{-1}-11a+a^3)z^7+(-1+3a^2)z^8+(2a^{-1}+3a)z^9+z^{10}\Big)$$

$$+\Lambda(L_0^{N-1})\Big((a^{-4}+3a^{-2}+3+a^2)z+(-a^{-5}+2a^{-1}-a^3)z^2+(-5a^{-4}-11a^{-2}-12-6a^2)z^3+(a^{-5}-4a^{-3}-9a^{-1}-4a+2a^3)z^4+(4a^{-4}+9a^{-2}+12+8a^2)z^5+(5a^{-3}+12a^{-1}+8a-a^3)z^6+(a^{-2}-3a^2)z^7+(-2a^{-1}-3a)z^8-z^9\Big)$$

$$+\Lambda(L_\infty^{N-1})\Big((-a^{-3}-2a^{-1}-a)z^2+(a^{-4}+a^{-2}+1+a^2)z^3+(5a^{-3}+9a^{-1}+5a)z^4+(-2a^{-4}+a^{-2}+1-2a^2)z^5+(-7a^{-3}-11a^{-1}-7a)z^6+(a^{-4}-5a^{-2}-5+a^2)z^7+(3a^{-3}+3a^{-1}+3a)z^8+(3a^{-2}+3)z^9+a^{-1}z^{10}\Big)$$

$\boldsymbol{\Lambda(L_\infty^N)}$

$$= \Lambda(L_1^{N-1})\Big((a^{-1}+a)z^{-1}-1+(-a^{-3}-12a^{-1}-9a+2a^3)z^3+(-4+6a^2-2a^4)z^4+(2a^{-3}+23a^{-1}+20a-8a^3)z^5+(3a^{-2}+13-13a^2+a^4)z^6+(-a^{-3}-12a^{-1}-18a+3a^3)z^7+(-2a^{-2}-11+4a^2)z^8+(a^{-1}+4a)z^9+2z^{10}\Big)$$

$$+\Lambda(L_0^{N-1})\Big((3a^{-3}+5a^{-1}+a-a^3)z^2+(-2a^{-4}+3a^{-2}+1-3a^2+a^4)z^3+(-9a^{-3}-14a^{-1}-8a+5a^3)z^4+(a^{-4}-10a^{-2}-7+9a^2-a^4)z^5+(4a^{-3}+8a^{-1}+13a-3a^3)z^6+(5a^{-2}+9-4a^2)z^7-4az^8-2z^9\Big)$$

$$+\Lambda(L_\infty^{N-1})\Big((-3a^{-2}-2+a^2)z^3+(3a^{-3}+3a^{-1}+3a-a^3)z^4+(12a^{-2}+9-4a^2)z^5+(-3a^{-3}+2a^{-1}-6a+a^3)z^6+(-10a^{-2}-9+3a^2)z^7+(a^{-3}-4a^{-1}+4a)z^8+(3a^{-2}+4)z^9+2a^{-1}z^{10}\Big)$$

2×2 matrix $M_{\Lambda(L_m^N(T))}$

$$\Lambda(L_0^N) - a\Lambda(L_1^N)$$

$$= \big(\Lambda(L_0^{N-1}) - a\Lambda(L_1^{N-1})\big)\Big((a^{-4}+3a^{-2}+3+a^2)z + (-a^{-5}+2a^{-1}-a^3)z^2 + (-5a^{-4}-11a^{-2}-12-6a^2)z^3 + (a^{-5}-4a^{-3}-9a^{-1}-4a+2a^3)z^4 + (4a^{-4}+9a^{-2}+12+8a^2)z^5 + (5a^{-3}+12a^{-1}+8a-a^3)z^6 + (a^{-2}-3a^2)z^7 + (-2a^{-1}-3a)z^8 - z^9\Big)$$

$$+\Big(\Lambda(L_\infty^{N-1}) - a\Lambda(L_1^{N-1})\big((a^{-1}+a)z^{-1}-1\big)\Big)\Big((-a^{-3}-2a^{-1}-a)z^2 + (a^{-4}+a^{-2}+1+a^2)z^3 + (5a^{-3}+9a^{-1}+5a)z^4 + (-2a^{-4}+a^{-2}+1-2a^2)z^5 + (-7a^{-3}-11a^{-1}-7a)z^6 + (a^{-4}-5a^{-2}-5+a^2)z^7 + (3a^{-3}+3a^{-1}+3a)z^8 + (3a^{-2}+3)z^9 + a^{-1}z^{10}\Big)$$

$$\Lambda(L_\infty^N) - a\Lambda(L_1^N)\big((a^{-1}+a)z^{-1}-1\big)$$

$$= \big(\Lambda(L_0^{N-1}) - a\Lambda(L_1^{N-1})\big)\Big((3a^{-3}+5a^{-1}+a-a^3)z^2 + (-2a^{-4}+3a^{-2}+1-3a^2+a^4)z^3 + (-9a^{-3}-14a^{-1}-8a+5a^3)z^4 + (a^{-4}-10a^{-2}-7+9a^2-a^4)z^5 + (4a^{-3}+8a^{-1}+13a-3a^3)z^6 + (5a^{-2}+9-4a^2)z^7 - 4az^8 - 2z^9\Big)$$

$$+\Big(\Lambda(L_\infty^{N-1}) - a\Lambda(L_1^{N-1})\big((a^{-1}+a)z^{-1}-1\big)\Big)\Big((-3a^{-2}-2+a^2)z^3 + (3a^{-3}+3a^{-1}+3a-a^3)z^4 + (12a^{-2}+9-4a^2)z^5 + (-3a^{-3}+2a^{-1}-6a+a^3)z^6 + (-10a^{-2}-9+3a^2)z^7 + (a^{-3}-4a^{-1}+4a)z^8 + (3a^{-2}+4)z^9 + 2a^{-1}z^{10}\Big)$$

$$\begin{pmatrix} \Lambda(L_0^N) - a\Lambda(L_1^N) \\ \Lambda(L_\infty^N) - a\Lambda(L_1^N)\big((a^{-1}+a)z^{-1}-1\big) \end{pmatrix} = M_{\Lambda(L_m^N(T))} \begin{pmatrix} \Lambda(L_0^{N-1}) - a\Lambda(L_1^{N-1}) \\ \Lambda(L_\infty^{N-1}) - a\Lambda(L_1^{N-1})\big((a^{-1}+a)z^{-1}-1\big) \end{pmatrix}$$

$$\begin{pmatrix} \Lambda(L_0^N) - a\Lambda(L_1^N) \\ \Lambda(L_\infty^N) - a\Lambda(L_1^N)\big((a^{-1}+a)z^{-1}-1\big) \end{pmatrix} = \big(M_{\Lambda(L_m^N(T))}\big)^N \begin{pmatrix} \Lambda(L_0^0) - a\Lambda(L_1^0) \\ \Lambda(L_\infty^0) - a\Lambda(L_1^0)\big((a^{-1}+a)z^{-1}-1\big) \end{pmatrix}$$

The 1st and 2nd minimum terms on z of $\left(M_{\Lambda(L_m^N(T))}\right)^N$

(1, 1)-entry: $[(1+a^2)^{3N}a^{-4N}z^N + A_N z^{N+1}, *]$

(1, 2)-entry: $[-(1+a^2)^{3N-1}a^{-4N+1}z^{N+1} + B_N z^{N+2}, *]$

(2, 1)-entry: $[(3-a^2)(1+a^2)^{3N-1}a^{-4N+1}z^{N+1} + C_N z^{N+2}, *]$

(2, 2)-entry: $[-(3-a^2)(1+a^2)^{3N-2}a^{-4N+2}z^{N+2} + D_N z^{N+3}, *]$

$A_1 = -a^{-5} + 2a^{-1} - a^3$, $B_1 = a^{-4} + a^{-2} + 1 + a^2$, $C_1 = -2a^{-4} + 3a^{-2} + 1 - 3a^2 + a^4$,
$D_1 = 3a^{-3} + 3a^{-1} + 3a - a^3$.

$N \geq 2$.

$$A_N = (-a^{-5} + 2a^{-1} - a^3)\left(a^{-4}(1+a^2)^3\right)^{N-1} + \frac{1-\left(a^{-4}(1+a^2)^3\right)^{N-1}}{1-a^{-4}(1+a^2)^3}\left(-a^{-4N-1}(1-a^4)^2(1+a^2)^{3N-3}\right),$$

$$B_N = (a^{-4} + a^{-2} + 1 + a^2)\left(a^{-4}(1+a^2)^3\right)^{N-1} + \frac{1-\left(a^{-4}(1+a^2)^3\right)^{N-1}}{1-a^{-4}(1+a^2)^3}\left(a^{-4N}(1-a^4)^2(1+a^2)^{3N-4}\right),$$

$$C_N = (3a^{-3} + 5a^{-1} + a - a^3)A_{N-1} + (-2a^{-4} + 3a^{-2} + 1 - 3a^2 + a^4)\left(a^{-4N+4}(1+a^2)^{3N-3}\right),$$

$$D_N = (3a^{-3} + 5a^{-1} + a - a^3)B_{N-1} + (-2a^{-4} + 3a^{-2} + 1 - 3a^2 + a^4)\left(-a^{-4N+5}(1+a^2)^{3N-4}\right).$$

The 1st and 2nd maximum terms on z of $\left(M_{\Lambda(L_m^N(T))}\right)^N$

$(1,1)$-entry: $[*, A_N z^{10N-2} - 2^{N-1}a^{-N+1}z^{10N-1}]$

$(1,2)$-entry: $[*, B_N z^{10N-1} + 2^{N-1}a^{-N}z^{10N}]$

$(2,1)$-entry: $[*, C_N z^{10N-2} - 2^{N}a^{-N+1}z^{10N-1}]$

$(2,2)$-entry: $[*, D_N z^{10N-1} + 2^{N}a^{-N}z^{10N}]$

$A_1 = -2a^{-1} - 3a,\ B_1 = 3a^{-2} + 3,\ C_1 = -4a,\ D_1 = 3a^{-2} + 4.$

$N \geq 2.$

$$A_N = a^{-1}C_{N-1} + (6a^{-2} + 5)(-2^{N-2}a^{-N+2}),$$

$$B_N = a^{-1}D_{N-1} + (6a^{-2} + 5)(2^{N-2}a^{-N+1}),$$

$$C_N = -4a(2a^{-1})^{N-1} + \frac{1-(2a^{-1})^{N-1}}{1-2a^{-1}}(6a^{-2} + 6)(-2^{N-2}a^{-N+2}),$$

$$D_N = (3a^{-2} + 4)(2a^{-1})^{N-1} + \frac{1-(2a^{-1})^{N-1}}{1-2a^{-1}}(6a^{-2} + 6)(2^{N-2}a^{-N+1}).$$

The 1st and 2nd minimum terms on a of $\left(M_{\Lambda(L_m^N(T))}\right)^N$

$(1,1)$-entry: $[(-1)^N(1-z^2)^N z^{2N} a^{-5N} + (-1)^{N+1}(N-4Nz^2)(1-z^2)^N z^{2N-1} a^{-5N+1}, *]$

$(1,2)$-entry:
$[(-1)^{N+1}(1-z^2)^{N+1} z^{2N+1} a^{-5N+1} + (-1)^N\left(N-(4N-1)z^2\right)(1-z^2)^{N+1} z^{2N} a^{-5N+2}, *]$

$(2,1)$-entry:
$[(-1)^N(2-z^2)(1-z^2)^{N-1} z^{2N+1} a^{-5N+1} + (-1)^{N+1}(2N+1-9Nz^2+4Nz^4)(1-z^2)^{N-1} z^{2N} a^{-5N+2}, *]$

$(2,2)$-entry:
$$\begin{cases} [(3z^4-3z^6+z^8)a^{-3} + (-3z^3+12z^5-10z^7+3z^9)a^{-2}, *] & (N=1) \\ [(-1)^{N+1}(2-z^2)(1-z^2)^N z^{2N+2} a^{-5N+2} & \\ +(-1)^N\left(2N+1-(9N-2)z^2+(4N-1)z^4\right)(1-z^2)^N z^{2N+1} a^{-5N+3}, *] & (N \geq 2) \end{cases}$$

The 1st and 2nd maximum terms on a of $\left(M_{\Lambda(L_m^N(T))}\right)^N$

$(1,1)$-entry:
$[*, (-1)^{N+1}\big(N - 5Nz^2 + (4N-1)z^4\big)(1-z^2)^N z^{2N-1} a^{3N-1} + (-1)^N (1-z^2)^{N+1} z^{2N} a^{3N}]$

$(1,2)$-entry: $[*, (-1)^N\big(N - (4N-1)z^2\big)(1-z^2)^{N+1} z^{2N} a^{3N-2} + (-1)^{N+1}(1-z^2)^{N+1} z^{2N+1} a^{3N-1}]$

$(2,1)$-entry:
$[*, (-1)^N\big(N - 5Nz^2 + (4N-1)z^4\big)(1-z^2)^{N-1} z^{2N} a^{3N} + (-1)^{N+1}(1-z^2)^N z^{2N+1} a^{3N+1}]$

$(2,2)$-entry: $[*, (-1)^{N+1}\big(N - (4N-1)z^2\big)(1-z^2)^N z^{2N+1} a^{3N-1} + (-1)^N (1-z^2)^N z^{2N+2} a^{3N}]$

By the 1st and 2nd minimum terms on z of $\left(M_{\Lambda(L_m^N(T))}\right)^N$, we have

$$\Lambda(L_0^N) - a\Lambda(L_1^N) = \begin{cases} [\gamma(a)z^{N-r+1}, *] & \text{(I, V)}, \\ [\gamma(a)z^{N-r}, *] \ (\gamma(a) \neq 0) & \text{(II, VI)}, \\ [\gamma(a)z^{N-r+2}, *] \ (\gamma(a) \neq 0) & \text{(III, IV, VII, VIII)}, \end{cases}$$

$$\Lambda(L_\infty^N) - a\Lambda(L_1^N)\big((a^{-1}+a)z^{-1} - 1\big) = \begin{cases} [\delta(a)z^{N-r+2}, *] & \text{(I, V)}, \\ [\delta(a)z^{N-r+1}, *] \ (\delta(a) \neq 0) & \text{(II, VI)}, \\ [\delta(a)z^{N-r+3}, *] \ (\delta(a) \neq 0) & \text{(III, IV, VII, VIII)}. \end{cases}$$

Here, $\gamma(a)$ and $\delta(a)$ are defined by the following:

$$\gamma(a) = \begin{cases} a^{-4N-r+1}(1+a^2)^{3N-1}\big((1+a^2)a^{w(L_0^0)}f_0(L_0^0) - a^{w(L_\infty^0)}f_0(L_\infty^0)\big) & \text{(I, V)}, \\ a^{-4N-r}(1+a^2)^{3N}a^{w(L_0^0)}f_0(L_0^0) & \text{(II, VI)}, \\ a^{-4N-r+2}(1+a^2)^{3N-1}\big((1+a^2)a^{w(L_0^0)}f_0(L_0^0) - 3a^{w(L_1^0)+1}f_0(L_1^0)\big) & \text{(III, IV, VII, VIII)}, \end{cases}$$

$$\delta(a) =$$
$$\begin{cases} a^{-4N-r+2}(3-a^2)(1+a^2)^{3N-2}\big((1+a^2)a^{w(L_0^0)}f_0(L_0^0) - a^{w(L_\infty^0)}f_0(L_\infty^0)\big) & \text{(I, V)}, \\ a^{-4N-r+1}(3-a^2)(1+a^2)^{3N-1}a^{w(L_0^0)}f_0(L_0^0) & \text{(II, VI)}, \\ a^{-r-2}(1+a^2)\big((3-a^2)(1+a^2)a^{w(L_0^0)+1}f_0(L_0^0) - (1+11a^2-2a^4)a^{w(L_1^0)}f_0(L_1^0)\big) & \text{(III, IV, VII, VIII) } N=1, \\ a^{-4N-r+3}(3-a^2)(1+a^2)^{3N-2}\big((1+a^2)a^{w(L_0^0)}f_0(L_0^0) - 3a^{w(L_1^0)+1}f_0(L_1^0)\big) & \text{(III, IV, VII, VIII) } N \geq 2. \end{cases}$$

In the cases (I, V),

$$\gamma(a) = 0 \iff \delta(a) = 0 \iff (1+a^2)a^{w(L_0^0)}f_0(L_0^0; a) = a^{w(L_\infty^0)}f_0(L_\infty^0; a).$$

By $w(L_m^N(T)) = \begin{cases} -m-N+w(T) & \text{(I, III, V, VII)}, \\ -m-N+2+w(T) & \text{(II, IV, VI, VIII)} \end{cases}$, we have

$$F(L_m^N) = a^{-w(L_m^N)}\Lambda(L_m^N)$$

$$= \begin{cases} a^{m+N-w(T)}\big(\sigma_m\Lambda(L_1^N) - \sigma_{m-1}\Lambda(L_0^N) + \tau_m\Lambda(L_\infty^N)\big) & \text{(I, III, V, VII)}, \\ a^{m+N-2-w(T)}\big(\sigma_m\Lambda(L_1^N) - \sigma_{m-1}\Lambda(L_0^N) + \tau_m\Lambda(L_\infty^N)\big) & \text{(II, IV, VI, VIII)} \end{cases}$$

$$= \begin{cases} a^{m+N-w(T)}\Big(\sigma_m\Lambda(L_1^N) - \sigma_{m-1}\big(a\Lambda(L_1^N) + [\gamma(a)z^{N-r+1}, *]\big) \\ +\tau_m\Big(a\Lambda(L_1^N)\big((a^{-1}+a)z^{-1}-1\big) + [\delta(a)z^{N-r+2}, *]\Big)\Big) & \text{(I, V)}, \\ a^{m+N-w(T)}\Big(\sigma_m\Lambda(L_1^N) - \sigma_{m-1}\big(a\Lambda(L_1^N) + [\gamma(a)z^{N-r+2}, *]\big) \\ +\tau_m\Big(a\Lambda(L_1^N)\big((a^{-1}+a)z^{-1}-1\big) + [\delta(a)z^{N-r+3}, *]\Big)\Big) & \text{(III, VII)}, \\ a^{m+N-2-w(T)}\Big(\sigma_m\Lambda(L_1^N) - \sigma_{m-1}\big(a\Lambda(L_1^N) + [\gamma(a)z^{N-r}, *]\big) \\ +\tau_m\Big(a\Lambda(L_1^N)\big((a^{-1}+a)z^{-1}-1\big) + [\delta(a)z^{N-r+1}, *]\Big)\Big) & \text{(II, VI)}, \\ a^{m+N-2-w(T)}\Big(\sigma_m\Lambda(L_1^N) - \sigma_{m-1}\big(a\Lambda(L_1^N) + [\gamma(a)z^{N-r+2}, *]\big) \\ +\tau_m\Big(a\Lambda(L_1^N)\big((a^{-1}+a)z^{-1}-1\big) + [\delta(a)z^{N-r+3}, *]\Big)\Big) & \text{(IV, VIII)} \end{cases}$$

By $\Lambda(L_1^N) = a^{-N}\Lambda(L_1^0)$,

$$= \begin{cases} a^{m+N-w(T)}\Big(\sigma_m a^{-N}\Lambda(L_1^0) - \sigma_{m-1}\big(a^{-N+1}\Lambda(L_1^0) + [\gamma(a)z^{N-r+1}, *]\big) \\ +\tau_m\Big(a^{-N+1}\Lambda(L_1^0)\big((a^{-1}+a)z^{-1}-1\big) + [\delta(a)z^{N-r+2}, *]\Big)\Big) & \text{(I, V)}, \\ a^{m+N-w(T)}\Big(\sigma_m a^{-N}\Lambda(L_1^0) - \sigma_{m-1}\big(a^{-N+1}\Lambda(L_1^0) + [\gamma(a)z^{N-r+2}, *]\big) \\ +\tau_m\Big(a^{-N+1}\Lambda(L_1^0)\big((a^{-1}+a)z^{-1}-1\big) + [\delta(a)z^{N-r+3}, *]\Big)\Big) & \text{(III, VII)}, \\ a^{m+N-2-w(T)}\Big(\sigma_m a^{-N}\Lambda(L_1^0) - \sigma_{m-1}\big(a^{-N+1}\Lambda(L_1^0) + [\gamma(a)z^{N-r}, *]\big) \\ +\tau_m\Big(a^{-N+1}\Lambda(L_1^0)\big((a^{-1}+a)z^{-1}-1\big) + [\delta(a)z^{N-r+1}, *]\Big)\Big) & \text{(II, VI)}, \\ a^{m+N-2-w(T)}\Big(\sigma_m a^{-N}\Lambda(L_1^0) - \sigma_{m-1}\big(a^{-N+1}\Lambda(L_1^0) + [\gamma(a)z^{N-r+2}, *]\big) \\ +\tau_m\Big(a^{-N+1}\Lambda(L_1^0)\big((a^{-1}+a)z^{-1}-1\big) + [\delta(a)z^{N-r+3}, *]\Big)\Big) & \text{(IV, VIII)} \end{cases}$$

By $F(L) = \begin{cases} a^{1-w(T)}\Lambda(L_1^0) & \text{(I, III, V, VII)}, \\ a^{-1-w(T)}\Lambda(L_1^0) & \text{(II, IV, VI, VIII)} \end{cases}$,

$$
= \begin{cases}
F(L)\Big(\sigma_m a^{m-1} - \sigma_{m-1}a^m + \tau_m a^m\big((a^{-1}+a)z^{-1}-1\big)\Big) \\
-\sigma_{m-1}a^{m+N-w(T)}[\gamma(a)z^{N-r+1},*] + \tau_m a^{m+N-w(T)}[\delta(a)z^{N-r+2},*] & \text{(I, V)}, \\
F(L)\Big(\sigma_m a^{m-1} - \sigma_{m-1}a^m + \tau_m a^m\big((a^{-1}+a)z^{-1}-1\big)\Big) \\
-\sigma_{m-1}a^{m+N-w(T)}[\gamma(a)z^{N-r+2},*] + \tau_m a^{m+N-w(T)}[\delta(a)z^{N-r+3},*] & \text{(III, VII)}, \\
F(L)\Big(\sigma_m a^{m-1} - \sigma_{m-1}a^m + \tau_m a^m\big((a^{-1}+a)z^{-1}-1\big)\Big) \\
-\sigma_{m-1}a^{m+N-2-w(T)}[\gamma(a)z^{N-r},*] + \tau_m a^{m+N-2-w(T)}[\delta(a)z^{N-r+1},*] & \text{(II, VI)}, \\
F(L)\Big(\sigma_m a^{m-1} - \sigma_{m-1}a^m + \tau_m a^m\big((a^{-1}+a)z^{-1}-1\big)\Big) \\
-\sigma_{m-1}a^{m+N-2-w(T)}[\gamma(a)z^{N-r+2},*] + \tau_m a^{m+N-2-w(T)}[\delta(a)z^{N-r+3},*] & \text{(IV, VIII)}
\end{cases}
$$

$\sigma_m a^{m-1} - \sigma_{m-1}a^m + \tau_m a^m\big((a^{-1}+a)z^{-1}-1\big) = 1$ より,

$$
= \begin{cases}
F(L) - \sigma_{m-1}a^{m+N-w(T)}[\gamma(a)z^{N-r+1},*] + \tau_m a^{m+N-w(T)}[\delta(a)z^{N-r+2},*] & \text{(I, V)}, \\
F(L) - \sigma_{m-1}a^{m+N-w(T)}[\gamma(a)z^{N-r+2},*] + \tau_m a^{m+N-w(T)}[\delta(a)z^{N-r+3},*] & \text{(III, VII)}, \\
F(L) - \sigma_{m-1}a^{m+N-2-w(T)}[\gamma(a)z^{N-r},*] + \tau_m a^{m+N-2-w(T)}[\delta(a)z^{N-r+1},*] & \text{(II, VI)}, \\
F(L) - \sigma_{m-1}a^{m+N-2-w(T)}[\gamma(a)z^{N-r+2},*] + \tau_m a^{m+N-2-w(T)}[\delta(a)z^{N-r+3},*] & \text{(IV, VIII)}
\end{cases}
$$

By $\sigma_{m-1} = \begin{cases} [(-1)^{\frac{m+1}{2}}\frac{m-1}{2}z, *] & m\text{: odd } (\neq 1), \\ [(-1)^{\frac{m+2}{2}}, *] & m\text{: even} \end{cases}$ and $\tau_m = [\varepsilon(a)z, *]$ $\big(\varepsilon(a) \neq 0\big)$ $(m \neq 0, 1)$,

$$= \begin{cases} F(L) - (-1)^{\frac{m+1}{2}}\frac{m-1}{2}a^{m+N-w(T)}z^{-r+1}[\gamma(a)z^{N+1}, *] & \text{(I)}, \\ F(L) - (-1)^{\frac{m+1}{2}}\frac{m-1}{2}a^{m+N-2-w(T)}z^{-r+1}[\gamma(a)z^{N}, *] & \text{(II)}, \\ F(L) - (-1)^{\frac{m+1}{2}}\frac{m-1}{2}a^{m+N-w(T)}z^{-r+1}[\gamma(a)z^{N+2}, *] & \text{(III)}, \\ F(L) - (-1)^{\frac{m+1}{2}}\frac{m-1}{2}a^{m+N-2-w(T)}z^{-r+1}[\gamma(a)z^{N+2}, *] & \text{(IV)}, \\ F(L) - (-1)^{\frac{m+2}{2}}a^{m+N-w(T)}z^{-r+1}[\gamma(a)z^{N}, *] & \text{(V)}, \\ F(L) - (-1)^{\frac{m+2}{2}}a^{m+N-2-w(T)}z^{-r+1}[\gamma(a)z^{N-1}, *] & \text{(VI)}, \\ F(L) - (-1)^{\frac{m+2}{2}}a^{m+N-w(T)}z^{-r+1}[\gamma(a)z^{N+1}, *] & \text{(VII)}, \\ F(L) - (-1)^{\frac{m+2}{2}}a^{m+N-2-w(T)}z^{-r+1}[\gamma(a)z^{N+1}, *] & \text{(VIII)}. \end{cases}$$

Main result: Coefficient polynomials of the Kauffman polynomial of $L_m^N(T)$

Let L be a link. Let N be a positive integer. There exists an infinite family of links $\{L_m^N(T)\}_{m\in\mathbb{Z}}$ such that $f_i(L_m^N(T);a) = f_i(L;a)$ for $0 \le i \le s$,

where $s = \begin{cases} \text{(I)} & N, \\ \text{(II)} & N-1, \\ \text{(III)} & N+1, \\ \text{(IV)} & N+1, \\ \text{(V)} & N-1, \\ \text{(VI)} & N-2\ (N \neq 1), \\ \text{(VII)} & N, \\ \text{(VIII)} & N. \end{cases}$ In particular, $L_1^N(T)$ is equivalent to L.

Moreover, we have $f_{s+1}(L_m^N(T);a) \neq f_{s+1}(L_{m'}^N(T);a)$ for $m \neq m'$ except in the following case.

A link $L_1^0(T)$ with a tangle T of Type-0 satisfies

$$f_0(L_\infty^0(T);a) = (1+a^2)a^{-2\nu} f_0(L_0^0(T);a). \ \cdots (\dagger)$$

Remark

We consider a link $L_1^0(T)$ with a tangle T of Type-0 which satisfies

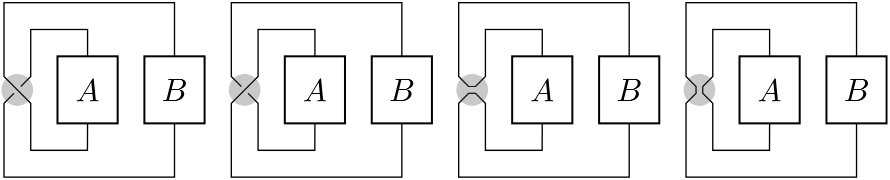
$$\begin{cases} F(L_0^0(T); a, z) = a^{2\nu} F(L_1^0(T); a, z), \\ F(L_\infty^0(T); a, z) = \big((a^{-1} + a)z^{-1} - 1\big) F(L_1^0(T); a, z). \end{cases} \qquad \cdots (\dagger\dagger)$$

Then, we see that the link $L_1^0(T)$ satisfies $(\dagger)$ and we have $F(L_1^0(T); a, z) = F(L_{-1}^0(T); a, z)$ and $F(L_m^N(T); a, z) = F(L; a, z)$ for any $m \in \mathbb{Z}$ and $N \in \mathbb{Z}_{>0}$.

A link $L_1^0(T)$ with a nugatory crossing and a tangle T of Type-0 as in the figure below satisfies the condition $(\dagger\dagger)$. In this case, we see that the link $L_m^N(T)$ is equivalent to the link L.

Question: Is there a link $L_1^0(T)$ with a non-nugatory crossing and a tangle T of Type-0 which satisfies the condition $(\dagger\dagger)$?

(We can find such a link $L_1^0(T)$ for the condition $(\dagger)$ easily.)



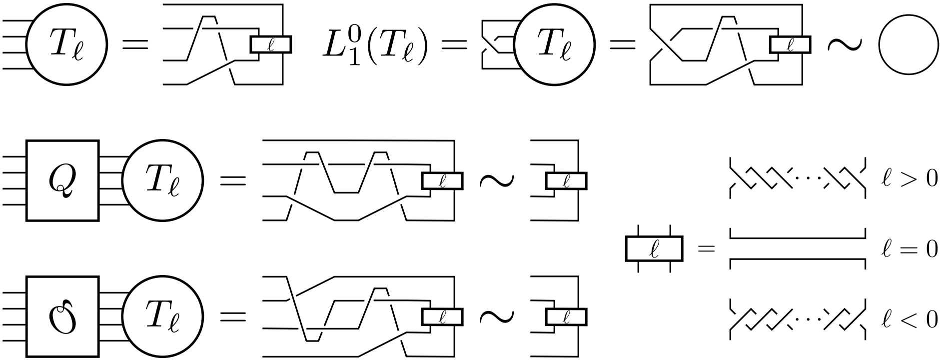
$L_1^0(T)$ $\quad L_{-1}^0(T)$ $\quad L_0^0(T)$ $\quad L_\infty^0(T)$

5. Knot $K_{m,\ell}^{N}$

Tangle T_ℓ

We consider a tangle T_ℓ with ℓ half twists for $\ell \in \mathbb{Z}$ as in the figure below.

We see that a knot $L_1^0(T_\ell)$ is the unknot.



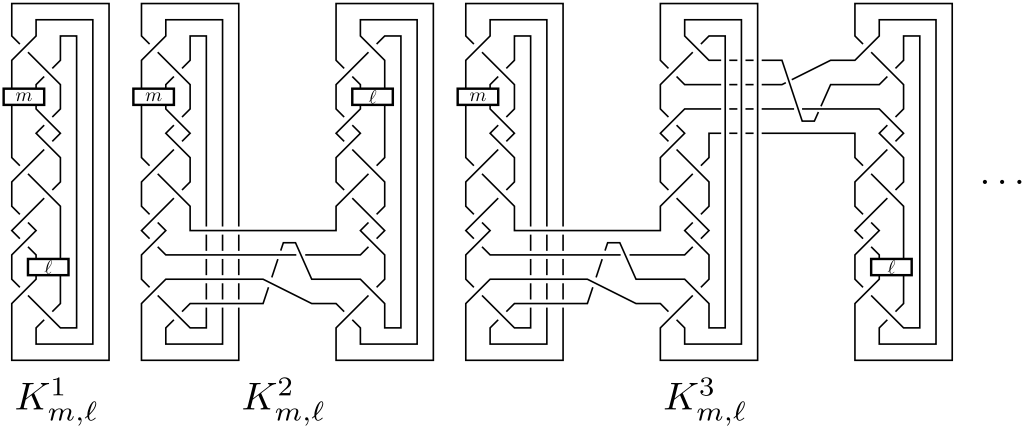
Knot $K^N_{m,\ell}$

We obtain a knot $K^N_{m,\ell}$ as in the figure below, which is equivalent to a knot $L^N_m(T_\ell)$.

By a property of the tangle S^N_m, we see that $K^N_{1,\ell}$ and $K^N_{m,-1}$ are the unknot.

By a property of the tangle Q, we see that $(K^N_{m,\ell})^*$ is equivalent to $-K^N_{-\ell,-m}$.

Therefore, $K^N_{m,-m}$ is a negative amphicheiral knot for any $m \in \mathbb{Z}$.



The HOMFLYPT polynomial of $K_{m,\ell}^N$

$$P(K_{m,\ell}^N; v, z) =$$

$$\begin{cases} 1-\zeta^N(1-v^{-m+1})(1-v^{-\ell-1})\left(1-\left(\dfrac{z}{v^{-1}-v}\right)^2\right) & (m,\ell)=(\text{odd},\text{odd}), \\[2ex] 1-z^2\zeta^{N-1}(v^{-2}+1)(1-v^{-m+1})\big(v^{-\ell}(v^{-1}-v)^2 \\ \quad +(1+v^2-2v^{-\ell})z^2\big)\left(1-\left(\dfrac{z}{v^{-1}-v}\right)^2\right) & (m,\ell)=(\text{odd},\text{even}), \\[2ex] 1-z^2\zeta^{N-1}(1+v^2)(1-v^{-\ell-1})\big(v^{-m}(v^{-1}-v)^2 \\ \quad +(v^{-2}+1-2v^{-m})z^2\big)\left(1-\left(\dfrac{z}{v^{-1}-v}\right)^2\right) & (m,\ell)=(\text{even},\text{odd}), \\[2ex] 1-\left(2z^2+z^4+v^{-m}\big((v^{-1}-v)^2-(3-v^2)z^2-z^4\big)\right) \\ \quad \left(2z^2+z^4+v^{-\ell}\big((v^{-1}-v)^2+(v^{-2}-3)z^2-z^4\big)\right)\left(1-\left(\dfrac{z}{v^{-1}-v}\right)^2\right) & (m,\ell)=[(\text{even},\text{even})\ N=1], \\[2ex] 1-z^4\zeta^{N-2}(v^{-1}+v)^2\big(v^{-m}(v^{-1}-v)^2+(v^{-2}+1-2v^{-m})z^2\big) \\ \quad \big(v^{-\ell}(v^{-1}-v)^2+(1+v^2-2v^{-\ell})z^2\big)\left(1-\left(\dfrac{z}{v^{-1}-v}\right)^2\right) & (m,\ell)=[(\text{even},\text{even})\ N\geq 2]. \end{cases}$$

The Kauffman polynomial of $K^N_{m,\ell}$

$$F(K^N_{m,\ell}; a, z) =$$

$$\begin{cases} 1-(-1)^{\frac{m+\ell-2}{2}\frac{m-1}{2}\frac{\ell+1}{2}}[(1+a^2)^{3N+1}a^{-3N+m+\ell-1}z^{N+1}, *] & (m,\ell)=(\text{odd},\text{odd}), \\ 1-(-1)^{\frac{m+\ell-1}{2}\frac{m-1}{2}}[(1+a^2)^{3N+1}a^{-3N+m+\ell-1}z^{N}, *] & (m,\ell)=(\text{odd},\text{even}), \\ 1-(-1)^{\frac{m+\ell-1}{2}\frac{\ell+1}{2}}[(1+a^2)^{3N+1}a^{-3N+m+\ell-1}z^{N}, *] & (m,\ell)=(\text{even},\text{odd}), \\ 1-(-1)^{\frac{m+\ell}{2}}[(1+a^2)^{3N+1}a^{-3N+m+\ell-1}z^{N-1}, *] & (m,\ell)=(\text{even},\text{even}). \end{cases}$$

The maximum term on z of $F(K^N_{m,\ell}; a, z)$

	$m \geq 2$	$m = 1$
$\ell \geq 1\ (N = 1)$	$[*, 3(1+a^2)a^{m+\ell-2}z^{m+\ell+8}]$	1
$\ell \geq 1\ (N \geq 2)$	$[*, 3 \times 2^{N-1}(1+a^2)a^{m+\ell-2}z^{10N+m+\ell-2}]$	1
$\ell = 0\ (N = 1)$	$[*, (1+a^2)a^{m-2}z^{m+8}]$	1
$\ell = 0\ (N \geq 2)$	$[*, 2^{N-1}(1+a^2)a^{m-2}z^{10N+m-2}]$	1
$\ell = -1$	1	1
$\ell \leq -2$	$[*, 2^N(1+a^2)a^{m+\ell-1}z^{10N+m-\ell-3}]$	1

	$m = 0$	$m \leq -1$
$\ell \geq 1\ (N = 1)$	$[*, (1+a^2)a^{\ell-1}z^{\ell+9}]$	$[*, 4(1+a^2)a^{m+\ell-1}z^{-m+\ell+9}]$
$\ell \geq 1\ (N \geq 2)$	$[*, 3 \times 2^{N-2}(1+a^2)a^{\ell-1}z^{10N+\ell-1}]$	$[*, 9 \times 2^{N-2}(1+a^2)a^{m+\ell-1}z^{10N-m+\ell-1}]$
$\ell = 0\ (N = 1)$	$[*, (a^{-1}+a)^2z^8]$	$[*, (1+a^2)a^{m-1}z^{-m+9}]$
$\ell = 0\ (N \geq 2)$	$[*, 2^{N-2}(a^{-1}+a)z^{10N-1}]$	$[*, 3 \times 2^{N-2}(1+a^2)a^{m-1}z^{10N-m-1}]$
$\ell = -1$	1	1
$\ell \leq -2$	$[*, 2^{N-1}(1+a^2)a^{\ell}z^{10N-\ell-2}]$	$[*, 3 \times 2^{N-1}(1+a^2)a^{m+\ell}z^{10N-m-\ell-2}]$

The minimum and maximum terms on a of $F(K^N_{m,\ell}; a, z)$

	$m \geq 3$	$m = 2$
$\ell \geq 1$	$[(-1)^N(2-z^2)(1-z^2)^{N-1}z^{2N+3}a^{-4N+1}, *]$	$[(-1)^{N+1}(1-3z^2+z^4)(1-z^2)^{N-1}z^{2N+1}a^{-4N+1}, *]$
$\ell = 0$	$[(-1)^{N+1}(2-z^2)(1-z^2)^N z^{2N+1}a^{-4N+1}, *]$	$[(-1)^N(1-3z^2+z^4)(1-z^2)^N z^{2N-1}a^{-4N+1}, *]$
$\ell = -1$	1	1
$\ell \leq -2$	$[(-1)^{N+1}\sigma_{\ell+1}(2-z^2)(1-z^2)^{N-1}z^{2N+1}a^{-4N+\ell+1}, *]$	$[(-1)^N\sigma_{\ell+1}(1-3z^2+z^4)(1-z^2)^{N-1}z^{2N-1}a^{-4N+\ell+1}, *]$

	$m = 1$	$m \leq 0$
$\ell \geq 1$	1	$[(-1)^{N+1}\sigma_{m-1}(1-z^2)^N z^{2N+1}a^{-4N+m-1}, *]$
$\ell = 0$	1	$[(-1)^N\sigma_{m-1}(1-z^2)^{N+1}z^{2N-1}a^{-4N+m-1}, *]$
$\ell = -1$	1	1
$\ell \leq -2$	1	$[(-1)^N\sigma_{m-1}\sigma_{\ell+1}(1-z^2)^N z^{2N-1}a^{-4N+m+\ell-1}, *]$

	$m \geq 2$	$m = 1$
$\ell \geq 0$	$[*, (-1)^N\sigma_{m-1}\sigma_{\ell+1}(1-z^2)^N z^{2N-1}a^{4N+m+\ell+1}]$	1
$\ell = -1$	1	1
$\ell = -2$	$[*, (-1)^{N+1}\sigma_{m-1}(1-3z^2+z^4)(1-z^2)^{N-1}z^{2N-1}a^{4N+m-1}]$	1
$\ell \leq -3$	$[*, (-1)^N\sigma_{m-1}(2-z^2)(1-z^2)^{N-1}z^{2N+1}a^{4N+m-1}]$	1

	$m = 0$	$m \leq -1$
$\ell \geq 0$	$[*, (-1)^{N+1}\sigma_{\ell+1}(1-z^2)^{N+1}z^{2N-1}a^{4N+\ell+1}]$	$[*, (-1)^N\sigma_{\ell+1}(1-z^2)^N z^{2N+1}a^{4N+\ell+1}]$
$\ell = -1$	1	1
$\ell = -2$	$[*, (-1)^N(1-3z^2+z^4)(1-z^2)^N z^{2N-1}a^{4N-1}]$	$[*, (-1)^{N+1}(1-3z^2+z^4)(1-z^2)^{N-1}z^{2N+1}a^{4N-1}]$
$\ell \leq -3$	$[*, (-1)^{N+1}(2-z^2)(1-z^2)^N z^{2N+1}a^{4N-1}]$	$[*, (-1)^N(2-z^2)(1-z^2)^{N-1}z^{2N+3}a^{4N-1}]$

Remark

Let $(m,\ell) \neq (m',\ell')$, $m, m' \neq 1$, $\ell, \ell' \neq -1$.

$P(K^N_{m,\ell}; v, z) = P(K^{N'}_{m',\ell'}; v, z) \iff$

$(N', m', \ell') = (N, \ell+2, m-2)$, $(m,\ell) = (\text{odd}, \text{odd})$, $(\text{odd}, \text{even})$, $[(\text{even}, \text{even})\ N \geq 2]$.

By the above information of $F(K^N_{m,\ell}; a, z)$ and the 2nd minimum and maximum terms on a, we see that all the knots $K^N_{m,\ell}$ $(m \neq 1, \ell \neq -1)$ are mutually inequivalent.

Thank you for your attention.